2. Digital Filter Design (A)

任何可以用來去除 noise 作用的 operation，皆被稱為 filter

甚至有部分的 operation，雖然主要功用不是用來去除 noise，但是可以用 FT + multiplication + IFT 來表示，也被稱作是 filter

FT + multiplication + IFT = convolution, LTI system

Reference

[1] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, London: Prentice-Hall, 3rd ed., 2010.

[2] D. G. Manolakis and V. K. Ingle, *Applied Digital Signal Processing*, Cambridge University Press, Cambridge, UK. 2011.

[3] B. A. Shenoi, *Introduction to Digital Signal Processing and Filter Design*, Wiley-Interscience, N. J., 2006.

[4] A. A. Khan, *Digital Signal Processing Fundamentals*, Da Vinci Engineering Press, Massachusetts, 2005.

[5] S. Winder, *Analog and Digital Filter Design*, 2nd Ed., Amsterdam, 2002.

◎ 2-A Classification for Filters

- filter
  - digital filter
    - IIR (infinite impulse response)
      - bilinear transform
      - impulse invariant
      - step invariant
      - others
    - FIR (finite impulse response)
      - (1) MSE (mean square error) 關心平均誤差
      - (2) Minimax (minimize the maximal error) 最小化最大誤差
      - (3) frequency sampling
      - (4) others
  - analog filter

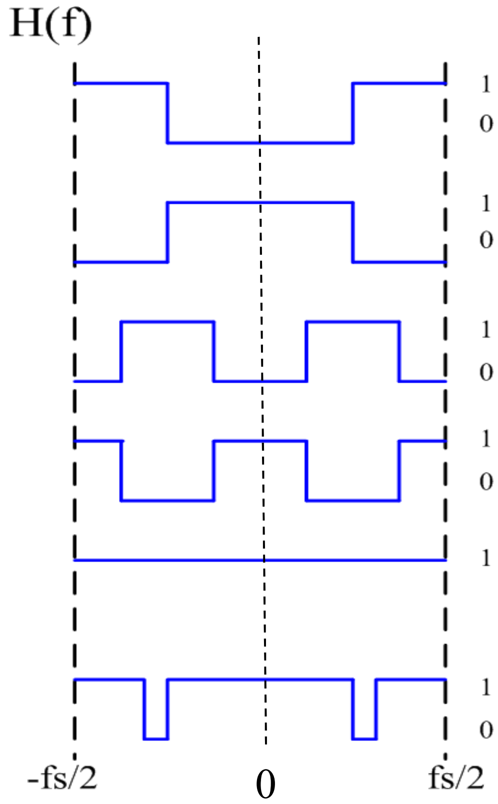
由 數位類比分類

由 response 是否有限長分類

由設計方法分類

Classification for filters (依型態分)

- pass-stop filter
  - high pass filter
  - low pass filter
  - band pass filter
  - band stop filter
  - all pass filter
  - Notch filter
- matched filter
- Wiener filter
- equalizer filter
- others
  - differentiation
  - integration
  - Hilbert transform
  - smoother
  - edge detection
  - Kalman filter
  - particle filter



⦿ 2-B FIR Filter Design

FIR filter: impulse response is nonzero at finite number of points

$$h[n] = 0 \text{ for } n < 0 \text{ and } n \geq N$$

($h[n]$ has N points, N is a finite number)

$h[n]$ is causal

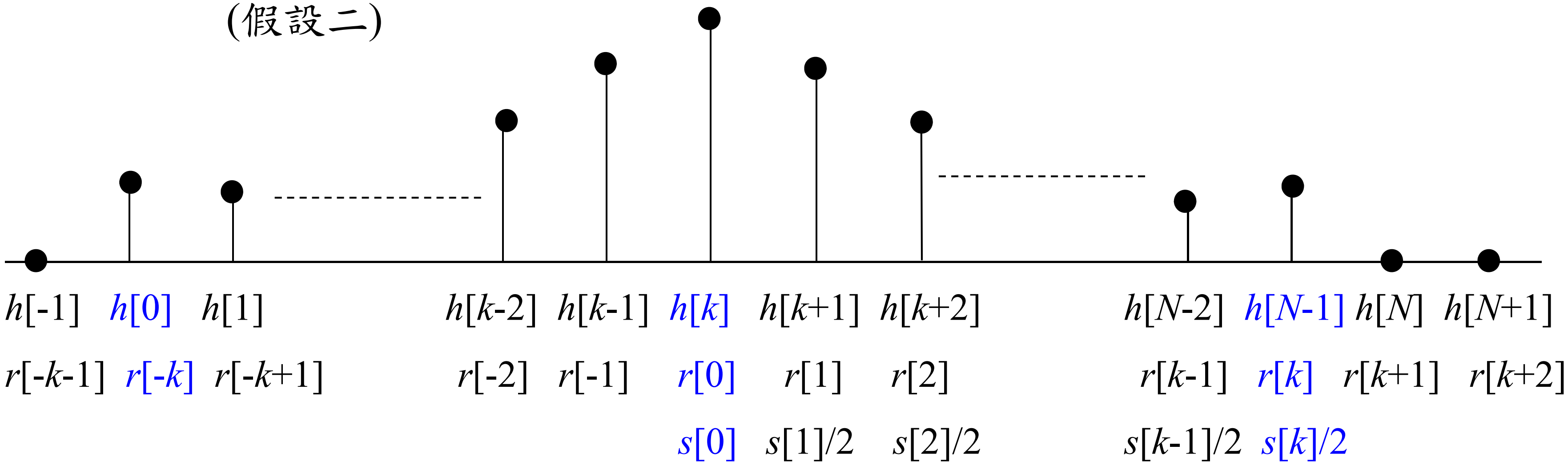
- FIR is more popular because its impulse response is finite.

(假設一)

Specially, when $h[n]$ is even symmetric $h[n] = h[N-1-n]$

and N is an odd number

(假設二)



(a) $r[n] = h[n + k]$, where $k = (N-1)/2$.

(b) $s[0] = r[0]$, $s[n] = 2r[n]$ for $0 < n \leq k$.

Impulse Response of the FIR Filter:

$h[n]$ $(h[n] \neq 0$ for $0 \leq n \leq N-1)$
$r[n] = h[n+k]$, $k = (N-1)/2$ $(r[n] \neq 0$ for $-k \leq n \leq k$, see page 45$)$

Suppose that the filter is **even**, $r[n] = r[-n]$.

Set $s[0] = r[0]$, $s[n] = 2r[n]$ for $n \neq 0$.

Then, the discrete-time Fourier transform of the filter is

$$H(F) = \sum_{n=-\infty}^{\infty} h[n] e^{-j2\pi F n}$$

($F = f\Delta_t$ is the **normalized frequency**)

See page 25

$$H(F) = e^{-j2\pi F k} R(F)$$

$$R(F) = \sum_{n=-k}^{k} r[n] e^{-j2\pi F n}$$

$$= \sum_{n=-k}^{-1} r[n] e^{-j2\pi F n} + r[0] + \sum_{n=1}^{k} r[n] e^{-j2\pi F n}$$

$$= \sum_{n=1}^{k} r[-n] e^{j2\pi F n} + r[0] + \sum_{n=1}^{k} r[n] e^{-j2\pi F n}$$

$$R(F) = \sum_{n=0}^{k} s[n] \cos(2\pi n F)$$

⦿ 2-C Least MSE Form and Minimax Form FIR Filters

(1) least MSE (mean square error) form

(關心 平均 誤差)

MSE = $f_s^{-1}\int_{-f_s/2}^{f_s/2}\left|H(f)-H_d(f)\right|^2 df$, f_s: sampling frequency

