## ADSP Final Presentation

# Diffusion Model for Inverse Problems

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# Generative AI

- **Deep learning**
  - Use NNs to learn the information of data and generate new content
  - Nowadays, high-resolution and photorealistic images can be produced



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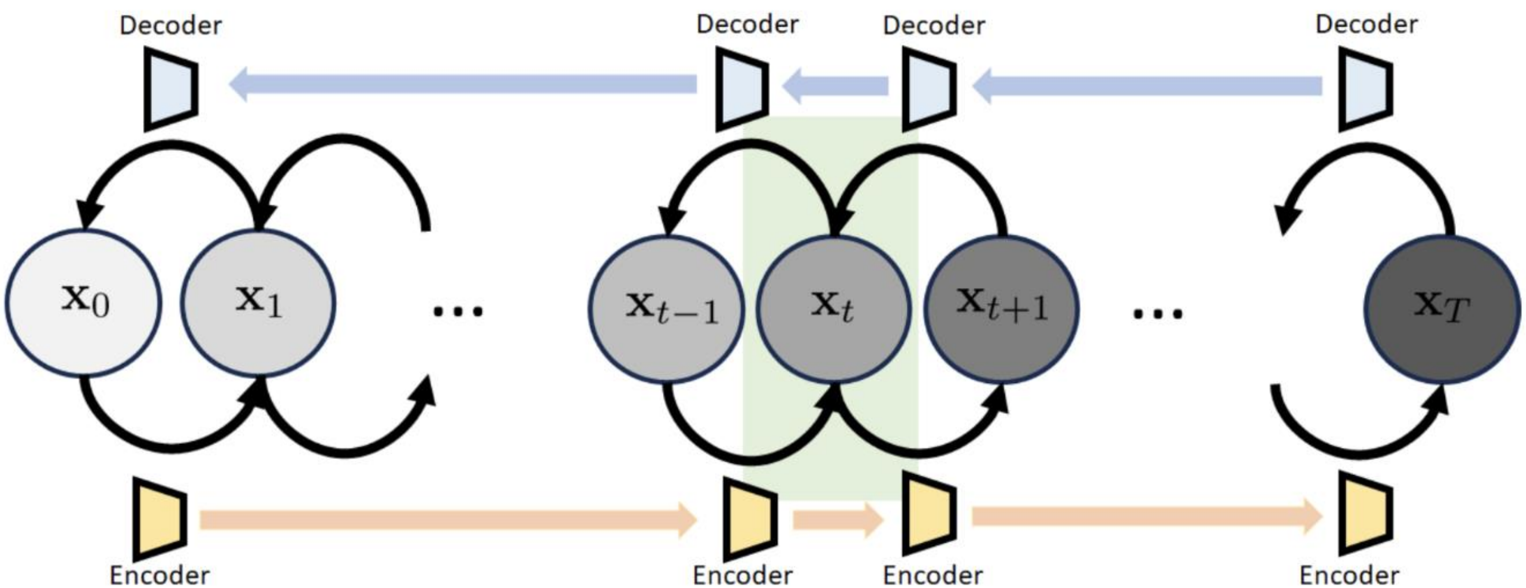
# Denoising Diffusion Probabilistic Model (DDPM) (1/2)

- **Forward diffusion process and reverse diffusion process**
  - Use Markov process to gradually add Gaussian noise or denoise

