

© 10-D Prime Factor Algorithm

[Ref] A. V. Oppenheim, *Discrete-Time Signal Processing*, London: Prentice-Hall, 3rd ed., 2010.

N 可以是任意整數

If $N = P_1^{k_1} P_2^{k_2} \dots P_M^{k_M}$

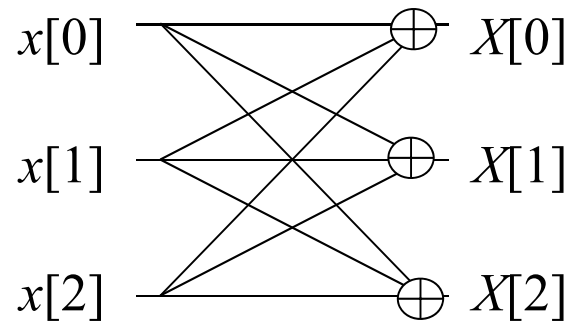
$P_1, P_2, P_3, \dots, P_M$ 不一定是 prime number,
但彼此互質

P_1, P_2, \dots, P_M are small integers and prime to each other

the powers k_1, k_2, \dots, k_M are small

then using the prime factor FFT to implement the N -point DFT may require fewer real multiplications.

3-point DFT butterfly:



Needs 4 complex multiplications (12 real multiplications)

N -point DFT butterfly: needs $3(N-1)(N-1)$ real multiplications

然而，可以使用特殊的方法，讓 N -point DFT 的乘法量大幅減少
(即使 $N \neq 2^k$)

例如 pages 346, 347, 353, 354

- Detail of the implementation method of the prime factor algorithm

$$F[m] = \sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N}mn} \quad \begin{array}{l} n = 0, 1, \dots, N-1, \\ m = 0, 1, \dots, N-1 \end{array}$$

Case 1: Suppose that $N = P_1 \times P_2$, P_1 is prime to P_2



拆成 P_2 個 P_1 -point DFTs , 和 P_1 個 P_2 -point DFTs

當 P_1, P_2 互質時，必可找到 n_1, n_2 使得

$$n = ((n_1 P_1 + n_2 P_2))_N \quad m = ((m_1 P_1 + m_2 P_2))_N \quad ((\))_N: \text{除以 } N \text{ 的餘數}$$

$$n_1, m_1 = 0, 1, \dots, P_2 - 1, \quad n_2, m_2 = 0, 1, \dots, P_1 - 1$$

且每一個 n_1, n_2 對應到唯一一個 n

(Proof):

First, if P_1 is prime to P_2 , we can always find two integers d_1 and d_2 such that

$$d_1P_1 + d_2P_2 = 1$$

Therefore, for any n ,

$$d_1nP_1 + d_2nP_2 = n$$

Suppose that

$$\left((d_1n) \right)_{P_2} = n_1, \quad \left((d_2n) \right)_{P_1} = n_2$$

then

$$d_1n = n_1 + k_1P_2, \quad d_2n = n_2 + k_2P_1$$

$$d_1nP_1 + d_2nP_2 = n_1P_1 + n_2P_2 + (k_1 + k_2)P_1P_2 = n$$

$$n_1P_1 + n_2P_2 = n - (k_1 + k_2)N$$

If $0 \leq n \leq N-1$, then

$$\left((n_1P_1 + n_2P_2) \right)_N = n$$

例子：當 $N = 15, P_1 = 3, P_2 = 5,$

$$0 = ((0 \cdot P_1 + 0 \cdot P_2))_{15}$$

$$10 = ((0 \cdot P_1 + 2 \cdot P_2))_{15}$$

$$1 = ((2 \cdot P_1 + 2 \cdot P_2))_{15}$$

$$11 = ((2 \cdot P_1 + 1 \cdot P_2))_{15}$$

$$2 = ((4 \cdot P_1 + 1 \cdot P_2))_{15}$$

$$12 = ((4 \cdot P_1 + 0 \cdot P_2))_{15}$$

$$3 = ((1 \cdot P_1 + 0 \cdot P_2))_{15}$$

$$13 = ((1 \cdot P_1 + 2 \cdot P_2))_{15}$$

$$4 = ((3 \cdot P_1 + 2 \cdot P_2))_{15}$$

$$14 = ((3 \cdot P_1 + 1 \cdot P_2))_{15}$$

$$5 = ((0 \cdot P_1 + 1 \cdot P_2))_{15}$$

$$6 = ((2 \cdot P_1 + 0 \cdot P_2))_{15}$$

$$7 = ((4 \cdot P_1 + 2 \cdot P_2))_{15}$$

$$8 = ((1 \cdot P_1 + 1 \cdot P_2))_{15}$$

$$9 = ((3 \cdot P_1 + 0 \cdot P_2))_{15}$$

$$F[m] = \sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N}mn} \quad N = P_1 \times P_2$$