$H(f)$: the spectrum of the filter we obtain

$H_d(f)$: the spectrum of the desired filter

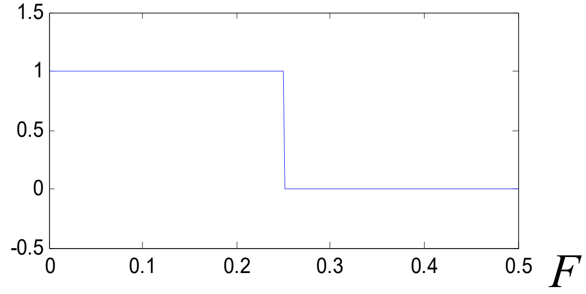
(2) mini-max (minimize the maximal error) form

(關心 最大 誤差)

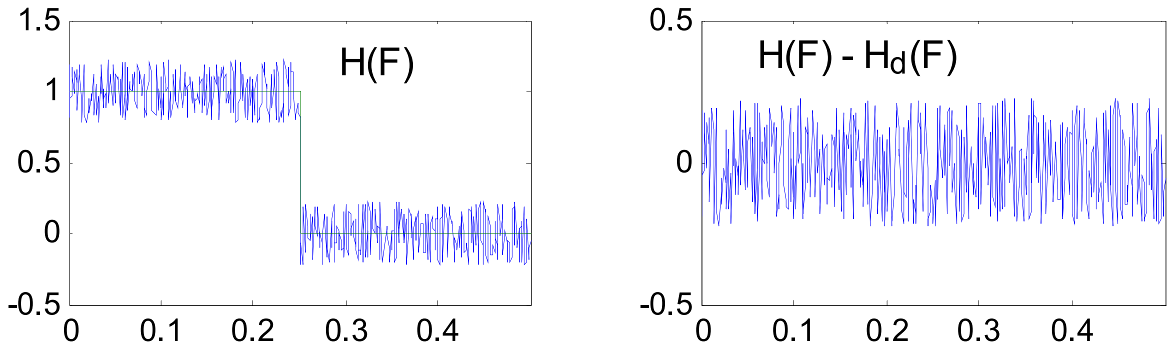
maximal error: $\underset{f}{Max}\left|H(f)-H_d(f)\right|$

The transition band is always ignored

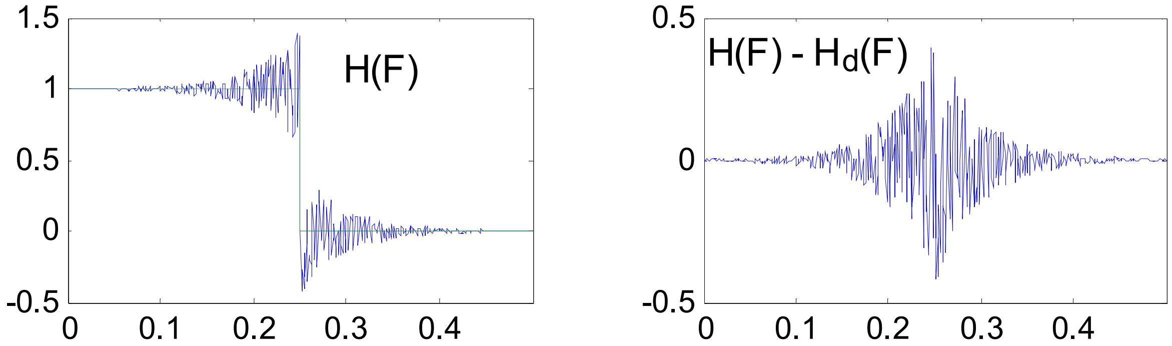
Example: desired output $H_d(F)$



(A) larger MSE, but smaller maximal error



(B) smaller MSE, but larger maximal error



⦿ 2-D Review: FIR Filter Design in the MSE Sense

$$R(F)=\sum_{n=0}^{k}s[n]\cos(2\pi nF)$$

$$MSE=f_s^{-1}\int_{-f_s/2}^{f_s/2}\left|R(f)-H_d(f)\right|^2 df=\int_{-1/2}^{1/2}\left|R(F)-H_d(F)\right|^2 dF$$

$$F=f/f_s$$

$$=\int_{-1/2}^{1/2}\left|\sum_{n=0}^{k}s[n]\cos(2\pi nF)-H_d(F)\right|^2 dF$$

$$=\int_{-1/2}^{1/2}\left(\sum_{v=0}^{k}s[v]\cos(2\pi vF)-H_d(F)\right)\left(\sum_{\tau=0}^{k}s[\tau]\cos(2\pi\tau F)-H_d(F)\right)dF$$

$$\frac{\partial MSE}{\partial s[n]}=\int_{-1/2}^{1/2}\cos(2\pi nF)\left(\sum_{\tau=0}^{k}s[\tau]\cos(2\pi\tau F)-H_d(F)\right)dF$$

$$+\int_{-1/2}^{1/2}\left(\sum_{v=0}^{k}s[v]\cos(2\pi vF)-H_d(F)\right)\cos(2\pi nF)dF=0$$

$$\frac{\partial MSE}{\partial s[n]}=2\sum_{\tau=0}^{k}s[\tau]\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi\tau F)dF-2\int_{-1/2}^{1/2}H_d(F)\cos(2\pi nF)dF=0$$

$$\frac{\partial MSE}{\partial s[n]} = 2\sum_{\tau=0}^{k} s[\tau]\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi\tau F)\,dF - 2\int_{-1/2}^{1/2} H_d(F)\cos(2\pi nF)\,dF = 0$$

From the facts that

$$\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi\tau F)\,dF = 0 \qquad \text{when } n \neq \tau,$$

$$\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi\tau F)\,dF = 1/2 \qquad \text{when } n = \tau,\ n \neq 0,$$

$$\int_{-1/2}^{1/2}\cos(2\pi nF)\cos(2\pi\tau F)\,dF = 1 \qquad \text{when } n = \tau,\ n = 0.$$

Therefore,

$$\frac{\partial MSE}{\partial s[0]} = 2s[0] - 2\int_{-1/2}^{1/2} H_d(F)\,dF = 0$$

$$\frac{\partial MSE}{\partial s[n]} = s[n] - 2\int_{-1/2}^{1/2}\cos(2\pi nF)H_d(F)\,dF = 0 \qquad \text{for } n \neq 0.$$

Minimize MSE $\rightarrow$ Make $\dfrac{\partial MSE}{\partial s[n]} = 0$ for all n's

$$\therefore \quad \boxed{s[0] = \int_{-1/2}^{1/2} H_d(F)\,dF}\,, \qquad \boxed{s[n] = 2\int_{-1/2}^{1/2} \cos(2\pi n F) H_d(F)\,dF}\,.$$

Finally, set $h[k]$ = s[0],

$h[k+n]$ = s[n]/2, $h[k-n]$ = s[n]/2 for n = 1, 2, 3, …., k,

$h[n] = 0$ for $n < 0$ and $n \geq N$.

Then, $h[n]$ is the impulse response of the designed filter.

◎ 2-E FIR Filter Design in the Mini-Max Sense

It is also called "Remez-exchange algorithm"

or "Parks-McClellan algorithm"

References

[1] T. W. Parks and J. H. McClellan, "Chebychev approximation for nonrecursive digital filter with linear phase", *IEEE Trans. Circuit Theory*, vol. 19, no. 2, pp. 189-194, March 1972.

[2] J. H. McClellan, T. W. Parks, and L. R. Rabiner "A computer program for designing optimum FIR linear phase digital filter", *IEEE Trans. Audio-Electroacoustics*, vol. 21, no. 6, Dec. 1973.

[3] F. Mintz and B. Liu, "Practical design rules for optimum FIR bandpass digital filter", *IEEE Trans. ASSP*, vol. 27, no. 2, Apr. 1979.

[4] E. Y. Remez, "General computational methods of Chebyshev approximation: The problems with linear real parameters," AEC-TR-4491. ERDA Div. Phys. Res., 1962.

Two constraints

Suppose that:

① Filter length = N, N is odd, $N = 2k+1$.

