

2-4 Exact Equations

2-4-1 方法的條件

任何 first order DE 皆可改寫成

$$M(x, y)dx + N(x, y)dy = 0 \quad \text{的型態}$$

(1) 當 $\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$ 成立時，

可以用本節的 **Exact Equation** 的方法來解

(2) 當 $\frac{\frac{\partial}{\partial y} M(x, y) - \frac{\partial}{\partial x} N(x, y)}{M(x, y)}$ is independent of x

或 $\frac{\frac{\partial}{\partial y} M(x, y) - \frac{\partial}{\partial x} N(x, y)}{N(x, y)}$ is independent of y

可以用 **Modified Exact Equation Method** 來解 (見講義 2-4-5)

2-4-2 方法的来源

- Review the concept of partial differentiation

$$df(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- Specially, when $f(x, y) = c$ where c is some constant,

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

補充：

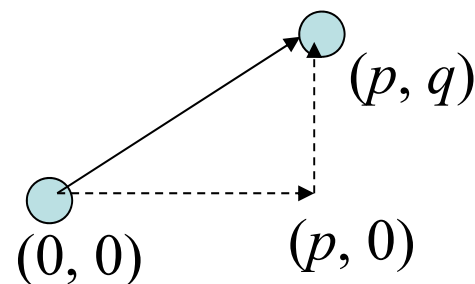
$$df(x, y, z) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$df(x_1, x_2, x_3, \dots, x_k) = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 + \dots + \frac{\partial f}{\partial x_k} dx_k$$

思考： 假設一個人在山坡的某處。若往東走，每走 1 公尺，高度會增加 a 公尺。若往北走，每走 1 公尺，高度會增加 b 公尺。假設這人現在所在的位置是 $(0, 0)$ 。那麼這人的東北方，座標為 (p, q) 的地方，高度會比 $(0, 0)$ 高多少？

$$a \times p + b \times q$$

$$df(x, y) = \underbrace{\frac{\partial f(x, y)}{\partial x}}_a \underbrace{dx}_p + \underbrace{\frac{\partial f(x, y)}{\partial y}}_b \underbrace{dy}_q$$



[Definition 2.4.1] We can express any 1st order DE as

$$M(x, y)dx + N(x, y)dy = 0$$

- If there exists some function $f(x, y)$ that satisfies

$$\frac{\partial f(x, y)}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y} = N(x, y) \quad ,$$

then we call the 1st order DE the exact equation.

- The method for checking whether the DE is an exact equation:

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

(Proof): If $\frac{\partial f(x, y)}{\partial x} = M(x, y)$ and $\frac{\partial f(x, y)}{\partial y} = N(x, y)$,

$$\text{then } \frac{\partial M(x, y)}{\partial y} = \frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x, y) = \frac{\partial N(x, y)}{\partial x}$$