$$m = ((m_1 P_1 + m_2 P_2))_N = m_1 P_1 + m_2 P_2 + c_1 N$$

$$n = ((n_1 P_1 + n_2 P_2))_N = n_1 P_1 + n_2 P_2 + c_2 N$$

$$\begin{aligned} e^{-j\frac{2\pi}{N}mn} &= e^{-j\frac{2\pi}{N}(m_1 P_1 + m_2 P_2 + c_1 N)(n_1 P_1 + n_2 P_2 + c_2 N)} \\ &= e^{-j\frac{2\pi}{N}[(m_1 P_1 + m_2 P_2)(n_1 P_1 + n_2 P_2)]} e^{-j\frac{2\pi}{N}[c_1 N(n_1 P_1 + n_2 P_2)]} e^{-j\frac{2\pi}{N}[c_2 N(m_1 P_1 + m_2 P_2)]} e^{-j\frac{2\pi}{N}[c_1 c_2 N^2]} \\ &= e^{-j\frac{2\pi}{N}[(m_1 P_1 + m_2 P_2)(n_1 P_1 + n_2 P_2)]} \end{aligned}$$

$$\begin{aligned}
F[((m_1 P_1 + m_2 P_2))_N] &= \sum_{n=0}^{N-1} f[((n_1 P_1 + n_2 P_2))_N] e^{-j \frac{2\pi}{P_1 P_2} (m_1 P_1 + m_2 P_2)(n_1 P_1 + n_2 P_2)} \\
&= \sum_{n=0}^{N-1} f[((n_1 P_1 + n_2 P_2))_N] e^{-j \frac{2\pi}{P_1 P_2} (m_1 n_1 P_1 P_1 + m_2 n_2 P_2 P_2 + m_1 n_2 P_1 P_2 + m_2 n_1 P_2 P_1)} \\
&= \sum_{n=0}^{N-1} f[((n_1 P_1 + n_2 P_2))_N] e^{-j \frac{2\pi}{P_2} m_1 P_1 n_1} e^{-j \frac{2\pi}{P_1} m_2 P_2 n_2} \\
&= \sum_{n_2=0}^{P_1-1} \left\{ \sum_{n_1=0}^{P_2-1} f[((n_1 P_1 + n_2 P_2))_N] e^{-j \frac{2\pi}{P_2} m_1 P_1 n_1} \right\} e^{-j \frac{2\pi}{P_1} m_2 P_2 n_2}
\end{aligned}$$

Step 2

Step 3

$$n_1, m_1 = 0, 1, \dots, P_2 - 1, \quad n_2, m_2 = 0, 1, \dots, P_1 - 1$$

Step 1 令 $g[n_1, n_2] = f[((n_1 P_1 + n_2 P_2))_N]$

Step 2 固定 n_2 ，對 n_1 做 P_2 -point DFT

$$\hat{G}_1[m_3, n_2] = \sum_{n_1=0}^{P_2-1} g[n_1, n_2] e^{-j \frac{2\pi}{P_2} m_3 n_1}$$

$$G_1[m_1, n_2] = \hat{G}_1[((P_1 m_1))_{P_2}, n_2]$$

n_2 有 P_1 個值，所以有 P_1 個 P_2 -point DFTs

Step 3 固定 m_3 ，對 n_2 做 P_1 -point DFT

$$\hat{G}_2[m_1, m_4] = \sum_{n_2=0}^{P_1-1} G_1[m_1, n_2] e^{-j \frac{2\pi}{P_1} m_4 n_2}$$

$$G_2[m_1, m_2] = \hat{G}_2[m_1, ((P_2 m_2))_{P_1}]$$

m_3 有 P_2 個值，所以有 P_2 個 P_1 -point DFTs

Step 4 $F[((m_1 P_1 + m_2 P_2))_N] = G_2[m_1, m_2]$

$$F[m] = \sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N}mn} \quad n = 0, 1, \dots, N-1, \quad m = 0, 1, \dots, N-1$$

Case 2: Suppose that $N = P_1 \times P_2$, P_1 is not prime to P_2



拆成 P_2 個 P_1 -point DFTs, P_1 個 P_2 -point DFTs, 和 twiddle factors

$$\text{令 } n = n_1 P_1 + n_2 \quad m = m_1 + m_2 P_2$$

$$n_1, m_1 = 0, 1, \dots, P_2 - 1, \quad n_2, m_2 = 0, 1, \dots, P_1 - 1$$

$$\begin{aligned} F[m_1 + m_2 P_2] &= \sum_{n=0}^{N-1} f[n_1 P_1 + n_2] e^{-j\frac{2\pi}{N}(m_1 + m_2 P_2)(n_1 P_1 + n_2)} \\ &= \sum_{n=0}^{N-1} f[n_1 P_1 + n_2] e^{-j\frac{2\pi}{P_1 P_2}(m_1 n_1 P_1 + m_1 n_2 + m_2 n_1 P_1 P_2 + m_2 n_2 P_2)} \\ &= \sum_{n=0}^{N-1} f[n_1 P_1 + n_2] e^{-j\frac{2\pi}{P_2} m_1 n_1} e^{-j\frac{2\pi}{P_1} m_2 n_2} e^{-j\frac{2\pi}{P_1 P_2} m_1 n_2} \\ &= \sum_{n_2=0}^{P_1-1} \left\{ \sum_{n_1=0}^{P_2-1} f[n_1 P_1 + n_2] e^{-j\frac{2\pi}{P_2} m_1 n_1} \right\} e^{-j\frac{2\pi}{N} m_1 n_2} e^{-j\frac{2\pi}{P_1} m_2 n_2} \end{aligned}$$

$$F[m_1 + m_2 P_2] = \sum_{n_2=0}^{P_1-1} \left\{ \sum_{n_1=0}^{P_2-1} f[n_1 P_1 + n_2] e^{-j \frac{2\pi}{P_2} m_1 n_1} \right\} e^{-j \frac{2\pi}{N} m_1 n_2} e^{-j \frac{2\pi}{P_1} m_2 n_2}$$