② Frequency response of the **desired filter**: $H_d(F)$ is an even function

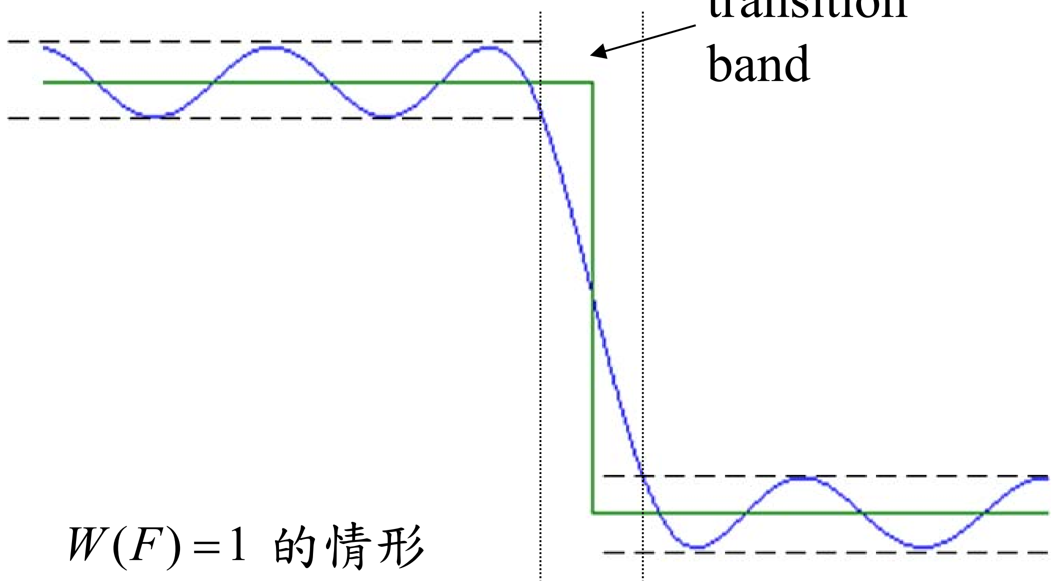
(*F* is the normalized frequency)

③ The weighting function is $W(F)$

用Mini-Max方法所設計出的filters，一定會滿足以下二個條件

(1) 有k+2個以上的extreme points → Error 的 local maximal / local minimum

(2) 在extreme points上，$W(F_m)\,|R(F_m)-H_d(F_m)|$ 是定值



證明可參考

T. W. Parks and J. H. McClellan, "Chebychev approximation for nonrecursive digital filter with linear phase", *IEEE Trans. Circuit Theory*, vol. 19, no. 2, pp. 189-194, March 1972.

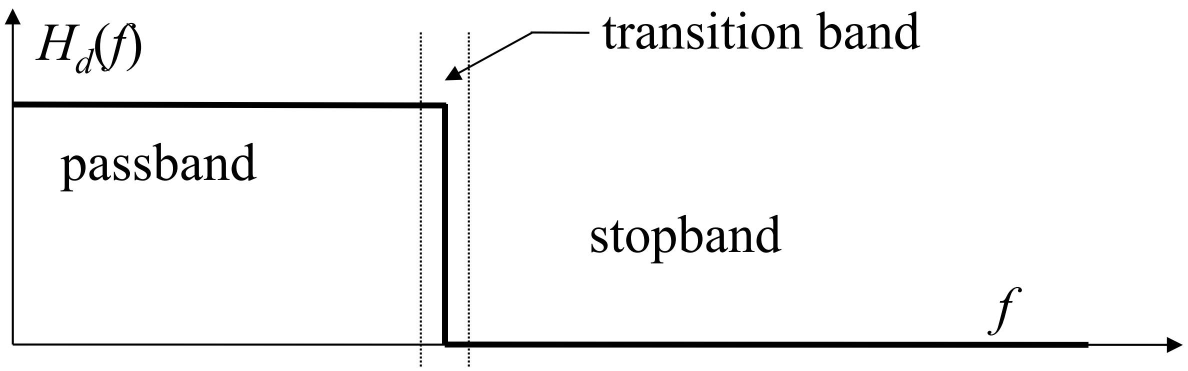
- Generalization for Mini-Max Sense by weight function

maximal error: $\underset{f,\, f \notin transition\ band}{Max} \left| R(f) - H_d(f) \right|$

weighted maximal error: $\underset{f,\, f \notin transition\ band}{Max} \left| W(f)[R(f) - H_d(f)] \right|$

where $W(f)$ is the weight function.

The weight function is designed according to which band is more important.



Q: How do we choose $W(f)$ When SNR ↑ ?

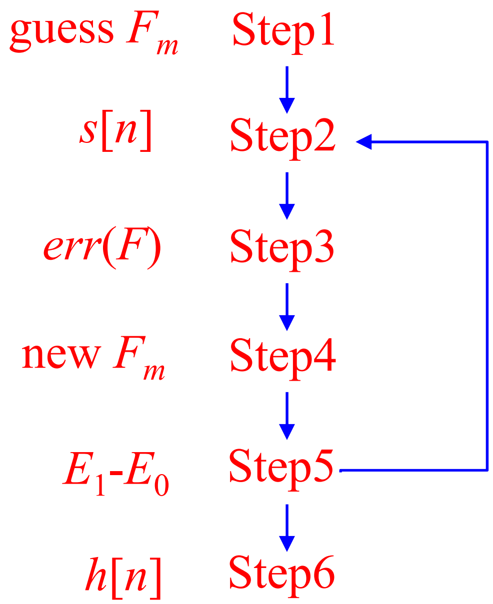
Example: If we treat the passband the same important as the stopband.
$W(f) = 1$ in the passband, $W(f) = 1$ in the stopband

Q1: $W(f) = 1$ in the passband, $W(f) < 1$ in the stopband 代表什麼？

Q2: $W(f) < 1$ in the passband, $W(f) = 1$ in the stopband 代表什麼？

Q3: 如何用來壓縮特定區域 (如 transition band 附近) 的 error？

Q4: Weighting function 的概念可否用在 MSE sense ？



guess F_m
Step1
$s[n]$
Step2
$err(F)$
Step3
new F_m
Step4
E_1-E_0
Step5
$h[n]$
Step6

◎ 2-F Mini-Max Design Process

(Step 1): Choose arbitrary k+2 extreme frequencies in the range of

$0 \le F \le 0.5$, (denoted by F_0, F_1, F_2, ….., F_{k+1})

Note: (1) Exclude the transition band.

(2) The extreme points cannot be all in the stop band.

Set E_1 (error) $\rightarrow \infty$

Extreme frequencies:

The locations where the error is maximal.

$$[R(F_0) - H_d(F_0)]W(F_0) = -e \qquad [R(F_1) - H_d(F_1)]W(F_1) = e$$

$$[R(F_2) - H_d(F_2)]W(F_2) = -e \qquad [R(F_3) - H_d(F_3)]W(F_3) = e$$

$$\vdots$$

$$[R(F_{k+1}) - H_d(F_{k+1})]W(F_{k+1}) = (-1)^{k+2} e$$

(參考 page 54)

(Step 2): From page 46, $[R(F_m) - H_d(F_m)]W(F_m) = (-1)^{m+1}e$ (where m = 0, 1, 2, ….. , k+1) can be written as

$$\sum_{n=0}^{k} s[n]\cos(2\pi F_m n) + (-1)^m W^{-1}(F_m)e = H_d(F_m)$$

where m = 0, 1, 2, ….. , k+1.