For the exact equation,

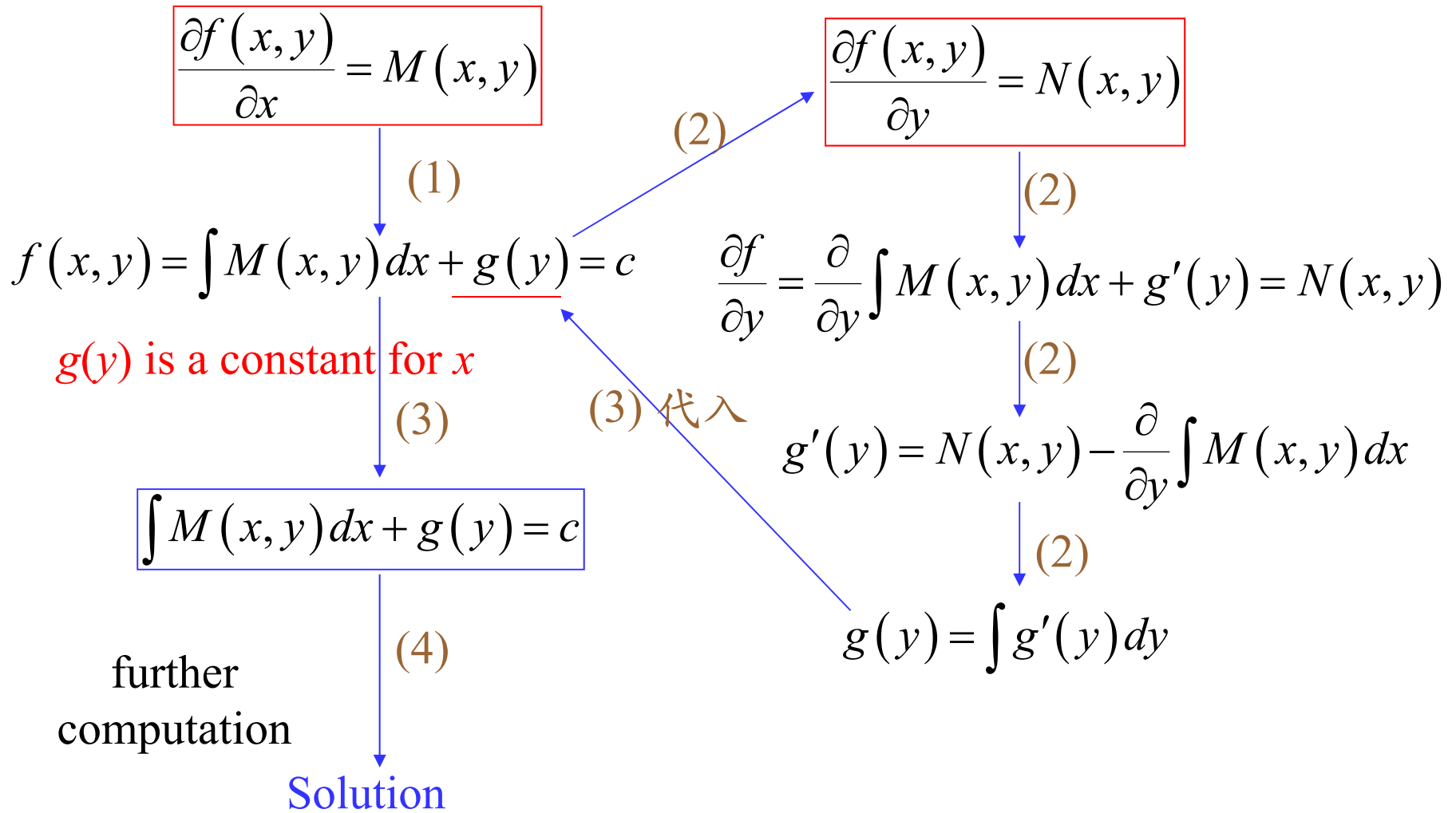
$$M(x, y)dx + N(x, y)dy = 0$$

$$\frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y} dy = 0$$

可改寫成 $df(x, y) = 0,$

$$f(x, y) = c$$

The Method for Solving the Exact Equation (A):



(0) Check $\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$

Previous Step: Check whether $\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$ is satisfied.

Step 1: Solve $\frac{\partial f(x, y)}{\partial x} = M(x, y) \longrightarrow f(x, y) = \int M(x, y) dx + g(y)$

Step 2: 將 $f(x, y)$ 代入 $\frac{\partial f(x, y)}{\partial y} = N(x, y)$ ，以解出 $g(y)$