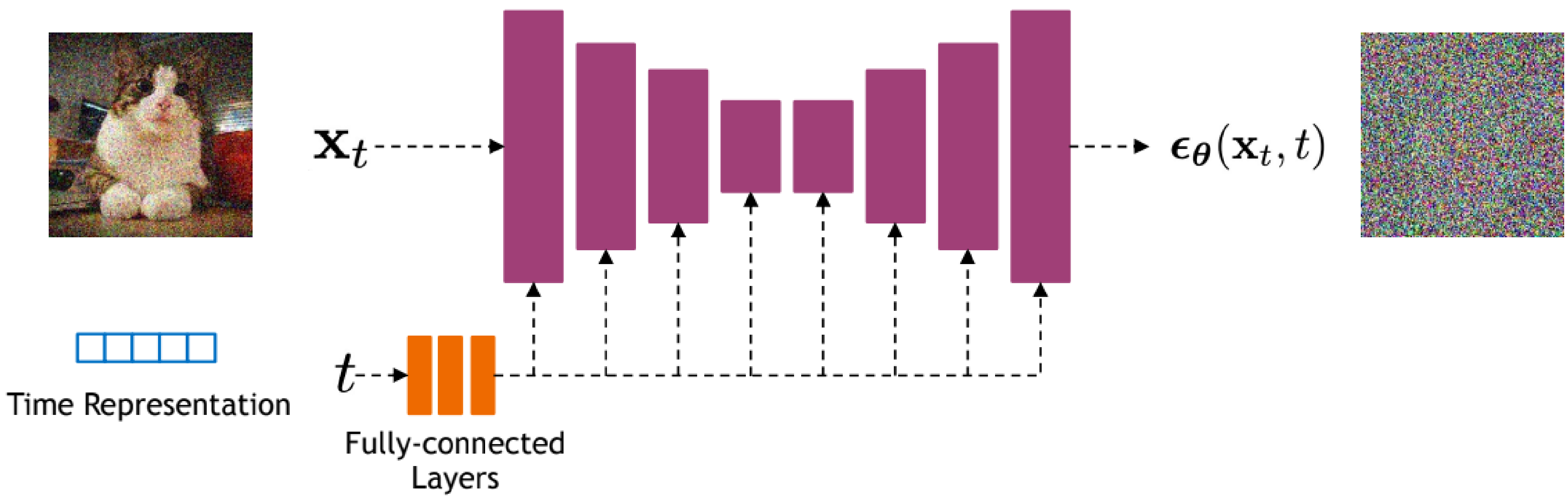
$$q(\mathbf{x}_t|\mathbf{x}_{t-1}) := \mathcal{N}(\mathbf{x}_t; \sqrt{1-\beta_t}\mathbf{x}_{t-1}, \beta_t\mathbf{I}) \quad p_\theta(\mathbf{x}_{t-1}|\mathbf{x}_t) := \mathcal{N}(\mathbf{x}_{t-1}; \boldsymbol{\mu}_\theta(\mathbf{x}_t, t), \boldsymbol{\Sigma}_\theta(\mathbf{x}_t, t))$$



# Denoising Diffusion Probabilistic Model (DDPM) (2/2)

- **Three equivalent objectives to optimize DDPM**
  - Predict the original image $\boldsymbol{x}_{\mathbf{0}}$, source noise $\boldsymbol{\epsilon}_{\mathbf{0}}$, score function $\boldsymbol{s}_{\boldsymbol{\theta}}(\boldsymbol{x})$

$$q(\mathbf{x}_t|\mathbf{x}_0) = \mathcal{N}(\mathbf{x}_t; \sqrt{\bar{\alpha}_t}\mathbf{x}_0, (1-\bar{\alpha}_t)\mathbf{I})$$