Expressed by the matrix form:

$$\begin{bmatrix} 1 & \cos(2\pi F_0) & \cos(4\pi F_0) & \cdots & \cos(2\pi k F_0) & 1/W(F_0) \\ 1 & \cos(2\pi F_1) & \cos(4\pi F_1) & \cdots & \cos(2\pi k F_1) & -1/W(F_1) \\ 1 & \cos(2\pi F_2) & \cos(4\pi F_2) & \cdots & \cos(2\pi k F_2) & 1/W(F_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \cos(2\pi F_k) & \cos(4\pi F_k) & \cdots & \cos(2\pi k F_k) & (-1)^k/W(F_k) \\ 1 & \cos(2\pi F_{k+1}) & \cos(4\pi F_{k+1}) & \cdots & \cos(2\pi k F_{k+1}) & (-1)^{k+1}/W(F_{k+1}) \end{bmatrix} \begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ \vdots \\ s[k] \\ e \end{bmatrix} = \begin{bmatrix} H_d[F_0] \\ H_d[F_1] \\ H_d[F_2] \\ \vdots \\ H_d[F_k] \\ H_d[F_{k+1}] \end{bmatrix}$$

Square matrix

Solve $s[0]$, $s[1]$, $s[2]$, ….. , $s[k]$ from the above matrix

(performing the matrix inversion).

(Step 3): Compute err(F) for $0 \le F \le 0.5$, exclude the transition band.

$$err(F) = [R(F) - H_d(F)]W(F) = \{\sum_{n=0}^{k} s[n]\cos(2\pi nF) - H_d(F)\}W(F)$$

Set $W(F) = 0$ at the transition band.

(Step 4): Find k+2 **local maximal (or minimal) points** of err(F)

local maximal point: if $q(\tau) > q(\tau + \Delta_F)$ and $q(\tau) > q(\tau - \Delta_F)$,

then τ is a local maximal of $q(x)$.

local minimal point: if $q(\tau) < q(\tau + \Delta_F)$ and $q(\tau) < q(\tau - \Delta_F)$,

then τ is a local minimal of $q(x)$.

Other rules: Page 63

Denote the local maximal (or minimal) points by $P_0, P_1, \ldots\ldots, P_k, P_{k+1}$

These k+2 extreme points could include the boundary points of the transition band

(Step 5):

Set E_0 = Max(|err(F)|).

$$\begin{cases} E_0 : 現在的 Max\,|\,err(F)\,| \\ E_1 : 前一次 iteration 的 Max\,|\,err(F)\,| \end{cases}$$

(Case a) If $E_1 - E_0 > \Delta$, or $E_1 - E_0 < 0$ (or the first iteration) $\rightarrow$

set $F_n = P_n$ and $E_1 = E_0$, return to Step 2.

(Case b) If $0 \le E_1 - E_0 \le \Delta$ $\rightarrow$ continue to Step 6.

(Step 6):

Set $h[k] = s[0]$,

$h[k+n] = s[n]/2$, $h[k-n] = s[n]/2$ for n = 1, 2, 3, …., k

(referred to page 45)

Then $h[n]$ is the impulse response of the designed filter.

◎ 2-G Mini-Max FIR Filter 設計時需注意的地方

(1) Extreme points 不要選在 transition band

Initial guess的extreme points只要注意別取在transition band裡，即能保證 converge，不同的guess會影響converge的速度但不影響結果

(2) E_1 (error of the previous iteration) < E_0 (present error) 時，亦不為收斂

(3) Remember to update $W(F_m)$ and $H_d(F_m)$ according to the locations of F_m.

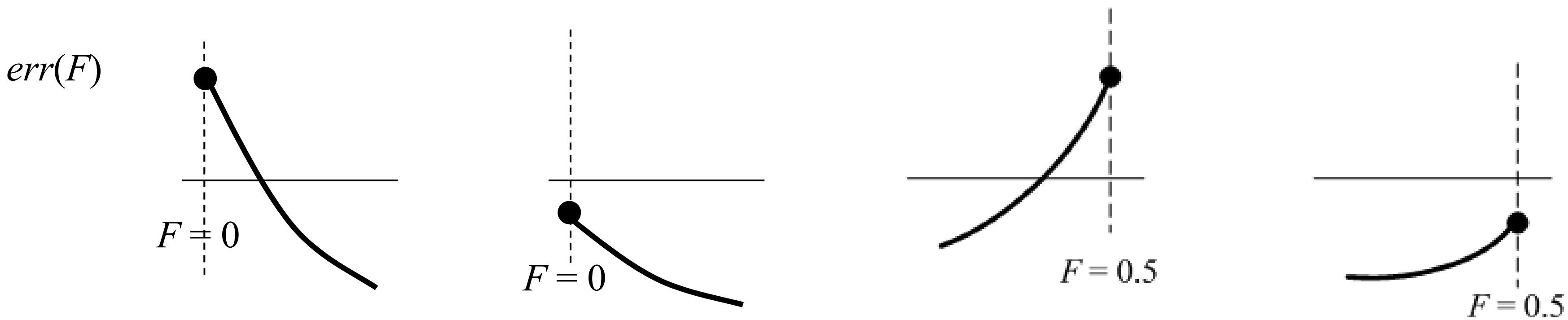
(4) Extreme points 判斷的規則：

(a) The local peaks or local dips that are not at boundaries must be extreme points.

Local peaks: $err(F) > err(F + \Delta_F)$ and $err(F) > err(F - \Delta_F)$

Local dips: $err(F) < err(F + \Delta_F)$ and $err(F) < err(F - \Delta_F)$

(b) For boundary points $(F = 0, F = 0.5)$



Add a zero to the outside and conclude whether the point is a local maximum or a local minimum.

(5) 有時，會找到多於 k+2 個 extreme points, 該如何選 $P_0, P_1, \ldots\ldots, P_k, P_{k+1}$

(a) 優先選擇不在 boundaries 的 extreme points

(b) 其次選擇 boundary extreme points 當中 |err(F)| 較大的，

直到湊足 k+2 個 extreme points 為止

(c) 找好 extreme points 之後，要記得重新依 F 值大小排序

◎ 2-H Examples for Mini-Max FIR Filter Design

- **Example 1:** Design a 9-length highpass filter in the mini-max sense

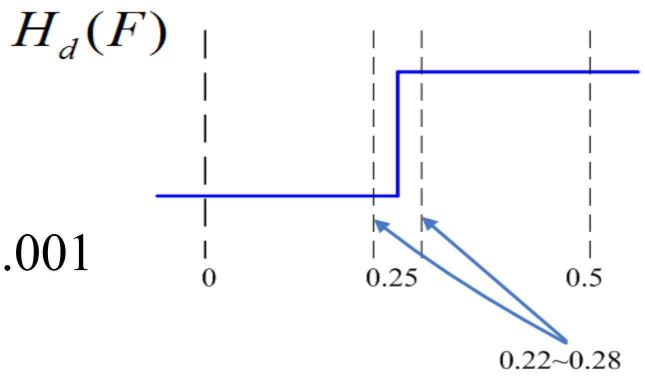
ideal filter: $H_d(F) = 0$ for $0 \le F < 0.25$,

$H_d(F) = 1$ for $0.25 < F \le 0.5$,

transition band: $0.22 < F < 0.28$ $\quad\quad \Delta = 0.001$

weighting function: $W(F) = 0.25$ for $0 \le F \le 0.22$,

$W(F) = 1$ for $0.28 \le F \le 0.5$,



(**Step 1**) Since $N = 9$, $k = (N-1)/2 = 4$, $k+2 = 6$,

$\rightarrow$ **Choose 6 extreme frequencies**

(e.g., $F_0 = 0$, $F_1 = 0.1$, $F_2 = 0.2$, $F_3 = 0.3$, $F_4 = 0.4$, $F_5 = 0.5$)