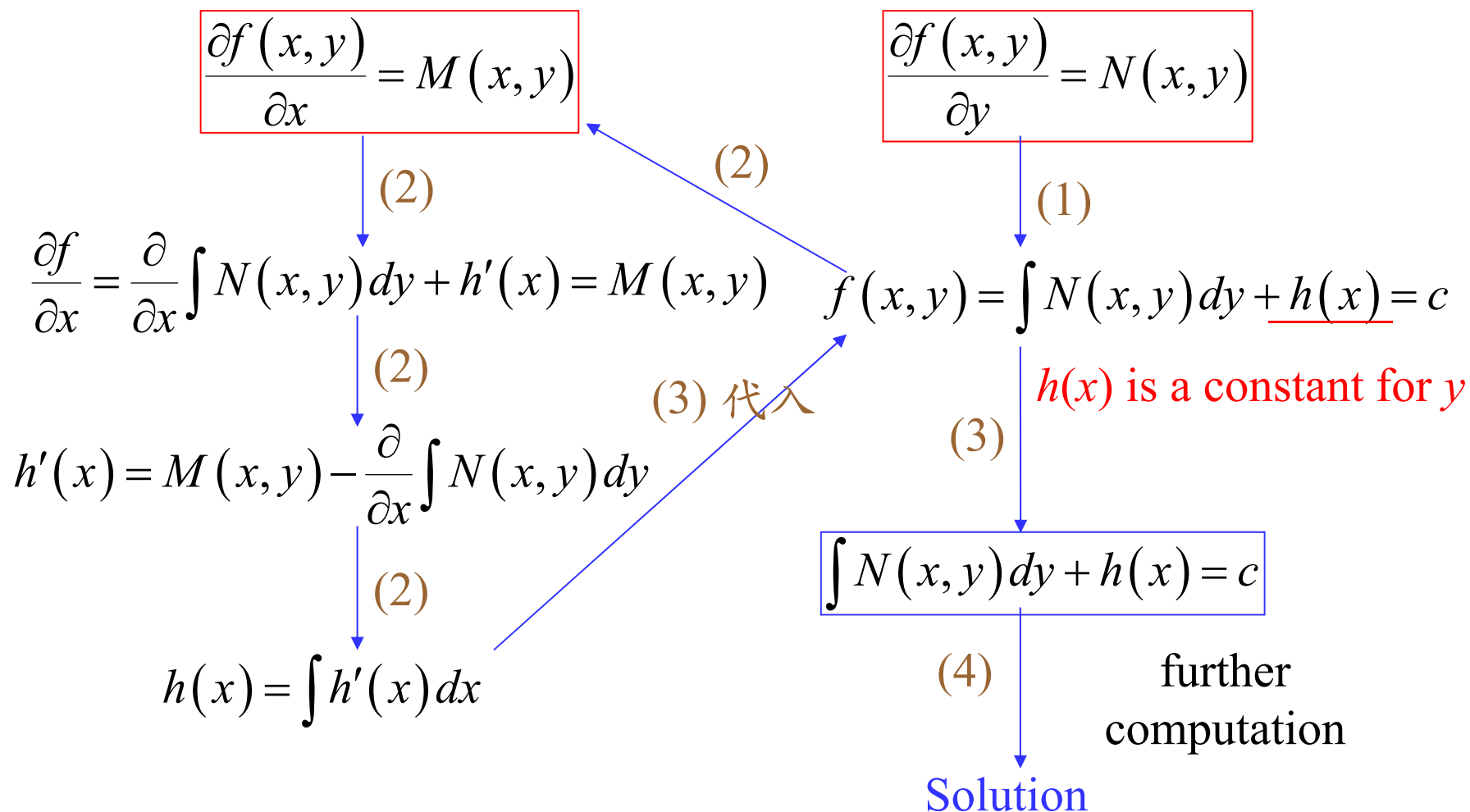
Step 3: Substitute $g(y)$ into

$$f(x, y) = \int M(x, y) dx + g(y) = c$$

Step 4: Further computation and obtain the solution

Extra Steps: (a) Consider the initial value problem

The Method for Solving the Exact Equation (B):



(0) Check $\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y)$

2-4-4 例子

[Example 1] (text page 67)

$$2xydx + (x^2 - 1)dy = 0$$

$$M(x, y) = 2xy$$

$$N(x, y) = x^2 - 1$$

Step 0: check whether it is exact

$$\frac{\partial f}{\partial x} = 2xy$$

$$\frac{\partial f}{\partial y} = x^2 - 1$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

Step 1

Step 2

Step 2

$$f(x, y) = x^2y + g(y)$$

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1$$

Step 3

Step 3

Step 2

$$x^2y - y = c$$

$$g'(y) = -1$$

Step 2

Step 4

$$g(y) = -y$$

$$y = c / (x^2 - 1)$$

思考: 是否有其他的方法可以解 Example 1?