$e^{-j \frac{2\pi}{N} m_1 n_2}$ 被稱為 **twiddle factor**，需要額外的乘法

$$n_1, m_1 = 0, 1, \dots, P_2 - 1, \quad n_2, m_2 = 0, 1, \dots, P_1 - 1$$

Number of twiddle factors: $P_1 \times P_2 = N$

excluding the case where $m_1 = 0$ or $n_2 = 0$

Number of twiddle factors: $(P_1 - 1) \times (P_2 - 1)$

Step 1 令 $g[n_1, n_2] = f[n_1 P_1 + n_2]$

Step 2 固定 n_2 ，對 n_1 作 P_2 -point DFT

$$G_1[m_1, n_2] = \sum_{n_1=0}^{P_2-1} g[n_1, n_2] e^{-j \frac{2\pi}{P_2} m_1 n_1}$$

n_2 有 P_1 個值，所以有 P_1 個 P_2 -point DFTs

Step 3 $G_2[m_1, n_2] = G_1[m_1, n_2] e^{-j \frac{2\pi}{N} m_1 n_2}$ twiddle factors

Step 4 固定 m_1 ，對 n_2 做 P_2 個 P_1 -point DFT

$$G_3[m_1, m_2] = \sum_{n_2=0}^{P_1-1} G_2[m_1, n_2] e^{-j \frac{2\pi}{P_1} m_2 n_2}$$

m_1 有 P_2 個值，所以有 P_2 個 P_1 -point DFTs

Step 5 $F[m_1 + m_2 P_2] = G_3[m_1, m_2]$

◎ 10-E FFT 的乘法量的計算

假設 $N = P_1 \times P_2$, P_1 is prime to P_2

P_1 -point DFT 的乘法量為 B_1 , P_2 -point DFT 的乘法量為 B_2

則 N -point DFT 的乘法量為

$$P_2 B_1 + P_1 B_2$$

假設 $N = P_1 \times P_2 \times \cdots \times P_K$ P_1, P_2, \dots, P_K 彼此互質

P_k -point DFT 的乘法量為 B_k

則 N -point DFT 可分解成 (N/P_1) 個 P_1 -point DFTs

(N/P_2) 個 P_2 -point DFTs

⋮

(N/P_K) 個 P_K -point DFTs

總乘法量為

$$\frac{N}{P_1} B_1 + \frac{N}{P_2} B_2 + \cdots + \frac{N}{P_k} B_k$$

假設 $N = P_1 \times P_2$, P_1 is not prime to P_2

P_1 -point DFT 的乘法量為 B_1 , P_2 -point DFT 的乘法量為 B_2

則 N -point DFT 的乘法量為

且 $m_1 n_2$ 當中 ($m_1 = 0, 1, \dots, P_2 - 1$, $n_2 = 0, 1, \dots, P_1 - 1$)

有 D_1 個值不為 $N/12$ 及 $N/8$ 的倍數

有 D_2 個值為 $N/12$ 或 $N/8$ 的倍數，但不為 $N/4$ 的倍數

則 N -point DFT 的乘法量為

$$P_2 B_1 + P_1 B_2 + 3D_1 + 2D_2$$

Note: $a \times \exp(j\theta)$, 當 a 為 complex , 需要 3 個乘法

然而，當 $\theta = \pi/4$, 只需 2 個乘法

當 $\theta = \pi/3$, 只需 2 個乘法

例子：16-point DFT, $16 = 8 \times 2$,

$$\text{乘法量} = 2 \times 4 + 8 \times 0 + 3 \times 4 + 2 \times 2 = 24$$

$$16 = 4 \times 4$$

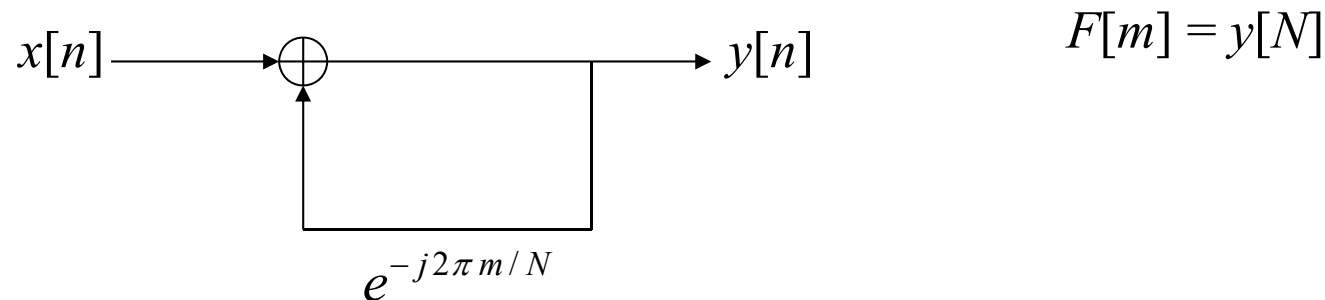
$$\text{乘法量} = 4 \times 0 + 4 \times 0 + 3 \times 4 + 2 \times 4 = 20$$