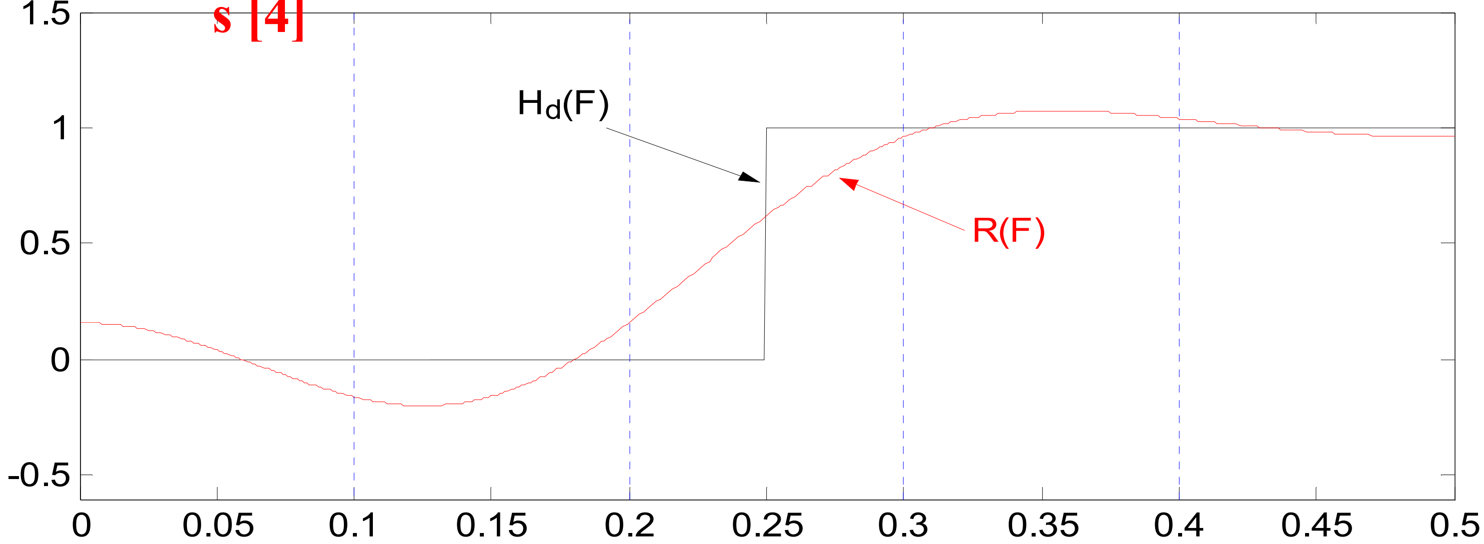
$$[R(F_n) - H_d(F_n)]W(F_n) = (-1)^{n+1}e, \qquad n = 0, 1, 2, 3, 4, 5.$$

(Step 2)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & 0.809 & 0.309 & -0.309 & -0.809 & -4 \\ 1 & 0.309 & -0.809 & -0.809 & 0.309 & 4 \\ 1 & -0.309 & -0.809 & 0.809 & 0.309 & -1 \\ 1 & -0.809 & 0.309 & 0.309 & -0.809 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ s[4] \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

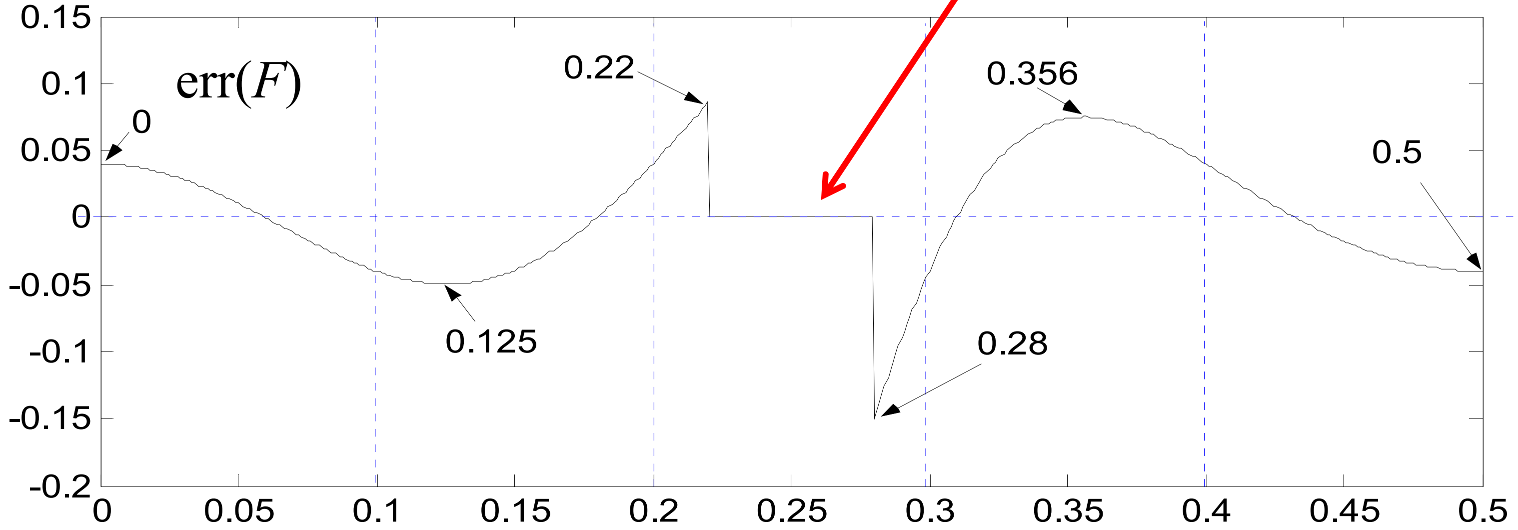
$$\begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ s[4] \\ e \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & 0.809 & 0.309 & -0.309 & -0.809 & -4 \\ 1 & 0.309 & -0.809 & -0.809 & 0.309 & 4 \\ 1 & -0.309 & -0.809 & 0.809 & 0.309 & -1 \\ 1 & -0.809 & 0.309 & 0.309 & -0.809 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5120 \\ -0.6472 \\ -0.0297 \\ 0.2472 \\ 0.0777 \\ -0.040 \end{bmatrix}$$

$$R(F) = \underset{s[0]}{0.5120} - \underset{s[1]}{0.6472}\cos(2\pi F) - \underset{s[2]}{0.0297}\cos(4\pi F) + \underset{s[3]}{0.2472}\cos(6\pi F) + \underset{s[4]}{0.0777}\cos(8\pi F)$$



(Step 3) $\mathbf{err}(F) = [R(F) - H_d(F)]W(F)$

$W(F) = 0$ for $0.22 < F < 0.28$



(Step 4) Extreme points:

$F_0 = 0$, $F_1 = 0.125$, $F_2 = 0.22$, $F_3 = 0.28$, $F_4 = 0.356$, $F_5 = 0.5$

(Step 5) E_0 **=** Max[|err(F)|] = $\underline{0.1501}$, return to Step 2.

Second iteration

(Step 2) Using $F_0 = 0$, $F_1 = 0.125$, $F_2 = 0.22$, $F_3 = 0.28$, $F_4 = 0.356$, $F_5 = 0.5$

$\rightarrow s[0] = 0.5018$, $s[1] = -0.6341$, $s[2] = -0.0194$, $s[3] = 0.3355$,
$s[4] = 0.1385$

(Step 3) $\text{err}(F) = [R(F) - H_d(F)]W(F)$,

(Step 4) extreme points : 0, 0.132, 0.22, 0.28, 0.336, 0.5

(Step 5) E_0 **=** Max[|err(F)|] = $\underline{0.0951}$, return to Step 2.

Third iteration

(Step 2), **(Step 3)**, **(Step 4)**, peaks : 0, 0.132, 0.22, 0.28, 0.334, 0.5

(Step 5) E_0 **=** $\underline{0.0821}$, return to Step 2.