[Example 2] (text page 67)

$$(e^{2y} - y \cos xy)dx + (2xe^{2y} - x \cos xy + 2y)dy = 0$$

$$M(x, y) = e^{2y} - y \cos xy \quad N(x, y) = 2xe^{2y} - x \cos xy + 2y$$

$$\frac{\partial f}{\partial x} = e^{2y} - y \cos xy$$

Step 2

$$e^{2y} - y \cos xy + h'(x)$$

$$= e^{2y} - y \cos xy$$

$$h'(x) = 0$$

Step 2

$$h(x) = \cancel{c_1} 0$$

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x \cos xy + 2y$$

Step 1

$$f(x, y) = xe^{2y} - \sin xy + y^2 + h(x)$$

Step 3

$$f(x, y) = xe^{2y} - \sin xy + y^2$$

Step 4

$$c = xe^{2y} - \sin xy + y^2$$

$$xe^{2y} - \sin xy + y^2 + c = 0$$

Step 0:

check for exact

$$\frac{\partial M}{\partial y}$$

$$= 2e^{2y} - \cos xy$$

$$+ xy \sin xy$$

$$= \frac{\partial N}{\partial x}$$

要注意

(a) 自行由另一個方向 $f(x, y) = \int M(x, y) dx + g(y)$ 來練習，

看是否得出同樣的結果。

(b) 得出的解 $xe^{2y} - \sin xy + y^2 + c = 0$ 為 implicit solution

(c) **思考**：何時用 $f(x, y) = \int M(x, y) dx + g(y)$

何時用 $f(x, y) = \int N(x, y) dy + h(x)$

[Example 3] (text page 68)

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)} \quad y(0) = 2$$

自修，但注意

(a) initial value problem,

(b) 何時用 $f(x, y) = \int M(x, y) dx + g(y)$

何時用 $f(x, y) = \int N(x, y) dy + h(x)$

(c) 得出的 implicit solution 為 $y^2(1-x^2) - \cos^2 x = 3$, 範圍： $x \in (-1, 1)$

而 explicit solution 為 $y = \sqrt{\frac{3 + \cos^2 x}{1 - x^2}}$, 範圍： $x \in (-1, 1)$

為何 $y = -\sqrt{\frac{3 + \cos^2 x}{1 - x^2}}$ 不為解？

2-4-5 Modified Exact Equation Method

Technique: Use the **integrating factor** $\mu(x, y)$ to convert the 1st order DE into the **exact equation**.

$$M(x, y)dx + N(x, y)dy = 0$$

$$\downarrow$$

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

such that
$$\frac{\partial \mu(x, y)M(x, y)}{\partial y} = \frac{\partial \mu(x, y)N(x, y)}{\partial x}$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\mu_x N - \mu_y M = (M_y - N_x)\mu$$

It is hard to find μ .

Simplification: To make $\mu_x = 0$ or $\mu_y = 0$.

$$\mu_x N - \mu_y M = (M_y - N_x) \mu$$

(1) If we want $\mu_x = 0$, i.e, μ is dependent on y alone, then

$$-\mu_y M = (M_y - N_x) \mu$$

$$\frac{\mu_y}{\mu} = -\frac{M_y - N_x}{M}$$

→ $(M_y - N_x) / M$ should be a function of y alone.

$$\frac{1}{\mu} \frac{d\mu}{dy} = -\frac{M_y - N_x}{M} \quad \text{用 separable variable 的方法}$$

$$\frac{d\mu}{\mu} = \frac{N_x - M_y}{M} dy$$

$$\mu(y) = e^{\int \frac{(N_x - M_y)}{M} dy}$$

(注意正負號)

$$\mu_x N - \mu_y M = (M_y - N_x) \mu$$

(1) If we want $\mu_y = 0$, i.e, μ is dependent on x alone, then

$$\mu_x N = (M_y - N_x) \mu$$

$$\frac{\mu_x}{\mu} = \frac{M_y - N_x}{N}$$

→ $(M_y - N_x) / N$ should be a function of x alone.