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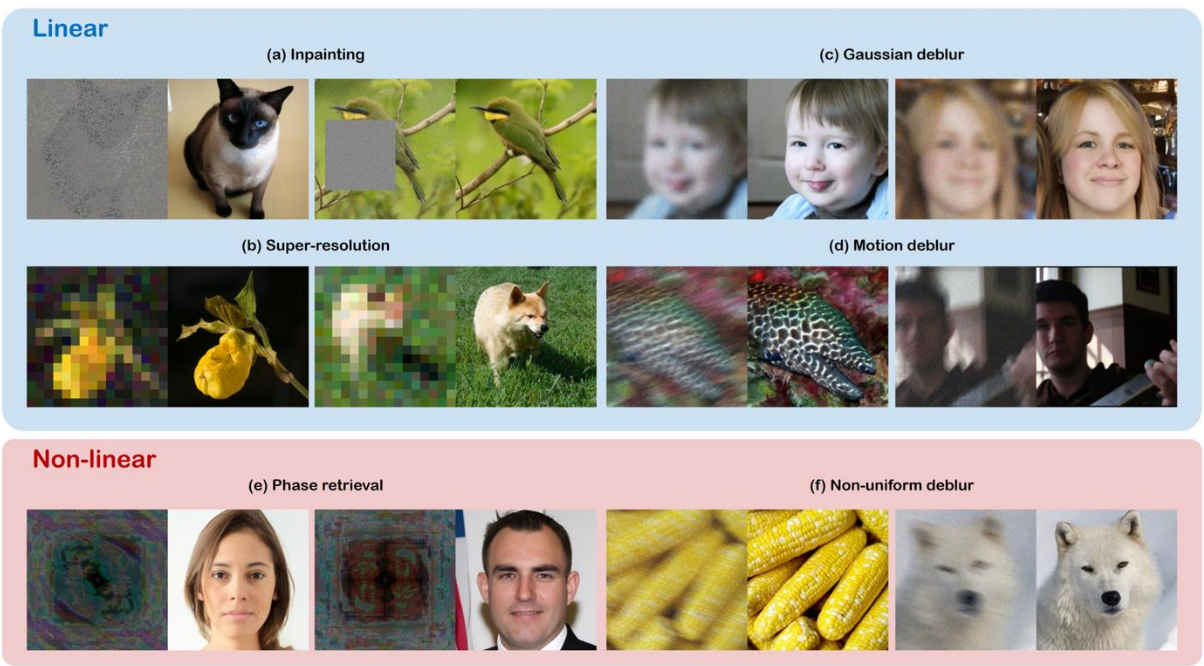
# Papers Overview

***Part I: Inverse Problems with DM***

- DPS: Diffusion Posterior Sampling for General Noisy Inverse Problems
- DDNM: Denoising Diffusion Null-Space Model



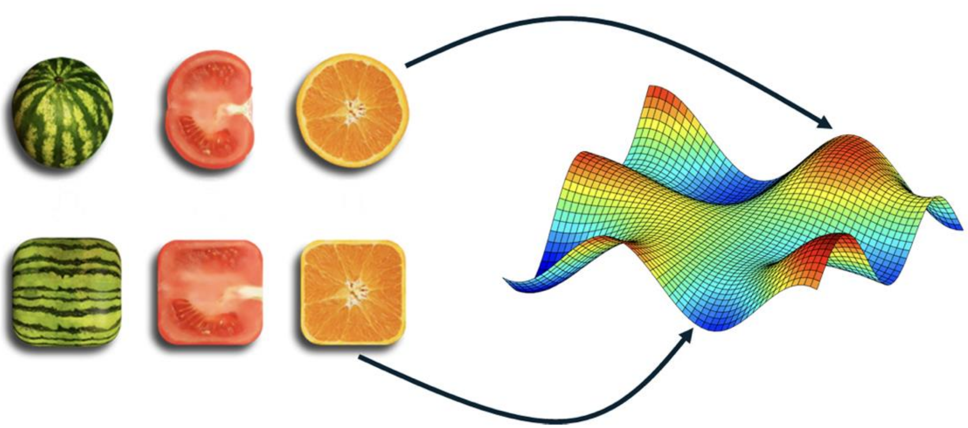
# Score Matching with Langevin Dynamics (SMLD)

- **Stein score function and Langevin dynamics in ML**
  - How to draw a sample from a given distribution function $p(x)$ ?

**Definition 3.1.** The (discrete-time) **Langevin equation** for sampling from a known distribution $p(\mathbf{x})$ is an iterative procedure for $t = 1, \ldots, T$:

$$\mathbf{x}_{t+1} = \mathbf{x}_t + \tau \nabla_{\mathbf{x}} \log p(\mathbf{x}_t) + \sqrt{2\tau}\mathbf{z}, \qquad \mathbf{z} \sim \mathcal{N}(0, \mathbf{I}), \tag{96}$$

where $\tau$ is the step size which users can control, and $\mathbf{x}_0$ is white noise.



