Selected Topics in Engineering Mathematics: Least Squares Problems

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Reference

1. R. A. Horn and C. R. Johnson, Matrix Analysis, 2nd ed., New York: Cambridge University Press, 2013.
   [HJ2013]
2. G. H. Golub and C. F. Van Loan, Matrix Computations, 4th ed., Baltimore: The Johns Hopkins University Press, 2013.
   [GVL2013]
3. J.-J. Ding. (2023). Selected Topics in Engineering Mathematics [PowerPoint slides].

Outline

Motivation

- Find a vector $\mathbf{x} \in \mathbb{C}^N$ such that

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{1}$$

  - The data matrix $\mathbf{A} \in \mathbb{C}^{M \times N}$ is given.
  - The observation vector $\mathbf{b} \in \mathbb{C}^M$ is given.
- The number of equations is M.
- The number of unknowns is N.
- Underdetermined systems: $M < N$
- Overdetermined systems: $M > N$

Questions

- How many solutions to (1)?

Examples of (1)

Underdetermined Systems

$$\underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{b}}. \tag{2}$$

The solutions to (2) are

$$\mathbf{x} = \begin{bmatrix} -2c \\ c \end{bmatrix},$$

where $c \in \mathbb{C}$.

Overdetermined Systems

$$\underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\mathbf{b}}. \tag{3}$$

There are no solutions to (3).

Usually, an overdetermine system has no exact solution.

The Least Squares Problem (1/2)

- We aim to find a solution such that

$$\mathbf{A}\mathbf{x} \approx \mathbf{b}. \tag{4}$$

- The vector p-norm measures the proximity of $\mathbf{A}\mathbf{x}$ to $\mathbf{b}$.

$$\left\|\mathbf{A}\mathbf{x} - \mathbf{b}\right\|_p , \tag{5}$$

where $p \in [1, \infty)$.

The Least Squares Problem (2/2)

The Least Squares (LS) Problem ($p = 2$)

$$\min_{\mathbf{x}\in\mathbb{C}^N} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \tag{6}$$

- The LS problem (6) is tractable for two reasons
  1. The solutions to (6) can be found readily.
     - Completion of squares
     - The (complex) derivatives of the objective function
  2. The ℓ_2 norm is invariant under unitary transformations. Namely,

$$\|\mathbf{U}\mathbf{v}\|_2 = \|\mathbf{v}\|_2 \,, \tag{7}$$

for a unitary matrix $\mathbf{U}$.

Outline

The LS Solution(s)

$$\min_{\mathbf{x}\in\mathbb{C}^N} \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2 \tag{8}$$

- Let $\mathbf{x}_{\mathrm{LS}}$ be a solution to the LS problem (6).

Questions

- Does $\mathbf{x}_{\mathrm{LS}}$ exist?
- How do we find $\mathbf{x}_{\mathrm{LS}}$?
- Is the LS solution $\mathbf{x}_{\mathrm{LS}}$ unique?

The Normal Equation

Normal Equation

If $\mathbf{A}$ has full column rank, then there is a unique LS solution $\mathbf{x}_{\mathrm{LS}}$, and it satisfies

$$\mathbf{A}^{\mathsf{H}}\mathbf{A}\mathbf{x}_{\mathrm{LS}} = \mathbf{A}^{\mathsf{H}}\mathbf{b}. \tag{9}$$

- See Section 5.3.1 in [GVL2013] for the complete arguments
- The minimum residual $\mathbf{r}_{\mathrm{LS}}$

$$\mathbf{r}_{\mathrm{LS}} \triangleq \mathbf{b} - \mathbf{A}\mathbf{x}_{\mathrm{LS}}. \tag{10}$$

- The size of $\mathbf{r}_{\mathrm{LS}}$

$$\rho_{\mathrm{LS}} \triangleq \left\|\mathbf{A}\mathbf{x}_{\mathrm{LS}} - \mathbf{b}\right\|_2 . \tag{11}$$