$$\frac{1}{\mu} \frac{d\mu}{dx} = \frac{M_y - N_x}{N}$$

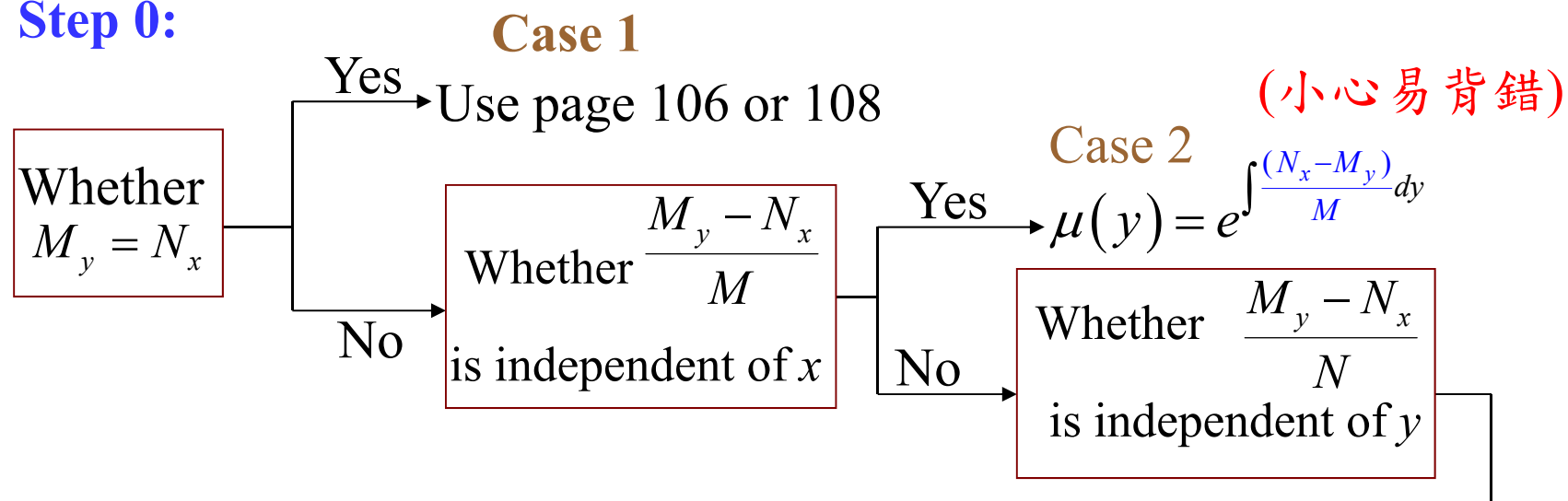
用 separable variable 的方法

$$\frac{d\mu}{\mu} = \frac{M_y - N_x}{N} dx$$

$$\mu(x) = e^{\int \frac{(M_y - N_x)}{N} dx}$$

前面 2-4-3 (pages 106~108) 的解法流程再加一個步驟：

Step 0:



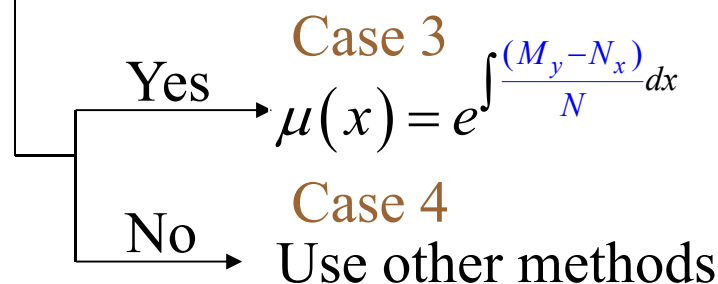
In Cases 2 and 3,

$$M(x, y) dx + N(x, y) dy = 0$$

$$\mu M(x, y) dx + \mu N(x, y) dy = 0$$

Use the process of page 106 or 108 but $M(x, y)$ is modified as $\mu M(x, y)$

$N(x, y)$ is modified as $\mu N(x, y)$



[Example 4] (text page 69)

$$xydx + (2x^2 + 3y^2 - 20)dy = 0$$

Step 0: $M = xy$ $N = 2x^2 + 3y^2 - 20$

$$M_y - N_x = x - 4x = -3x$$

$$\frac{M_y - N_x}{N} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

$$\frac{M_y - N_x}{M} = -\frac{3}{y}$$

(independent of x)
(Case 2)

$$\mu(y) = e^{\int \frac{3}{y} dy} = e^{3 \ln|y|} = y^3$$

Q: 為何 c 以及
 \pm 可省略?

$$xy^4 dx + (2x^2 y^3 + 3y^5 - 20y^3) dy = 0$$

double N

Steps 1~4:

$$\frac{1}{2} x^2 y^4 + \frac{1}{2} y^6 - 5y^4 = c$$

2-4-6 本節需要注意的地方

(1) 使本節方法時，要先將 DE 改成如下的型態

$$M(x, y)dx + N(x, y)dy = 0$$

並且假設

$$\frac{\partial}{\partial x} f(x, y) = M(x, y), \quad \frac{\partial}{\partial y} f(x, y) = N(x, y)$$