# Score-Based Generative Model

- **Score-based generative modeling with SDEs**
  - Score Matching with Langevin Dynamics (SMLD) & Denoising Diffusion Probabilistic Models (DDPM) are discretizations of two distinct SDEs
  - Diffusion process can be modeled as the solution to an Itō SDE
  - Generate samples by reverse-time SDE (estimate score function)
  - SMLD (NCSN) → Variance Exploding (VE) SDE

$$d\boldsymbol{x} = \sqrt{\frac{d[\sigma^2(t)]}{dt}}\, d\boldsymbol{w}$$

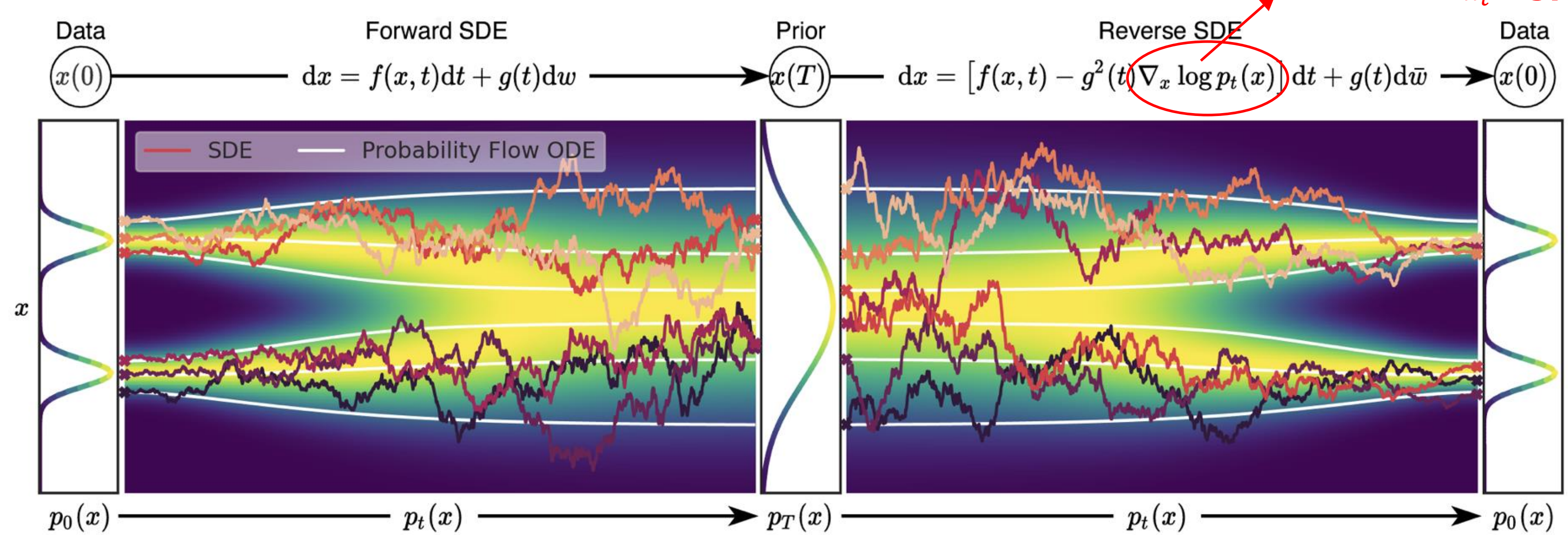
  - DDPM → Variance Preserving (VP) SDE

$$d\boldsymbol{x} = -\frac{1}{2}\beta(t)\boldsymbol{x}\, dt + \sqrt{\beta(t)}\, d\boldsymbol{w}$$