Remarks on the Normal Equation

- Assume that $\mathbf{A} \in \mathbb{C}^{M \times N}$ and $M \geq N$.
- If $\mathbf{A}$ has full column rank, then
  - $\operatorname{rank}(\mathbf{A}) = N$.
  - $\operatorname{rank}(\mathbf{A}^{\mathsf{H}}\mathbf{A}) = N$.
  - $\mathbf{A}^{\mathsf{H}}\mathbf{A}$ is invertible.
- If $\mathbf{A}$ has full column rank, then the LS solution can be uniquely found by

$$\mathbf{x}_{\mathrm{LS}} \triangleq \left(\mathbf{A}^{\mathsf{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}}\mathbf{b}. \tag{12}$$

- Interpretations of $\mathbf{x}_{\mathrm{LS}}$
  - Wiener-Hopf equation in Adaptive Signal Processing
  - Singular values and singular vectors of $\mathbf{A}$

The LS Solution and the SVD (1/4)

- We assume that $\text{rank}(\mathbf{A}) = N$.
- The SVD of $\mathbf{A}$ is denoted by

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{H}} = \sum_{i=1}^{N} \sigma_i \mathbf{u}_i \mathbf{v}_i^{\mathsf{H}}. \tag{13}$$

- The matrix $\boldsymbol{\Sigma}$ is

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_N \\ \mathbf{0}_{(M-N)\times N} \end{bmatrix}, \qquad \boldsymbol{\Sigma}_N = \text{diag}\left(\sigma_1, \sigma_2, \ldots, \sigma_N\right). \tag{14}$$

- The singular values satisfy $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_N > 0$.
- The unitary matrices $\mathbf{U}$ and $\mathbf{V}$ comprise left and right singular vectors.

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_M \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_N \end{bmatrix}. \tag{15}$$

The LS Solution and the SVD (2/4)

- The unitary matrices $\mathbf{U}$ and $\mathbf{V}$ satisfy

$$\mathbf{U}^{\mathsf{H}}\mathbf{U} = \mathbf{I}_M, \qquad \mathbf{V}^{\mathsf{H}}\mathbf{V} = \mathbf{I}_N, \tag{16}$$

- Substituting (13) into (12) leads to

$$\mathbf{x}_{\mathrm{LS}} = \left(\left(\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{H}}\right)^{\mathsf{H}} \left(\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{H}}\right) \right)^{-1} \left(\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{H}}\right)^{\mathsf{H}} \mathbf{b} \tag{17}$$

$$= \left(\mathbf{V}\mathbf{\Sigma}^{\mathsf{H}}\mathbf{U}^{\mathsf{H}}\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{H}}\right)^{-1} \mathbf{V}\mathbf{\Sigma}^{\mathsf{H}}\mathbf{U}^{\mathsf{H}}\mathbf{b} \tag{18}$$

$$= \mathbf{V}\left(\mathbf{\Sigma}^{\mathsf{H}}\mathbf{\Sigma}\right)^{-1} \mathbf{V}^{\mathsf{H}}\mathbf{V}\mathbf{\Sigma}^{\mathsf{H}}\mathbf{U}^{\mathsf{H}}\mathbf{b} \tag{19}$$

$$= \mathbf{V}\left(\mathbf{\Sigma}^{\mathsf{H}}\mathbf{\Sigma}\right)^{-1} \mathbf{\Sigma}^{\mathsf{H}}\mathbf{U}^{\mathsf{H}}\mathbf{b}. \tag{20}$$

The LS Solution and the SVD (3/4)

- From (14), the matrix associated with $\mathbf{\Sigma}$ can be expressed as

$$\left(\mathbf{\Sigma}^{\mathsf{H}}\mathbf{\Sigma}\right)^{-1}\mathbf{\Sigma}^{\mathsf{H}} = \left(\begin{bmatrix}\mathbf{\Sigma}_N \\ \mathbf{0}_{(M-N)\times N}\end{bmatrix}^{\mathsf{H}}\begin{bmatrix}\mathbf{\Sigma}_N \\ \mathbf{0}_{(M-N)\times N}\end{bmatrix}\right)^{-1}\begin{bmatrix}\mathbf{\Sigma}_N \\ \mathbf{0}_{(M-N)\times N}\end{bmatrix}^{\mathsf{H}} \tag{21}$$

$$= \left(\mathbf{\Sigma}_N^{\mathsf{H}}\mathbf{\Sigma}_N\right)^{-1}\begin{bmatrix}\mathbf{\Sigma}_N^{\mathsf{H}} & \mathbf{0}_{N\times(M-N)}\end{bmatrix} \tag{22}$$

$$= \begin{bmatrix}\mathbf{\Sigma}_N^{-1} & \mathbf{0}_{N\times(M-N)}\end{bmatrix} \tag{23}$$

$$= \begin{bmatrix}\sigma_1^{-1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^{-1} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_N^{-1} & 0 & \dots & 0\end{bmatrix}. \tag{24}$$