(2) 對 x 而言， $g(y)$ 是個常數；對 y 而言， $h(x)$ 是個常數

(3) 本節很少有 singular solution 的問題，

但是可能有 singular point 的問題

(4) 背熟三個判別式，二種情況的 integrating factor (小心勿背錯)

並熟悉解法的流程

2-5 Solutions by Substitutions

介紹 3 個特殊解法

Question: 尚有不少的 1st order DE 無法用 Sections 2-2~2-4 的方法來解

本節所提到的特殊解法的共通點：

用新的變數 u 來取代 y

對症下藥

2-5-1 特殊解法 1: Homogeneous Equations

If $g(tx, ty) = t^\alpha g(x, y)$,

then $g(x, y)$ is a **homogeneous function of degree α** .

Which one is homogeneous?

$$g(x, y) = x^3 + y^3$$

$$g(x, y) = x^3 + y^3 + 1$$

(key: The sum of powers should be the same).

Note: 課本中，**homogeneous** 有兩種定義

一種是 Section 2-3 的定義 (較常用)

一種是這裡的定義

兩者並不相同

■ For a 1st order DE:

$$M(x, y)dx + N(x, y)dy = 0$$

If $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree,
then the 1st order DE is homogeneous.

解法的限制條件



It can be solved by setting

$$y = xu, \quad dy = udx + xdu,$$

(from page 103)

and use the separable value method.

If $M(x, y)dx + N(x, y)dy = 0$ is homogeneous

$$M(tx, ty) = t^\alpha M(x, y) \quad N(tx, ty) = t^\alpha N(x, y)$$

then

$$M(1, u) = x^{-\alpha} M(x, y) \quad N(1, u) = x^{-\alpha} N(x, y)$$

$$M(x, y) = x^\alpha M(1, u) \quad N(x, y) = x^\alpha N(1, u)$$

以 $t = 1/x$ 得出

where $u = y/x, \quad y = xu$

$$dy = udx + xdu$$

$$x^\alpha M(1, u)dx + x^\alpha N(1, u)(udx + xdu) = 0$$

$$[M(1, u) + uN(1, u)]dx + xN(1, u)du = 0$$

$$\frac{dx}{x} = -\frac{N(1, u)du}{M(1, u) + uN(1, u)} \quad (\text{separable})$$

Procedure for solving the homogeneous 1st order DE

Previous Step: Conclude whether the DE is homogeneous

(快速判斷法：看 powers (指數) 之和是否一致)

Step 1: Set $u = y/x$ ($y = ux$), $dy = udx + xdu$

並化簡

Step 2: Convert into the separable 1st order DE

Step 3: Solve the separable 1st order DE (用 Sec. 2-2 的方法)

Step 4: Substitute $u = y/x$ (別忘了這個步驟)

[Example 1] (text page 73)

$$\underbrace{(x^2 + y^2)}_{M(x, y)} dx + \underbrace{(x^2 - xy)}_{N(x, y)} dy = 0$$

$$M(tx, ty) = t^2 M(x, y) \quad N(tx, ty) = t^2 N(x, y)$$

homogeneous DE

Previous Step:

Conclude whether the DE is homogeneous

Step 1: Set $y = ux$, $dy = udx + xdu$

原式

$$(x^2 + u^2 x^2) dx + (x^2 - ux^2)(udx + xdu) = 0$$

$$(1 + u^2) dx + (1 - u)(udx + xdu) = 0$$

$$(1 + u) dx + (1 - u) x du = 0 \longrightarrow \boxed{\left[\frac{1-u}{1+u} \right] du + \frac{dx}{x} = 0}$$