Fourth iteration

(Step 2), **(Step 3)** , **(Step 4)**, peaks : 0, 0.132, 0.22, 0.28, 0.334, 0.5

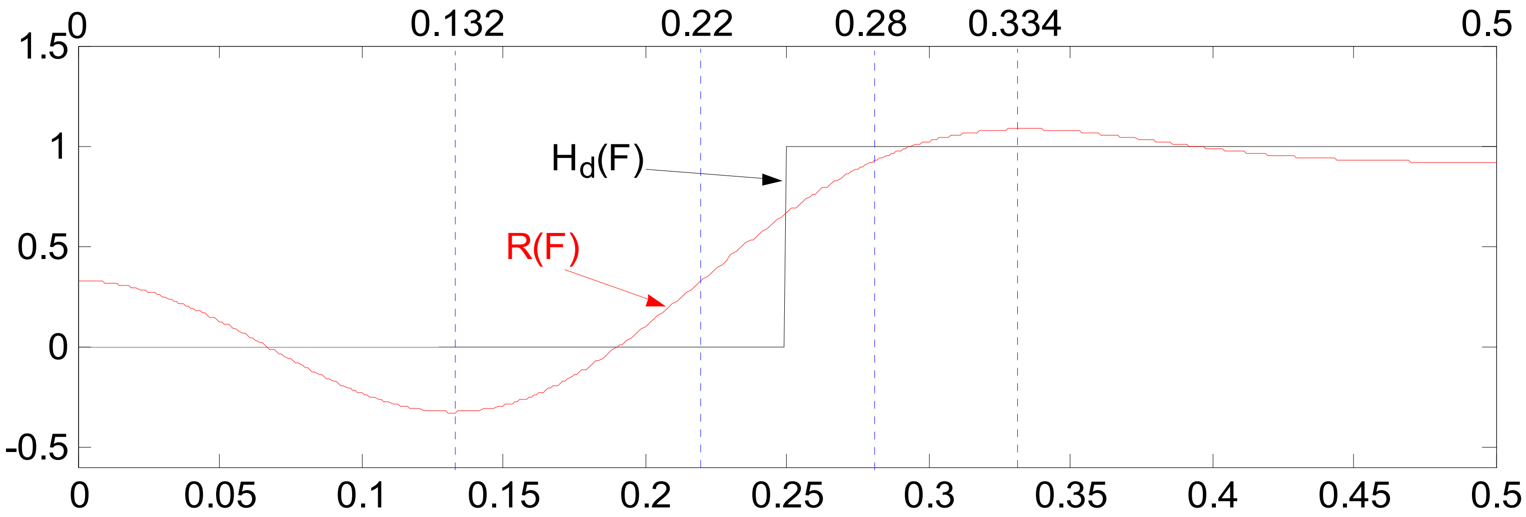
(Step 5) E_0 **=** $\underline{0.0820}$, $E_1 - E_0 = 0.0001 \leq \Delta$, continues to Step 6.

(Step 6) From $s[0] = 0.4990$, $s[1] = –0.6267$, $s[2] = –0.0203$, $s[3] = 0.3316$, $s[4] = 0.1442$

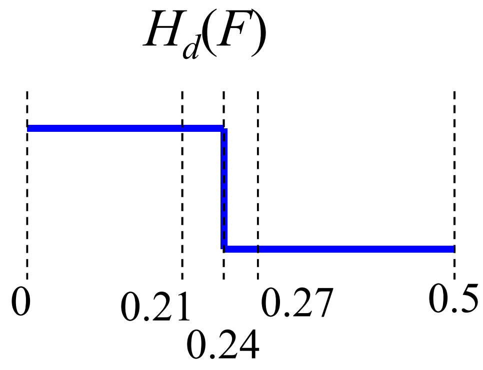
$h[4] = s[0] = 0.4990$, $h[3] = h[5] = s[1]/2 = –0.3134$,

$h[2] = h[6] = s[2]/2 = –0.0101$, $h[1] = h[7] = s[3]/2 = 0.1658$,

$h[0] = h[8] = s[4]/2 = 0.0721$.



- **Example 2:** Design a 7-length digital filter in the mini-max sense



ideal filter: $H_d(F) = 1$ for $0 \le F < 0.24$,

$H_d(F) = 0$ for $0.24 < F \le 0.5$,

transition band: $0.21 < F < 0.27$

weighting function: $W(F) = 1$ for $0 \le F \le 0.21$,

$W(F) = 0.5$ for $0.27 \le F \le 0.5$,

$\Delta = 0.001$

(**Step 1**) Since $N = 7$, $k = (N\text{-}1)/2 = 3$, $k+2 = 5$,

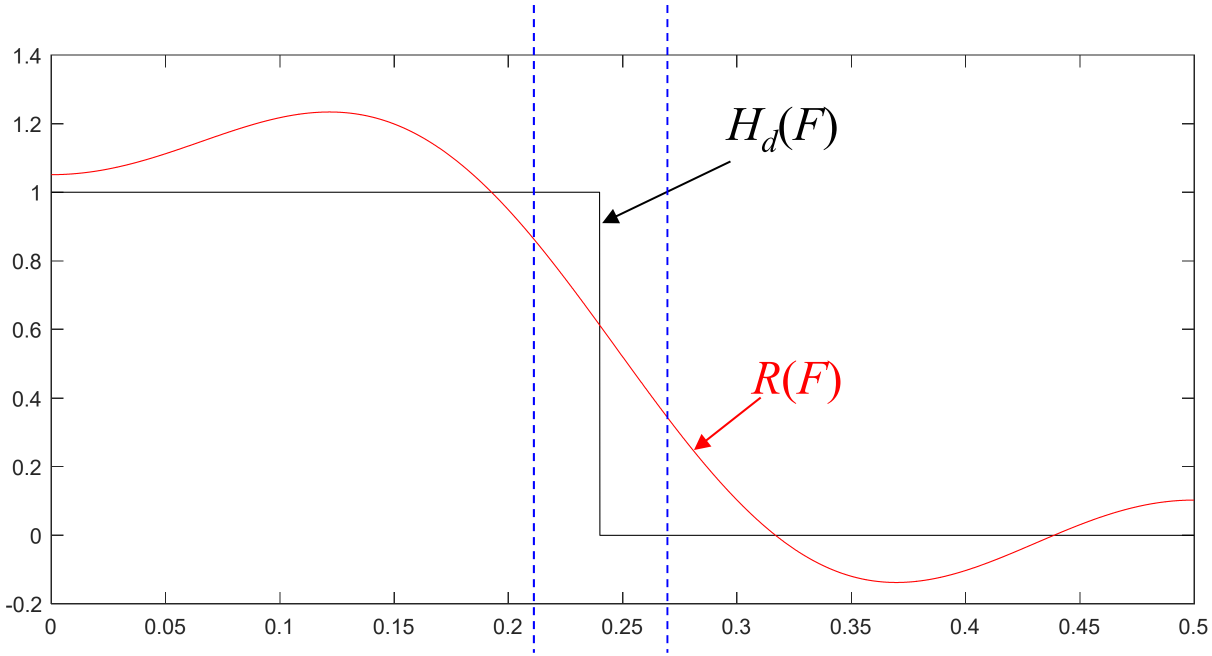
$\rightarrow$ Choose 5 extreme frequencies

(e.g., $F_0 = 0$, $F_1 = 0.2$, $F_2 = 0.3$, $F_3 = 0.4$, $F_4 = 0.5$)

(Step 2)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0.309 & -0.809 & -0.809 & -1 \\ 1 & -0.309 & -0.809 & 0.809 & 2 \\ 1 & -0.809 & 0.309 & 0.309 & -2 \\ 1 & -1 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} s[0] \\ s[1] \\ s[2] \\ s[3] \\ e \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow$ $s[0] = 0.5486$, $s[1] = 0.7215$, $s[2] = 0.0284$, $s[3] = -0.2472$, $e = -0.0514$