The LS Solution and the SVD (4/4)

- Substituting (24) and (15) into (20) gives

$$\mathbf{x}_{\mathrm{LS}} = \sum_{i=1}^{N} \frac{\mathbf{u}_i^{\mathsf{H}}\mathbf{b}}{\sigma_i}\mathbf{v}_i. \tag{25}$$

- $\mathbf{x}_{\mathrm{LS}}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N\}$.
- Two factors influence the combination coefficients
  1. The inner product $\langle \mathbf{b}, \mathbf{u}_i \rangle \triangleq \mathbf{u}_i^{\mathsf{H}}\mathbf{b}$
  2. The singular value σ_i

The Size of the Minimum Residual

- (Exercise) It can be shown that the size of the minimum residual (denoted by ρ_{LS}) satisfies

$$\rho_{\mathrm{LS}}^2 = \sum_{i=N+1}^{M} \left| \mathbf{u}_i^{\mathsf{H}} \mathbf{b} \right|^2. \tag{26}$$

Outline

Motivation

- (The normal equation of LS problems) If $\mathbf{A}$ has full column rank, then there is an unique LS solution $\mathbf{x}_{\mathrm{LS}}$ and

$$\mathbf{A}^{\mathsf{H}}\mathbf{A}\mathbf{x}_{\mathrm{LS}} = \mathbf{A}^{\mathsf{H}}\mathbf{b}. \tag{27}$$

- What if $\mathbf{A}$ is rank-deficient? Namely, $\mathbf{A} \in \mathbb{C}^{M\times N}$, and

$$\mathrm{rank}(\mathbf{A}) = r < N. \tag{28}$$

- Logical reasoning:

$$p \to q \qquad \equiv \qquad \sim q \to \sim p \tag{29}$$

Example 1

- We consider the following equations

$$\underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{\mathbf{b}}. \tag{30}$$

- The associated LS problem is cast as

$$\min_{\mathbf{x} \in \mathbb{C}^N} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 \tag{31}$$

Observations

- There are infinitely many solutions to (30).
- If $\mathbf{x}^\star$ is a solution to (30), then $\|\mathbf{A}\mathbf{x}^\star - \mathbf{b}\|_2 = 0$.
- The LS problem (31) has an infinite number of solutions.

The Minimum 2-Norm Solution

- We define the objective function

$$\psi(\mathbf{x}) \triangleq \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2 . \tag{32}$$

- The minimum of $\psi(\mathbf{x})$ is denoted by $\psi_{\min}$.
- The set of all minimizers

$$\mathcal{X} \triangleq \{\mathbf{x} \in \mathbb{C}^N \mid \psi(\mathbf{x}) = \psi_{\min}\}. \tag{33}$$

- The set $\mathcal{X}$ is convex [GVL2013, Section 5.5.1].
- Among the vectors in $\mathcal{X}$, we select the unique element with the minimum 2-norm:

$$\mathbf{x}_{\mathrm{LS}} \triangleq \arg\min_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x}\|_2 \tag{34}$$

The Rank-Deficient LS Solution with the Minimum 2-Norm

Theorem (Revised from Theorem 5.5.1 in [GVL2013])

Let the SVD of $\mathbf{A}$ *be* $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{H}} \in \mathbb{C}^{M\times N}$ *with* $\operatorname{rank}(\mathbf{A}) = r$. *The singular vectors satisfy*

$$\mathbf{U} \triangleq \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_M \end{bmatrix}, \qquad \mathbf{V} \triangleq \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_N \end{bmatrix}. \tag{35}$$

Assume that $\mathbf{b} \in \mathbb{C}^M$. *Then*

$$\mathbf{x}_{\mathrm{LS}} = \sum_{i=1}^{r} \frac{\mathbf{u}_i^{\mathsf{H}}\mathbf{b}}{\sigma_i}\mathbf{v}_i \tag{36}$$

minimizes $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2$ *and has the smallest* 2*-norm of all minimizers.*