# Inverse Problem with Diffusion Model

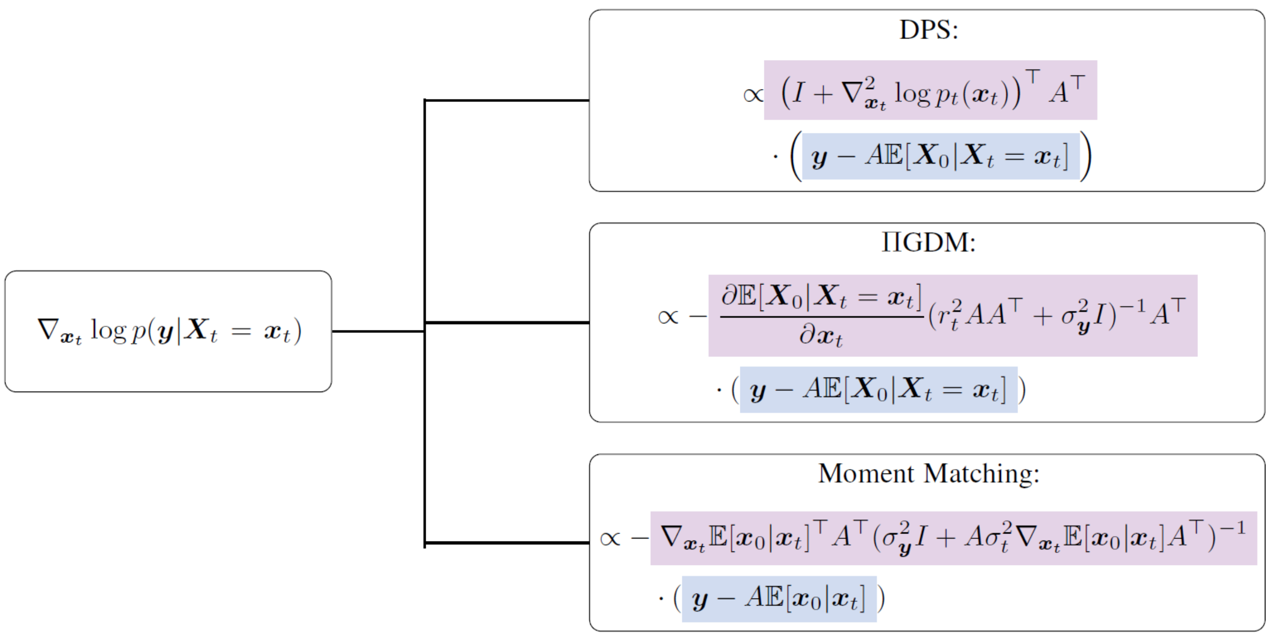
- **Inverse Problem setting $y = \mathcal{A}(x) + n$ (i.e. $n \sim \mathcal{N}(0, \sigma_y^2 \mathbb{I})$)**
  - Score function (Posterior) : $\nabla_x \log p(x|y) = \nabla_x \log p(x) + \nabla_x \log p(y|x)$
  - $\nabla_x \log p(x)$ is hard to estimate w/o diffusion model (view DM as a prior)
  - $\nabla_x \log p(y|x)$ is easy to compute, but $\nabla_{x_t} \log p(y|x_t)$ is intractable with DM (because of $p(y|x_t) = \int p(y|x_0) p(x_0|x_t)\, dx_0$)

modified: $\nabla_{x_t} \log p(x_t|y)$



# Explicit Measurements Matching term

- **Approximation of $p(x_0|x_t) \to p(y|x_t) = \int p(y|x_0)p(x_0|x_t)\,dx_0$**
  - DPS (Diffusion Posterior Sampling) : $\delta(x_0 - \mathbb{E}[X_0|X_t = x_t])$
  - ΠGDM (Pseudoinverse-guided Diffusion Models) : $\mathcal{N}(\mathbb{E}[X_0|X_t = x_t], r_t^2 I_n)$
  - Moment Matching : $\mathcal{N}\left(\mathbb{E}[X_0|X_t = x_t], \mathbb{V}[X_0|X_t = x_t] = \sigma_t^2 \nabla_{x_t}\mathbb{E}[X_0|X_t = x_t]\right)$



# Diffusion Posterior Sampling (DPS)

- **Posterior sampling with Jensen's inequality ( $\widehat{\boldsymbol{x}_0} = \mathbb{E}[X_0|X_t = \boldsymbol{x}_t]$)**
  - Jensen gap : $\mathcal{J} = \mathbb{E}_{\boldsymbol{x}_0 \sim p(\boldsymbol{x}_0|\boldsymbol{x}_t)}[p(\boldsymbol{y}|\boldsymbol{x}_0)] - p\left(\boldsymbol{y}\middle|\mathbb{E}_{\boldsymbol{x}_0 \sim p(\boldsymbol{x}_0|\boldsymbol{x}_t)}[\boldsymbol{x}_0]\right)$
  - Gaussian noise $\boldsymbol{n}$ (d-dim measurement) → Jensen gap has upper bound