After Step 2,

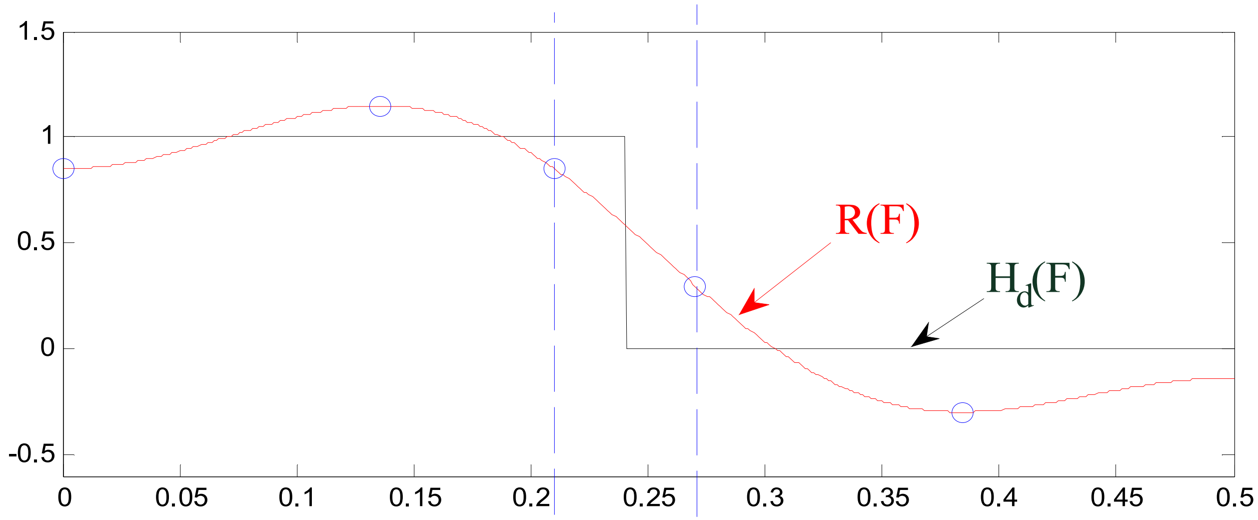
(Step 3) $\mathrm{err}(F) = [0.5486 + 0.7215\cos(2\pi F) + 0.0284\cos(4\pi F)$
$-0.2472\cos(6\pi F) - H_d(F)]W(F)$

(Step 4) extreme points: 0.1217, 0.21, 0.27, 0.3698, 0.5.

(Step 5) E_0 = Max[|err(F)|] = 0.2341, return to Step 2.

Iteration	1	2	3	4	5	6
Max[|err(F)|]	0.2341	0.3848	0.1685	0.1496	0.1493	0.1493

After 7 times of iteration



$s[0] = 0.4243$, $s[1] = 0.7559$, $s[2] = -0.0676$, $s[3] = -0.2619$, $e = 0.1493$

(Step 6):

$h[3] = 0.4243$, $h[2] = h[4] = s[1]/2 = 0.3780$,
$h[1] = h[5] = s[2]/2 = -0.0338$,
$h[0] = h[6] = s[3]/2 = -0.1309$, $h[n] = 0$ for $n < 0$ and $n > 6$

附錄二：Spectrum Analysis for Sampled Signals

(學信號處理的人一定要會的基本常識)

已知 $x[n]$ 是由一個 continuous signal $y(t)$ 取樣而得

$$x[n] = y(n\Delta_t)$$

DFT: $X[m] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nm/N}$ FT: $Y(f) = \int_{-\infty}^{\infty} e^{-j2\pi f t} y(t)\,dt$

Q: $x[n]$ 的 DFT 和 $y(t)$ 的 Fourier transform 之間有什麼關係？

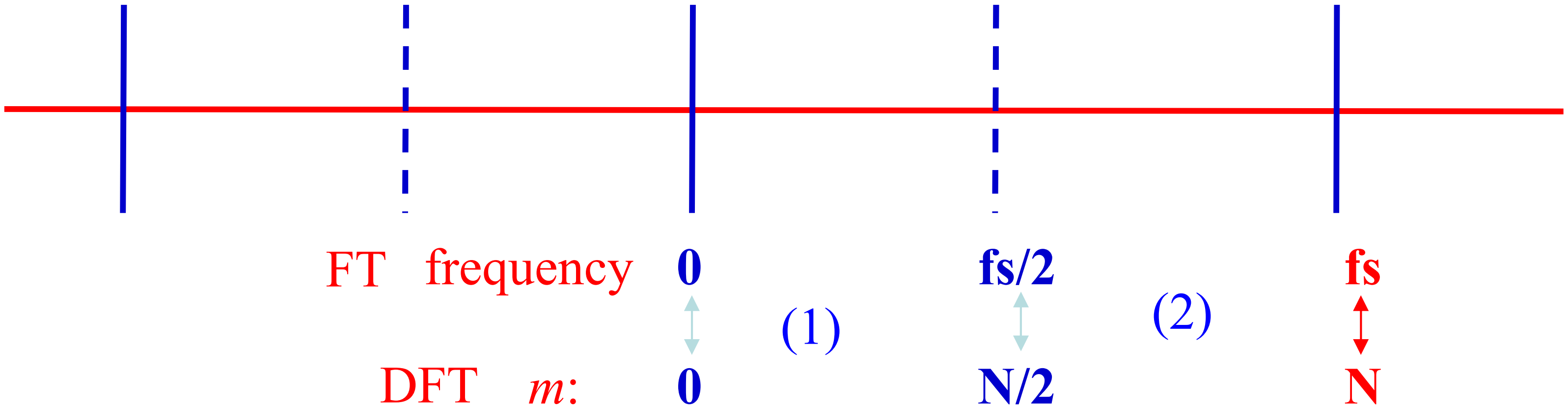
Basic rule：把間隔由 1 換成 f_s/N where $f_s = 1/\Delta_t$

$$f = m\frac{f_s}{N}$$

(Very important)

$$(1)\quad X[m]\Delta_t \cong Y\left(m\frac{f_s}{N}\right) \qquad f_s = 1/\Delta_t \qquad \text{for } m \le N/2$$

$$(2)\quad X[m]\Delta_t \cong Y\left((m-N)\frac{f_s}{N}\right) = Y\left(m\frac{f_s}{N} - f_s\right) \qquad \text{for } m > N/2$$



If the sampling frequency is f_s, the FT output has the period of f_s

The DFT output has the period of N

Proof： $$Y(f)=\int_{-\infty}^{\infty} e^{-j2\pi f t} y(t)\,dt$$

用 $t=n\Delta_t,\ f=m\Delta_f$ 代入

$$Y(m\Delta_f)\cong\sum_n e^{-j2\pi m\Delta_f n\Delta_t} y(n\Delta_t)\Delta_t=\Delta_t\sum_n e^{-j2\pi m\Delta_f n\Delta_t} x[n]$$

當 $\Delta_t\Delta_f=\dfrac{1}{N}$ i.e., $\Delta_f=\dfrac{1}{N\Delta_t}=\dfrac{f_s}{N}$