The LS Solution in Example 1

- We consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \end{bmatrix}$ in (30).
- The rank of $\mathbf{A}$ is 1.
- The SVD of $\mathbf{A}$

$$\mathbf{u}_1 = 1, \qquad \sigma_1 = \sqrt{5}, \qquad \mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}. \tag{37}$$

- The set of minimizers

$$\mathcal{X} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2 \;\middle|\; x_1 + 2x_2 = 1 \right\}. \tag{38}$$

The LS Solution in Example 1

- The rank-deficient LS solution with the minimum 2-norm

$$\mathbf{x}_{\mathrm{LS}} \triangleq \arg\min_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x}\|_2 = \arg\min_{\mathbf{x}\in\mathcal{X}} \sqrt{|x_1|^2 + |x_2|^2} \tag{39}$$

- We decompose the elements x_1 and x_2 into the real and imaginary parts:

$$x_1 = \mathrm{Re}\{x_1\} + \jmath\mathrm{Im}\{x_1\}, \tag{40}$$

$$x_2 = \mathrm{Re}\{x_2\} + \jmath\mathrm{Im}\{x_2\}. \tag{41}$$

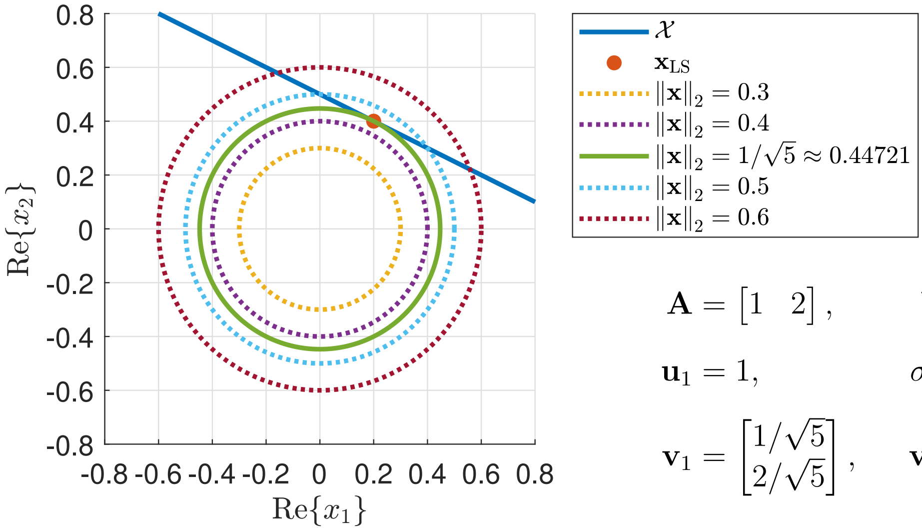
- The LS solution $\mathbf{x}_{\mathrm{LS}}$

$$\mathbf{x}_{\mathrm{LS}} \triangleq \arg\min_{\mathbf{x}\in\mathcal{X}} \sqrt{(\mathrm{Re}\{x_1\})^2 + (\mathrm{Im}\{x_1\})^2 + (\mathrm{Re}\{x_2\})^2 + (\mathrm{Im}\{x_2\})^2} \tag{42a}$$

$$\text{subject to} \quad \mathrm{Re}\{x_1\} + 2\mathrm{Re}\{x_2\} = 1, \tag{42b}$$

$$\mathrm{Im}\{x_1\} + 2\mathrm{Im}\{x_2\} = 0. \tag{42c}$$

Illustration of the LS Solution



$$\mathbf{A} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \qquad \mathbf{b} = 1,$$

$$\mathbf{u}_1 = 1, \qquad \sigma_1 = \sqrt{5},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}.$$

Outline

Pseudo-inverse Using the SVD

- Let $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{H}} \in \mathbb{C}^{M\times N}$ where $\mathrm{rank}(\mathbf{A}) = r \leq \min\{M, N\}$ (c.f. page 21).
- We define a matrix $\boldsymbol{\Sigma}^{\dagger}$ (c.f. page 14)

$$\boldsymbol{\Sigma}^{\dagger} \triangleq \begin{bmatrix} \sigma_1^{-1} & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^{-1} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_r^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{C}^{N\times M}. \tag{43}$$