◎ 10-F Goertzel Algorithm

$$\text{DFT: } F[m] = \sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N}mn}$$

$$x[n] = f[N-n], n = 1, 2, \dots, N$$

$$F[m] = x[1]e^{-j\frac{2\pi}{N}m(N-1)} + x[2]e^{-j\frac{2\pi}{N}m(N-2)} + \dots + x[N]e^{-j\frac{2\pi}{N}m(0)}$$

 $f[N-1]$
 $f[N-2]$
 $f[0]$


優點：Hardware 最為精簡

缺點：運算時間較長 ($N-1$ times of feedback)

◎ 10-G Chirp Z Transform

當 $\Delta_t \Delta_f = 1/N$ 時，Continuous Fourier transform 可以用 DFT 和 FFT 來做 implementation。

問題: 當 $\Delta_t \Delta_f \neq 1/N$ 時怎麼辦?

$$G(f) = \int e^{-j2\pi ft} g(t) dt \xrightarrow[f = m \Delta_f]{\text{取 } t = n \Delta_t} G(m\Delta_f) = \Delta_t \sum_n e^{-j2\pi mn\Delta_t\Delta_f} g(n\Delta_t)$$

$$G(m\Delta_f) = \Delta_t e^{-j\pi m^2 \Delta_t \Delta_f} \sum_n e^{j\pi(m-n)^2 \Delta_t \Delta_f} e^{-j\pi n^2 \Delta_t \Delta_f} g(n\Delta_t)$$

$$G(m\Delta_f) = \Delta_t e^{-j\pi m^2 \Delta_t \Delta_f} \left(e^{j\pi n^2 \Delta_t \Delta_f} * e^{-j\pi n^2 \Delta_t \Delta_f} g(n\Delta_t) \right)$$

↑
convolution

$$\text{Z-transform: } X(z) = \sum_{n=0}^{N-1} x[n]z^{-n} \longrightarrow X(k) = X(z) \Big|_{z=e^{j\frac{2\pi k}{N}}}$$

CZT algorithm:

Define $Z_k = AW^{-k}$, $k=0, 1, \dots, M-1$, 其中 M 為任意 output points
 A 和 W 為任意 complex number。

$$X_k = \sum_{n=0}^{N-1} x[n](AW^{-k})^{-n} = \sum_{n=0}^{N-1} x[n]A^{-n}W^{kn}, \quad k = 0, 1, \dots, M-1$$

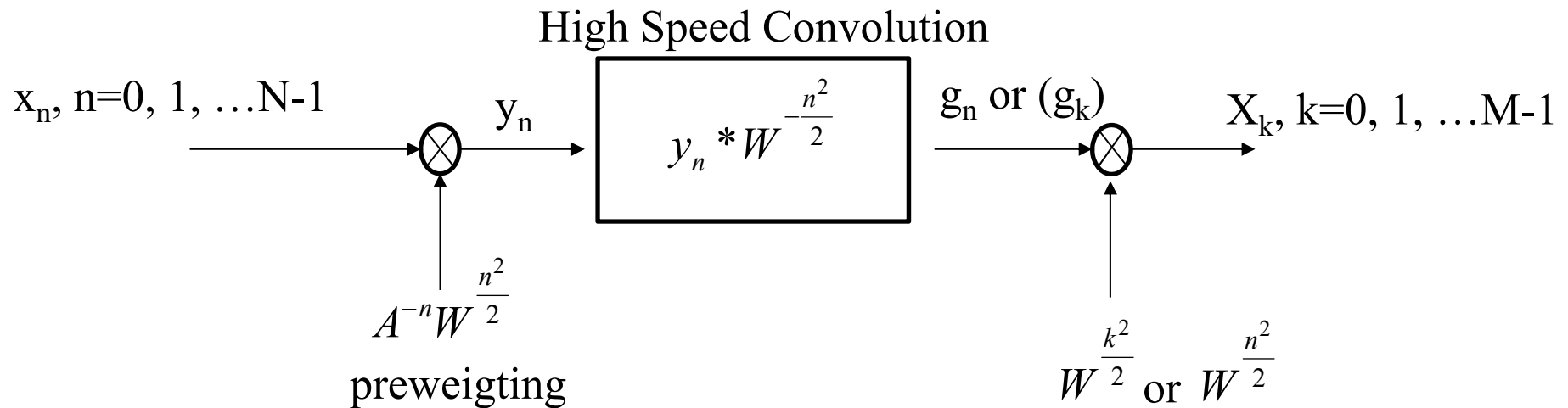
令 $nk = \frac{n^2 + k^2 - (k-n)^2}{2}$ 代入並整理得:

$$X_k = \sum_{n=0}^{N-1} (x[n]A^{-n}W^{\frac{n^2}{2}}) W^{\frac{k^2}{2}} W^{\frac{-(k-n)^2}{2}}, \quad k = 0, 1, \dots, M-1$$

\parallel y_n \parallel v_{k-n}

$$\Rightarrow X_k = W^{\frac{k^2}{2}} \sum_{n=0}^{N-1} y[n]v[k-n] = W^{\frac{k^2}{2}} (y[k] * v[k]), \quad k = 0, 1, \dots, M-1$$

Block diagram:



優點:

- (1) input/output point 可以不相同 ($N \neq M$), N 和 M 為任意整數
- (2) contour 不需要在單位圓上 (arc 即可)
- (3) 初始點任意 (arbitrary initial frequency), 而 DFT 必須要 DC 點開始

缺點: 運算量較大 (3 times)

© 10-H Winograd Algorithm for DFT Implementation

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Basic idea:

Except for the 1st row and the 1st column, the N -point DFT is equivalent to the $(N-1)$ -point circular convolution when N is a prime number.