$$Y\left(m\frac{f_s}{N}\right)\cong\Delta_t\sum_n e^{-j2\pi\frac{mn}{N}} x[n]$$

$$=\Delta_t DFT\{x[n]\}$$

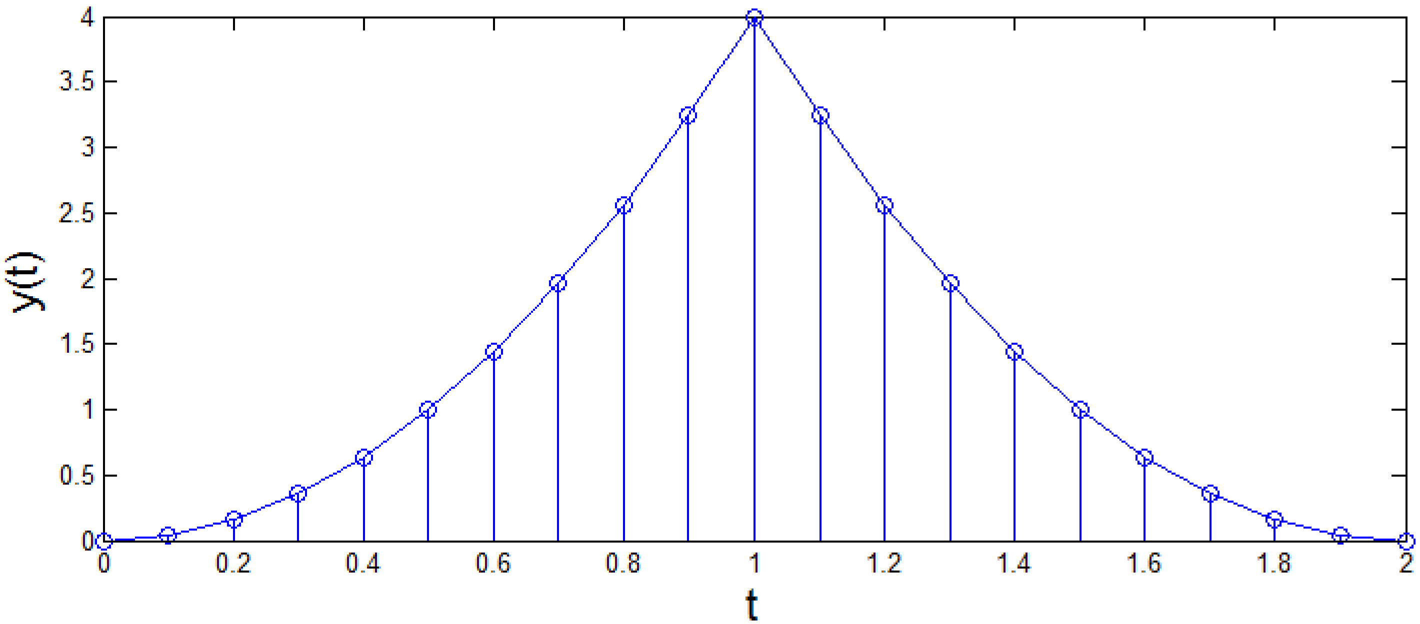
Example：已知

$$y(t) = (2t)^2 \quad \text{for } 0 \le t \le 1 \qquad y(t) = (4-2t)^2 \quad \text{for } 1 \le t \le 2$$

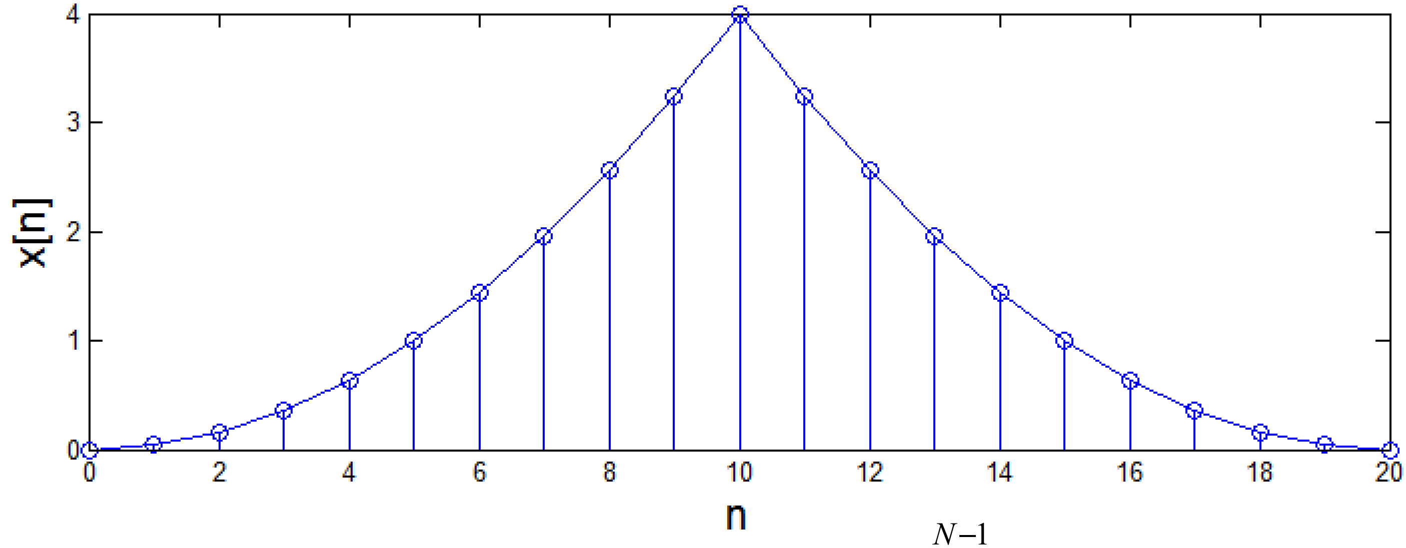
取樣間隔：$\Delta_t = 0.1$

$x[n] = y(n\,\Delta_t)$ for $0 \le n \le 20$

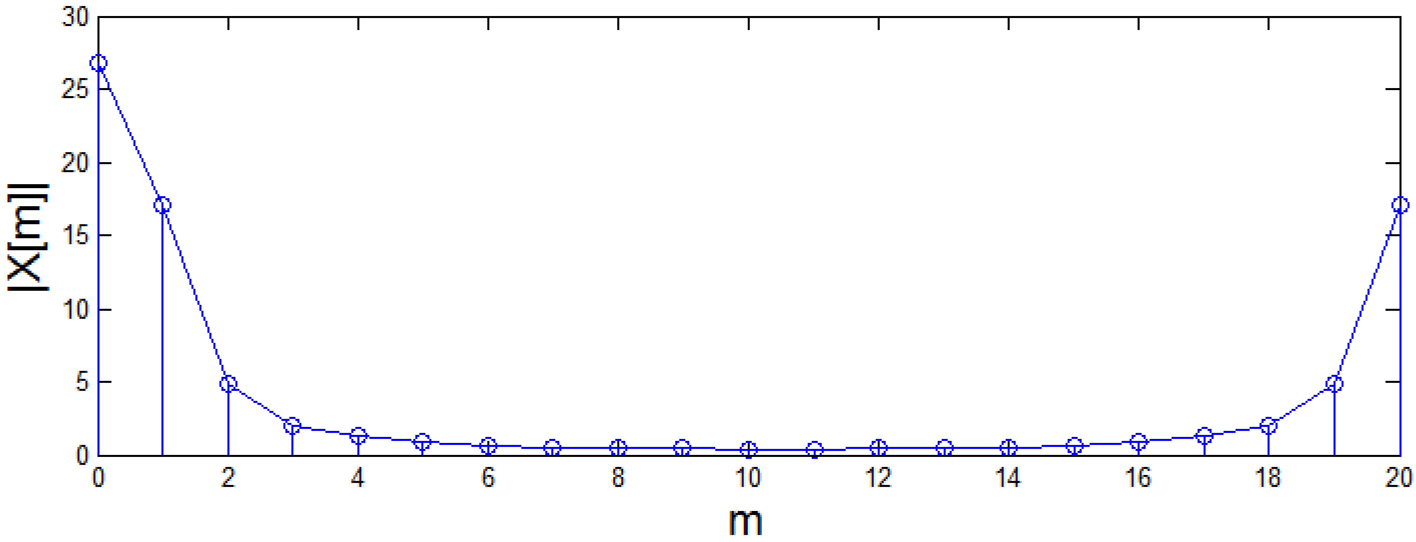
如何用 DFT 來正確的畫出 $y(t)$ 的頻譜？



$x[n] = y(n\,\Delta_t)$ for $0 \le n \le 20$



(Step 1) Perform the DFT for $x[n]$ $$X[m] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nm/N} \qquad N = 21$$



(Step 2-1) $Y\left(m\frac{f_s}{N}\right) \cong X[m]\Delta_t$ for $m \le N/2$

(Step 2-2) $Y\left((m-N)\frac{f_s}{N}\right) \cong X[m]\Delta_t$ for $m > N/2$

In this example, $\frac{f_s}{N} = \frac{1}{N\Delta_t} = \frac{1}{21 \cdot 0.1} = 0.4762$