$$\mathcal{J} \leq \frac{d}{\sqrt{2\pi\sigma_{\boldsymbol{y}}^2}} e^{-\frac{1}{2\sigma_{\boldsymbol{y}}^2}} (\max_{\boldsymbol{x}} \|\nabla_{\boldsymbol{x}} \mathcal{A}(\boldsymbol{x})\|) \int \|\boldsymbol{x}_0 - \widehat{\boldsymbol{x}_0}\| \, p(\boldsymbol{x}_0|\boldsymbol{x}_t) d\boldsymbol{x}_0$$

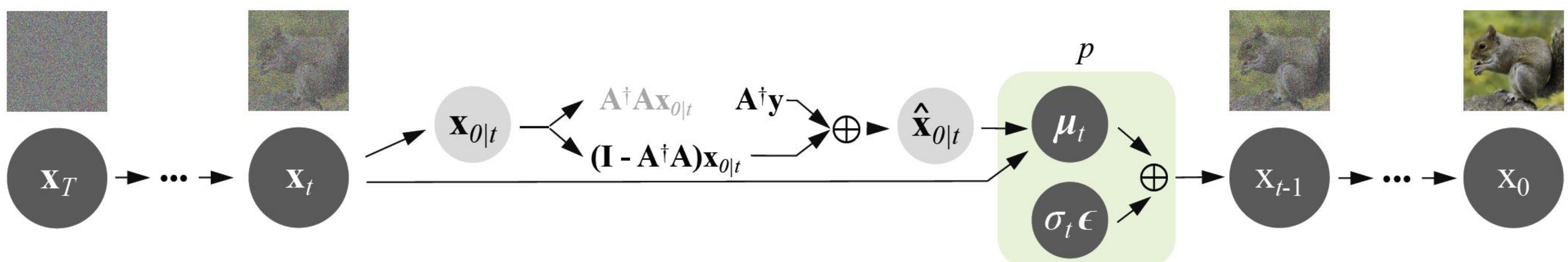
  - Approximate gradient of log likelihood $\nabla_{\boldsymbol{x}_t} \log p(\boldsymbol{y}|\boldsymbol{x}_t) \rightarrow \nabla_{\boldsymbol{x}_t} \log p(\boldsymbol{y}|\widehat{\boldsymbol{x}_0})$

$$\nabla_{\boldsymbol{x}_t} \log p(\boldsymbol{y}|\boldsymbol{x}_t) \approx -\frac{1}{\sigma_{\boldsymbol{y}}^2} \nabla_{\boldsymbol{x}_t} \|\boldsymbol{y} - \mathcal{A}(\widehat{\boldsymbol{x}_0})\|_2^2$$

$$\nabla_{\boldsymbol{x}_t} \log p(\boldsymbol{x}_t|\boldsymbol{y}) \approx s_{\theta^*}(\boldsymbol{x}_t, t) - \frac{1}{\sigma_{\boldsymbol{y}}^2} \nabla_{\boldsymbol{x}_t} \|\boldsymbol{y} - \mathcal{A}(\widehat{\boldsymbol{x}_0})\|_2^2$$

# Denoising Diffusion Null-Space Model (DDNM)

- **DDNM: Noise-Free Image restoration problem $y = Ax$**
  - (Matrix) Using singular value decomposition, we can compute $A^{\dagger} = V\Sigma^{\dagger}U^{*}$
  - Range-Null space decomposition $Ax = A\left(A^{\dagger}Ax + \left(I - A^{\dagger}A\right)x\right) = y$
  - Diffusion model should generate a solution $\hat{x}_{0|t}$ that satisfies $A\hat{x}_{0|t} = Ax$
  - Reverse process sampling with $p\left(x_{t-1} \middle| x_t, t, \hat{x}_{0|t} = A^{\dagger}y + \left(I - A^{\dagger}A\right)x_{0|t}\right)$



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# Summary

- DPS provides a differentiable guidance-loss method for solving inverse problems
- DDNM is more hardware-friendly for DSP processor design, as it uses a given operator to guide the synthesis direction

# Reference

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