Example: 5-point DFT

$$\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}, \quad \omega = \exp[-j\angle 72^\circ],$$

移除第一個 row 和第一個 column

$$\begin{bmatrix} V_1 - v_0 \\ V_2 - v_0 \\ V_3 - v_0 \\ V_4 - v_0 \end{bmatrix} = \begin{bmatrix} \omega & \omega^2 & \omega^3 & \omega^4 \\ \omega^2 & \omega^4 & \omega & \omega^3 \\ \omega^3 & \omega & \omega^4 & \omega^2 \\ \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

先將 3rd and 4th rows, 再 2nd, 3rd and 4th columns 作交換

$$\begin{bmatrix} V_1 - v_0 \\ V_2 - v_0 \\ V_4 - v_0 \\ V_3 - v_0 \end{bmatrix} = \begin{bmatrix} \omega & \omega^2 & \omega^4 & \omega^3 \\ \omega^3 & \omega & \omega^2 & \omega^4 \\ \omega^4 & \omega^3 & \omega & \omega^2 \\ \omega^2 & \omega^4 & \omega^3 & \omega \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \\ v_4 \\ v_2 \end{bmatrix}$$

變成 circular convolution 的
型態

Circular Convolution

$$z[n] = y[n] \otimes h[n] = \sum_{k=0}^{N-1} y[k] h[((n-k))_N]$$

$$\longrightarrow z[n] = \text{IFFT} \{ \text{FFT}(y[n]) \text{FFT}(h[n]) \}$$

$$\begin{bmatrix} V_1 - v_0 \\ V_2 - v_0 \\ V_4 - v_0 \\ V_3 - v_0 \end{bmatrix} = \text{IFFT} \left(\text{FFT}_4 \left\{ \begin{bmatrix} v_1 \\ v_3 \\ v_4 \\ v_2 \end{bmatrix} \right\} \cdot \text{FFT}_4 \left\{ \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_4 \\ \omega_3 \end{bmatrix} \right\} \right)$$

$$\text{FFT}_4 \left\{ \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_4 \\ \omega_3 \end{bmatrix} \right\} = \begin{bmatrix} -1 \\ -1.7156 - 1.9021j \\ 2.2361 \\ 1.7156 - 1.9021j \end{bmatrix}$$

當 N 為其他的 prime numbers 時，也可以運用 permutation 和 circular convolution 來計算 prime-number DFTs

(Step 1) Delete the 1st row and the 1st column.

(Step 2) Perform the row and column permutations.

Rows 和 columns 的順序相同

(a) 找出一個 **primitive root** a , 使得 $a^k \bmod N \neq 1$ when $k = 1, 2, \dots, N-2$,
 $a^{N-1} \bmod N = 1$ (Primitive root 的概念，會在後面講到數論時複習)

(b) Rows 和 columns 的順序，以 $p[n]$ 來表示，

$$p[n] = a^n \bmod N, \quad n = 0, 1, \dots, N-2$$

(Step 3) 變成 circular convolution 的型態

則 N -point DFT 可以用 $(N-1)$ -point DFTs 來 implementation

$$\begin{bmatrix} V_{p[0]} - v_0 \\ V_{p[1]} - v_0 \\ \vdots \\ V_{p[N-2]} - v_0 \end{bmatrix} = IDFT_{N-1} \left(DFT_{N-1} \left\{ \begin{bmatrix} v_{p[0]} \\ v_{p[N-2]} \\ \vdots \\ v_{p[1]} \end{bmatrix} \right\} DFT_{N-1} \left\{ \begin{bmatrix} w^{p[0]} \\ w^{p[1]} \\ \vdots \\ w^{p[N-2]} \end{bmatrix} \right\} \right)$$

重要理論：

Any N -point DFT can be implemented by the 2^k -point DFTs whatever the value of N is.

7-point DFT

123-point DFT

XI. Discrete Fourier Transform 的替代方案

◎ 11-A Why Should We Use Other Operations?

Discrete Fourier Transform (DFT):

$$X_F[m] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi mn}{N}}$$

優點：有 fast algorithm (complexity 為 $O(N\log_2 N)$).
適合做頻譜分析和 convolution implementation

問題：(1) complex output
(2) The exponential function is irrational.