- The pseudo-inverse of $\mathbf{A}$ is defined as

$$\mathbf{A}^{\dagger} \triangleq \mathbf{V}\boldsymbol{\Sigma}^{\dagger}\mathbf{U}^{\mathsf{H}} \in \mathbb{C}^{N\times M}. \tag{44}$$

Example of the Pseudo-Inverse

- We consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \end{bmatrix}$ in (30).
- The rank of $\mathbf{A}$ is 1.
- The SVD of $\mathbf{A}$

$$\mathbf{u}_1 = 1, \qquad \sigma_1 = \sqrt{5}, \qquad \mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}. \tag{45}$$

- The pseudo-inverse of $\mathbf{A}$

$$\mathbf{A}^\dagger = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} \sigma_1^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \end{bmatrix}^{\mathsf{H}} = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}. \tag{46}$$

Properties of the Pseudo-Inverse (1/5)

- Let $\mathbf{A} \in \mathbb{C}^{M \times N}$
- Let $\mathbf{A}^\dagger$ be the pseudo-inverse of $\mathbf{A}$
- Let $\mathbf{b} \in \mathbb{C}^M$.
- The LS solution $\mathbf{x}_{\mathrm{LS}}$ satisfies

$$\mathbf{x}_{\mathrm{LS}} = \mathbf{A}^\dagger \mathbf{b}. \tag{47}$$

- Remarks
  - Comparison: (25) and (36).
  - Initially, we aim to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Properties of the Pseudo-Inverse (2/5)

- If $\operatorname{rank}(\mathbf{A}) = N$, then

$$\mathbf{A}^{\dagger} = \left(\mathbf{A}^{\mathsf{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}}. \tag{48}$$

- If $M = N = \operatorname{rank}(\mathbf{A})$, then

$$\begin{aligned}
\mathbf{A}^{\dagger} &= \left(\mathbf{A}^{\mathsf{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathsf{H}} && (49)\\
&= \mathbf{A}^{-1}\left(\mathbf{A}^{\mathsf{H}}\right)^{-1}\mathbf{A}^{\mathsf{H}} && (50)\\
&= \mathbf{A}^{-1}. && (51)
\end{aligned}$$

Properties of the Pseudo-Inverse (3/5)

- The pseudo-inverse $\mathbf{A}^\dagger$ satisfies the four Moore-Penrose conditions:

$$\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}, \tag{52}$$

$$\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger, \tag{53}$$

$$\left(\mathbf{A}\mathbf{A}^\dagger\right)^{\mathsf{H}} = \mathbf{A}\mathbf{A}^\dagger, \tag{54}$$

$$\left(\mathbf{A}^\dagger\mathbf{A}\right)^{\mathsf{H}} = \mathbf{A}^\dagger\mathbf{A}. \tag{55}$$

- (Exercise) Prove the four Moore-Penrose conditions.

Properties of the Pseudo-Inverse (4/5)

- The matrix $\mathbf{A}\mathbf{A}^{\dagger}$ can be expressed as

$$\mathbf{A}\mathbf{A}^{\dagger} = \sum_{i=1}^{r} \mathbf{u}_i \mathbf{u}_i^{\mathsf{H}}, \tag{56}$$

  where $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ are the left singular vectors of $\mathbf{A}$.

- The matrix $\mathbf{A}^{\dagger}\mathbf{A}$ can be expressed as

$$\mathbf{A}^{\dagger}\mathbf{A} = \sum_{i=1}^{r} \mathbf{v}_i \mathbf{v}_i^{\mathsf{H}}, \tag{57}$$

  where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are the right singular vectors of $\mathbf{A}$.

Properties of the Pseudo-Inverse (5/5)

- The size of the minimum residual satisfies

$$\rho_{\mathrm{LS}} = \left\| \left(\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger} \right) \mathbf{b} \right\|_2 . \tag{58}$$

Outline

Concluding Remarks

- The LS problem

$$\min_{\mathbf{x}\in\mathbb{C}^N} \|\mathbf{Ax}-\mathbf{b}\|_2$$

  - Normal equations
  - Full-rank LS
  - Rank-deficient LS
  - Pseudo inverse
- Extensions
  - Weighted least squares (WLS)
  - Total least squares (TLS)
  - Constrained least squares (CLS)
  - Recursive least squares (RLS)