Step 2: Convert into the separable 1st order DE

$$\left[\frac{1-u}{1+u} \right] du + \frac{dx}{x} = 0$$

Step 3: Solve the separable 1st order DE

$$\int \left[-1 + \frac{2}{1+u} \right] du + \int \frac{dx}{x} = 0$$

$$-u + 2 \ln|1+u| + \ln|x| + c_1 = 0$$

$$\ln[(1+u)^2 |x|] = u - c_1$$

$$(1+u)^2 |x| = e^{u-c_1}$$

$$(1+u)^2 x = c_2 e^u \quad (c_2 = \pm e^{-c_1})$$

Step 4 代回 $u = y/x$

$$(1 + y/x)^2 x = c_2 e^{y/x} \longrightarrow \boxed{(x+y)^2 = c_2 x e^{y/x}}$$

2-5-2 特殊解法 2: Bernoulli's Equations

【定義】 Bernoulli's equation:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

$$\text{so } y^{-n} \frac{dy}{dx} + P(x)y^{1-n} = f(x)$$

We can set $u = y^{1-n}$, $\frac{dy}{dx} = \frac{1}{1-n} u^{\frac{n}{1-n}} \frac{du}{dx}$, and the method of solving the 1st order linear DE to solve the Bernoulli's equation.

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{du^{\frac{1}{1-n}}}{du} \frac{du}{dx} = \frac{1}{1-n} u^{\frac{n}{1-n}} \frac{du}{dx}$$

(Chain rule)

Procedure for solving the Bernoulli's equation

Previous Step : Conclude whether the DE is a Bernoulli's equation

Step 1: Set $u = y^{1-n}$, $y = u^{\frac{1}{1-n}}$, $\frac{dy}{dx} = \frac{1}{1-n} u^{\frac{n}{1-n}} \frac{du}{dx}$

Step 2: Convert the Bernoulli's equation into the 1st order linear DE

Step 3: Solve the 1st order linear DE (use the method in Sec. 2-3)

Step 4: Substitute $u = y^{1-n}$ (别忘了)

[Example 2] (text page 74)

$$\boxed{x \frac{dy}{dx} + y = x^2 y^2}$$



Previous Step: 判斷 (Bernoulli, $n = 2$)

Step 1: set $u = y^{-1}$ ($y = u^{-1}$) $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = -u^{-2} \frac{du}{dx}$

(Chain rule)

Step 2: Convert into the 1st order linear DE (standard form)

$$\text{原式} \longrightarrow -xu^{-2} \frac{du}{dx} + u^{-1} = x^2 u^{-2} \longrightarrow \frac{du}{dx} - \frac{1}{x} u = -x$$

Step 3: Obtain the solution of the 1st order DE

$$u = -x^2 + cx$$

Step 4: 代回 $u = y^{-1}$

$$\boxed{y = \frac{1}{-x^2 + cx}}$$

2-5-3 特殊解法 3

If the 1st order DE has the form,

$$\frac{dy}{dx} = f(Ax + By + C) \quad (B \neq 0)$$

(解法的限制條件)

we can set $u = Ax + By + C$ to solve it.

$$\frac{dy}{dx} = \frac{1}{B} \frac{du}{dx} - \frac{A}{B}$$

Since $du = Adx + Bdy$ (這式子也許較好記)

Procedure for solving $\frac{dy}{dx} = f(Ax + By + C)$

Previous Step: Conclude

Step 1: Set $u = Ax + By + C \longrightarrow du = A dx + B dy \longrightarrow \frac{dy}{dx} = \frac{1}{B} \frac{du}{dx} - \frac{A}{B}$

Step 2: Converting (轉化成用其他方法可以解出來的 DE
未必一定是轉化成 separable variable DE)

Step 3: Solving

Step 4: Substitute $u = Ax + By + C$ (別忘了)

[Example 3] (text page 74)

$$\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0$$

Previous Step: 判斷

Step 1: Set

$$u = -2x + y$$

$$du = -2dx + dy$$

$$\frac{dy}{dx} = \frac{du}{dx} + 2$$

Step 2: Converting

$$\text{原式} \longrightarrow \frac{du}{dx} + 2 = u^2 - 7 \longrightarrow \frac{du}{u^2 - 9} = dx$$