Applications of the DFT: ① Spectrum analysis ② Performing the convolution⁴⁰¹

For **spectrum analysis**, the DFT can be replaced by:

- ★ (1) DCT, (2) DST, (3) DHT,
- (4) Walsh (Hadamard) transform,
- (5) Haar transform,
- (6) orthogonal basis expansion,
(including orthogonal polynomials and CDMA),
- (7) wavelet transform,
- ★ (8) time-frequency distribution

When **performing the convolution**, the DFT can be replaced by:

- (1) DCT, (2) DST, (3) DHT,
- (4) Directly Computing,
- ★ (5) Sectioned DFT convolution,
- (6) Winograd algorithm,
- (7) number theoretic transform (NTT)
- ★ (8) Z-transform based recursive method

◎ 11-B Discrete Sinusoid Transforms

DCT (discrete cosine transform) has 8 types

DST (discrete sine transform) has 8 types

DHT (discrete Hartley transform) has 4 types

共通的特性：皆為 real, 且和 DFT 密切相關

Reference

- N. Ahmed, T. Natarajan, and K. R. Rao, “Discrete cosine transform,” *IEEE Trans. Comput.*, vol. C-23, pp. 90-93, Jan 1974.
- Z. Cvetkovic and M. V. Popovic, “New fast recursive algorithms for the computation of discrete cosine and sine transforms,” *IEEE Trans. Signal Processing*, vol. 40, pp. 2083-2086, Aug. 1992.
- R. N. Bracewell, *The Hartley Transform*, New York, Oxford University Press, 1986.
- S. C. Chan and K. L. Ho, “Prime factor real-valued Fourier, cosine and Hartley transform,” *Proc. Signal Processing VI*, pp. 1045-1048, 1992.

- **Case 1:** 當 $x[n]$ 為 even function , $x[n] = x[N-n]$

在做頻譜分析時，

N -point DFT 可以被 $(\text{floor}(N/2) + 1)$ -point DCT (type 1) 取代

$$X_C[m] = \sum_{n=0}^Q k_n x[n] \cos\left(\frac{\pi m n}{Q}\right), \quad Q = \text{floor}(N/2),$$

$$\begin{cases} k_n = 1 & , \text{when } n = 0 \text{ or } N/2 \\ k_n = 2 & , \text{otherwise} \end{cases}$$

可以證明，當 $x[n]$ 為 even , $X_C[m] = X_F[m]$

(運算量減少將近一半)

$$\text{Recover: } x[n] = \frac{1}{N} \sum_{m=0}^Q k_m X_C[m] \cos\left(\frac{\pi m n}{Q}\right)$$

注意：和 JPEG 所用的 DCT (type 2) 並不相同

$$F[m] = \sqrt{\frac{2}{N}} C_m \sum_{n=0}^{N-1} f[n] \cos \frac{(n+1/2)m\pi}{N} \quad \begin{array}{l} C_0 = 1/\sqrt{2} \\ C_m = 1 \quad \text{otherwise} \end{array}$$

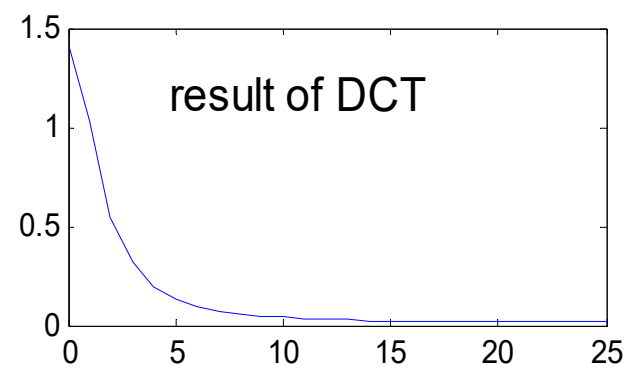
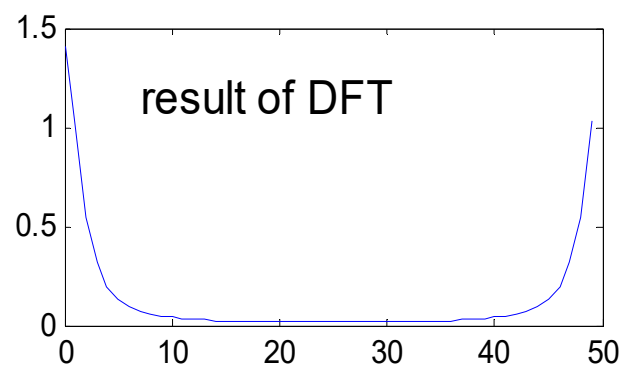
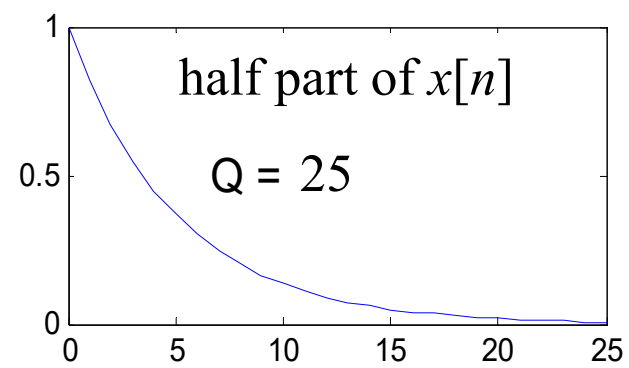
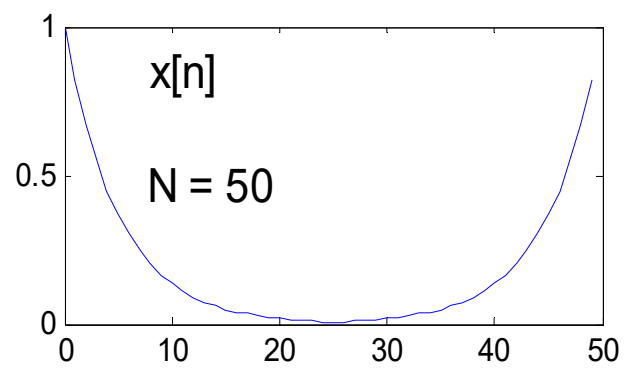
(Proof)

$$X_F[m] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi mn}{N}}$$

When $x[n] = x[N-n]$, N is even

(The case where N is odd can be proved in the similar way)

$$\begin{aligned} X_F[m] &= x[0] + \sum_{n=1}^{N/2-1} x[n] e^{-j\frac{2\pi mn}{N}} + x\left[\frac{N}{2}\right] e^{-j\pi m} + \sum_{n=1}^{N/2-1} x[N-n] e^{-j\frac{2\pi m(N-n)}{N}} \\ &= x[0] + \sum_{n=1}^{N/2-1} x[n] e^{-j\frac{2\pi mn}{N}} + x\left[\frac{N}{2}\right] (-1)^m + \sum_{n=1}^{N/2-1} x[n] e^{j\frac{2\pi m(n)}{N}} \\ &= x[0] + 2 \sum_{n=1}^{N/2-1} x[n] \cos\left(\frac{2\pi mn}{N}\right) + x\left[\frac{N}{2}\right] (-1)^m \\ &= \sum_{n=0}^{N/2} k_n x[n] \cos\left(\frac{2\pi mn}{N}\right) \quad \begin{cases} k_n = 1 & , \text{when } n = 0 \text{ or } N/2 \\ k_n = 2 & , \text{otherwise} \end{cases} \\ &= X_C[m] \end{aligned}$$



- **Case 2:** 當 $x[n]$ 為 odd function , $x[n] = -x[N-n]$

在做頻譜分析時，

N -point DFT 可以被 $(N/2 - 1)$ -point DST (type 1) 取代

$$X_S[m] = 2 \sum_{n=1}^{Q-1} x[n] \sin\left(\frac{\pi m n}{Q}\right), \quad Q = N/2.$$

可以證明，當 $x[n]$ 為 odd , $X_S[m] = jX_F[m]$

(運算量減少將近一半)

Recover:
$$x[n] = \frac{2}{N} \sum_{m=1}^{Q-1} X_S[m] \sin\left(\frac{\pi m n}{Q}\right)$$

- **Case 3:** 當 $x[n]$ 為 real function，在做頻譜分析時，

N -point DFT 可以被 N -point DHT (type 1) 取代

$$X_H[m] = \sum_{n=0}^{N-1} x[n] \text{cas}\left(\frac{2\pi mn}{N}\right), \quad \text{where } \text{cas}(k) = \cos(k) + \sin(k)$$

比較： $\exp(-jk) = \cos(k) - j\sin(k)$

可以證明，若 $x[n]$ 為 real， $X_H[m] = \text{real}\{X_F[m]\} - \text{imag}\{X_F[m]\}$

(運算量減少將近一半)

$$\text{Recover: } x[n] = \sum_{m=0}^{N-1} X_H[m] \text{cas}\left(\frac{2\pi mn}{N}\right)$$

- 大部分的 convolution 仍然使用 DFT 。

$$y[n] = x[n] * h[n]$$

$$y[n] = \text{IDFT} \{ \text{DFT}(x[n]) \times \{ \text{DFT}(h[n]) \} \}$$

思考：何時適合用 DCT 做 convolution ？

何時適合用 DST 做 convolution ？

何時適合用 DHT 做 convolution ？

附錄十二：論文英文常見的文法錯誤

(1) *** transform, *** equation, *** method, *** algorithm 在論文當中，當成是可數名詞，而非專有名詞(除非是所有格的形態)。

可數名詞單數時，前面要冠詞(a 或 the)

Fourier transform is important for signal processing. (錯誤)

The Fourier transform is important for signal processing. (正確)

A Fourier transform is important for signal processing. (正確)

Fourier transforms are important for signal processing. (正確)

I have written the Matlab program of Parks-McClellan algorithm (錯誤)

I have written the Matlab program of the Parks-McClellan algorithm (正確)

(2) 若是所有格的形態，不必加冠詞

I have written the Matlab program of the Parks-McClellan's algorithm (錯誤)

I have written the Matlab program of Parks-McClellan's algorithm (正確)

(3) 論文視同正式的文件，對 not, is, are 不用縮寫

they're (錯誤) they are (正確)

he's (錯誤) he is (正確)

aren't (錯誤) are not (正確)

don't (錯誤) do not (正確)

can't (錯誤) cannot (正確)

(4) Suppose, assume 後面要加關係代名詞

Suppose x is a large number. (錯誤)

Suppose **that** x is a large number. (正確)

(5) 每一個子句都有一個動詞，而且只有一個動詞

(6) In this paper, in this section, in this chapter 開頭的句子，應該用現在式，而非未來式

In this paper, the fast algorithm of DCT will be introduced. (錯誤)

In this paper, , the fast algorithm of DCT **is** introduced. (正確)

(7) 在 conclusion 當中回顧文章一內容，用過去式

(8) 敘述所引用的論文的內容，用過去式

In [10], the number theoretic transform **was** proposed.