Step 3: Obtain the solution (別忘了在運算過程中，代回 $u = Ax + By$)

$$\frac{1}{6} \left(\frac{1}{u-3} - \frac{1}{u+3} \right) du = dx$$

$$\ln|u-3| - \ln|u+3| = 6x + c_1$$

$$\frac{u-3}{u+3} = c_2 e^{6x} \quad c_2 = \pm e^{c_1}$$

$$\frac{u-3}{u+3} = c_2 e^{6x}$$

$$u = \frac{3c_2 e^{6x} + 3}{1 - c_2 e^{6x}}$$

Step 4: 代回 $u = Ax + By + C$

$$-2x + y = \frac{3c_2 e^{6x} + 3}{1 - c_2 e^{6x}}$$

$$y = 2x + \frac{3c_2 e^{6x} + 3}{1 - c_2 e^{6x}}$$

Extra: From $y(0) = 0$

$$0 = 0 + \frac{3c_2 + 3}{1 - c_2} \quad c_2 = -1$$

$$y = 2x + \frac{3(1 - e^{6x})}{1 + e^{6x}}$$

2-5-4 本節要注意的地方

- (1) 對症下藥，先判斷 DE 符合什麼樣的條件，再決定要什麼方法來解 (部分的 DE 可以用兩個以上的方法來解)
- (2) 別忘了，寫出解答時，要將 u 用 y/x , y^{1-n} , 或 $Ax + By + C$ 代回來
- (3) 本節方法皆有五大步驟

Previous Step: 判斷用什麼方法

Step 1: Set $u = \dots$, $du/dx = \dots$

Step 2: Converting ,

Step 3: Solving ,

Step 4: 將 u 用 x, y 代回來

附錄四 整理：Methods of solving the 1st order DE

(1) Direct integral

$$\text{條件：} \frac{dy}{dx} = f(x)$$

破解法：直接積分

$$y = \int f(x) dx + c$$

(2) Separable variable

$$\text{條件：} \frac{dy}{dx} = g(x)h(y)$$

破解法：x, y 各歸一邊後積分

$$\int \frac{dy}{h(y)} = \int g(x) dx$$

(3) Linear DE

$$\text{條件：} \frac{dy}{dx} = -P(x)y + f(x)$$

破解法：算 $e^{\int P(x) dx}$

$$\left(e^{\int P(x) dx} y(x) \right)' = e^{\int P(x) dx} f(x)$$

(4) Exact equation

$$\text{條件：} \frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

破解法：pages 106, 108 (double N)

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

$$\begin{aligned} \text{先處理} \quad \frac{\partial f}{\partial x} &= M(x, y) \\ &\quad \text{(或反過來)} \\ \text{再處理} \quad \frac{\partial f}{\partial y} &= N(x, y) \end{aligned}$$

(4-1) Exact equation 變型

$$\text{條件} : \frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

$(M_y - N_x) / M$ independent of x

(4-2) Exact equation 變型

$$\text{條件} : \frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

$(M_y - N_x) / N$ independent of y

(5) Homogeneous equation

$$\text{條件} : \frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

$$M(tx, ty) = t^\alpha M(x, y)$$

$$N(tx, ty) = t^\alpha N(x, y)$$

$$\text{破解法} : \mu(y) = e^{\int \frac{(N_x - M_y)}{M} dy}$$

$$\mu(y)M(x, y)dx + \mu(y)N(x, y)dy = 0$$

is exact

$$\text{破解法} : \mu(x) = e^{\int \frac{(M_y - N_x)}{N} dx}$$

$$\mu(x)M(x, y)dx + \mu(x)N(x, y)dy = 0$$

is exact

$$\text{破解法} : u = y/x, \quad (y = xu)$$

$$dy = udx + xdu$$

再用 separable variable method

(6) Bernoulli's Equation

$$\text{條件：} \frac{dy}{dx} = -P(x)y + f(x)y^n$$

破解法： $u = y^{1-n}$

$$\frac{dy}{dx} = \frac{1}{1-n} u^{\frac{n}{1-n}} \frac{du}{dx}$$

再用 linear DE 的方法

(7) $Ax + By + C$

$$\text{條件：} \frac{dy}{dx} = f(Ax + By + C)$$

破解法： $u = Ax + By + c$

$$\frac{dy}{dx} = \frac{1}{B} \frac{du}{dx} - \frac{A}{B}$$

注意 (a) 速度的訓練

(b) Exercises in Review 2 多練習

(c) 行有餘力，觀察 singular solution 和 singular point

練習題

Sec. 2-4: 3, 8, 13, 17, 20, 25, 29, 32, 34, 35, 38, 42

Sec. 2-5: 3, 5, 10, 13, 14, 17, 20, 22, 24, 25, 29, 31

Chap. 2 Review: 2, 13, 16, 17, 18, 19, 22, 23, 24, 26, 33