(9) time domain, frequency domain 前面也加冠詞

in time domain (錯誤) in **the** time domain (正確)

(10) 不以 “this paper”, “section *”, “Ref. [*”] 當主詞用

This paper describes several concepts. (錯誤)

In this paper, several concepts are described. (正確)

Ref. [1] proposed the method. (錯誤)

In Ref. [1], Parks and McClellan proposed the method. (正確)

(11) 提及某個 equation 時，直接括號加數字即可

in equation (3) (錯誤) in (3) (正確)

提及某個 section, table, or figure 時，前面不加冠詞，而且常用大寫

in the section 4 (錯誤) in Section 4 (正確)

in the table 5 (錯誤) in Table 4 (正確)

(12) 寫科技論文不是寫文學作品，不要用高明、漂亮、但沒有把握的文法。

儘量用簡單而有把握的文法。

(13) 科技論文英文講求「長話短說」，儘量用精簡的文字來表達意思

(14) 用字儘量避免重覆

(15) Equations 也當成是文章的一部分，所以通常也要加標點符號

The formula of Newton's 2nd law is

$$F = ma.$$

← 要加標點符號

(16) 解釋 parameters 和 symbols 時，用 **where** 當關係代名詞

$x = 10t$ **where** x is the location of the object and t is time.

(17) 很重要的論文，投稿至國際學術期刊，又對自己的英文文法沒有十足的把握時

可以用網路上的論文編修服務，來修改文法上的錯誤

本系以及台大語言中心也經常有英文論文寫作相關的訓練課程，有志將來在學術界奮鬥的同學，可以多參與相關的課程

附錄十三：論文的標準格式與編輯論文技巧

註：這裡指的是一般 journal papers 和 conference papers 的格式。

然而，不同的 journals 和 conferences，對於格式的規定，也會稍有不同。投稿前，還是要細讀相關的規定。

(1) 變數使用斜體，矩陣或向量使用粗體

$$f(x) = x^2 + 3x + 2. \quad (f, x \text{ 皆用斜體})$$

$$\mathbf{y} = [1 \quad 2 \quad 3] \quad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad (\mathbf{y}, \mathbf{A} \text{ 皆用粗體})$$

(2) 段落的經常用「左右對齊」的格式

如果使用 Word，可以按 常用 → 段落 → 對齊方式 → 左右對齊

或是按工具列中的 

(3) Equation 的標號，經常用「定位點」的功能，讓標號的位置固定
如果使用 Word，可以按 常用 → 段落 → 定位點 (在對話框左下角)
，再設定定位點的位置

(4) 至於 equations 本身，通常置於這一行的中間，例如

$$F = ma. \quad (1)$$

Equations 和前一行以及後一行，皆要有足夠的距離。而且，equations 的後方常常要加逗號或句號 (以下一行是否為新的句子而定)。

(5) 標題(包括 papers 的標題以及每個 chapters 和 sections 的標題) 當中，
每個單字的開頭一定要大寫，除了 (a) 介係詞 (b) 連詞 (c) 冠詞 以外。
若為第一個單字，即使是介係詞，連詞，或冠詞，也要大寫

The Applications of the Fourier Transform in Daily Life

Fast Algorithms of the Wavelet Transform and JPEG2000

(6) 文章一定要包括

(a) Abstract,

(b) Introduction (通常是第一個 section)

(c) 內文

(d) Conclusions 或 Conclusions and Future Works (通常是最後一個 section)

(e) References

(7) 每一張圖 (figures)，每一張表 (tables) 都要編號，而且要附加文字說明。

如 Fig. 3 The result of the Fourier transform for a chirp signal.

若一張圖當中有很多個小圖，每個小圖也要編號 (a), (b), (c), (d)

(8) 同一個 equation，同一張圖，要放在同一頁，不分散於兩頁。

(9) 一般而言，Journal papers 的初稿，是 one column, double space 的格式。

在 Word 當中，double space 可以用後下的方法設定

常用 → 段落 → 行距 → 2倍行高

但有時，2倍行高會讓初稿過於稀疏，在 Word 2007 當中可以用

版面配置 → 版面設定 → 文件格線 → 沒有格線

來讓文件看起來不會那麼稀疏，且不易超過規定的頁數。

(10) Conference papers 是 two columns, one space 的格式。有時 Journal papers 被接受後，也會要求改成 two columns, one space 的格式。

在 Word 2007，two columns 可以用

版面配置 → 欄 → 二 (W)

來設定

(11) References 的編號，通常是按照在文章中出現的順序來排序

或者也可按照第一作者的 last name 的英文字母順序排序

(12) Reference 的寫法

(以 IEEE Transactions on Signal Processing 為例)

(A) Journal papers and conference papers

Authors (first name 或 middle name 只用一個字母代表), “title,” name of the journal (縮寫為佳), vol. *, no. *, pp. **~**, month, year.

使用縮寫

逗號在引號之前

加句號

只有第一個字母、專有名詞
開頭、和縮為用大寫

範例：

S. Abe and J. T. Sheridan, “Optical operations on wave functions as the Abelian subgroups of the special affine Fourier transformation,” *Opt. Lett.*, vol. 19, no. 22, pp. 1801-1803, 1994.

(B) Books

Authors (first name 或 middle name 只用一個字母代表), *title* (斜體, 字開頭大寫, 不加引號), 第幾版 (非必需), 出版社, 出版地, year.

範例

H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, 1st Ed., John Wiley & Sons, New York, 2000.

(C) Websites

Authors, “title,” available in <http://網址>.

範例

張智星, “Utility toolbox,” available in <http://neural.cs.nthu.edu.tw/jang/matlab/toolbox/utility/>.