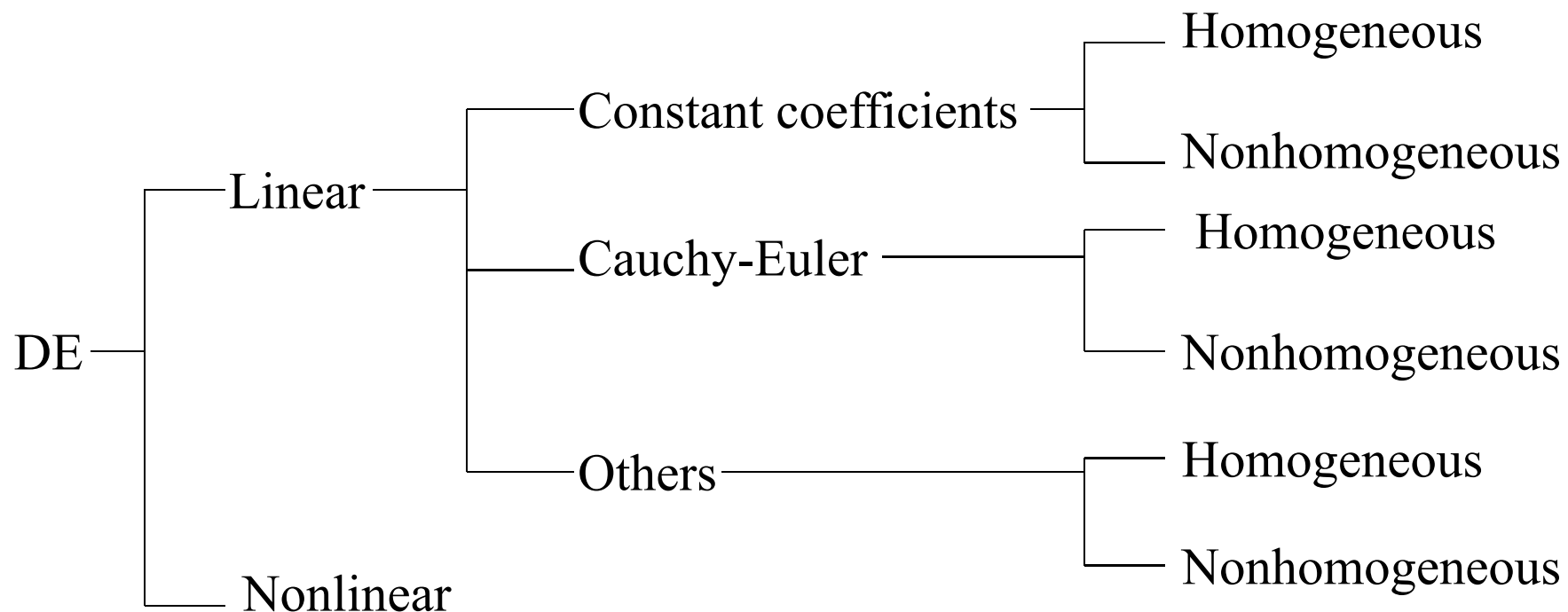


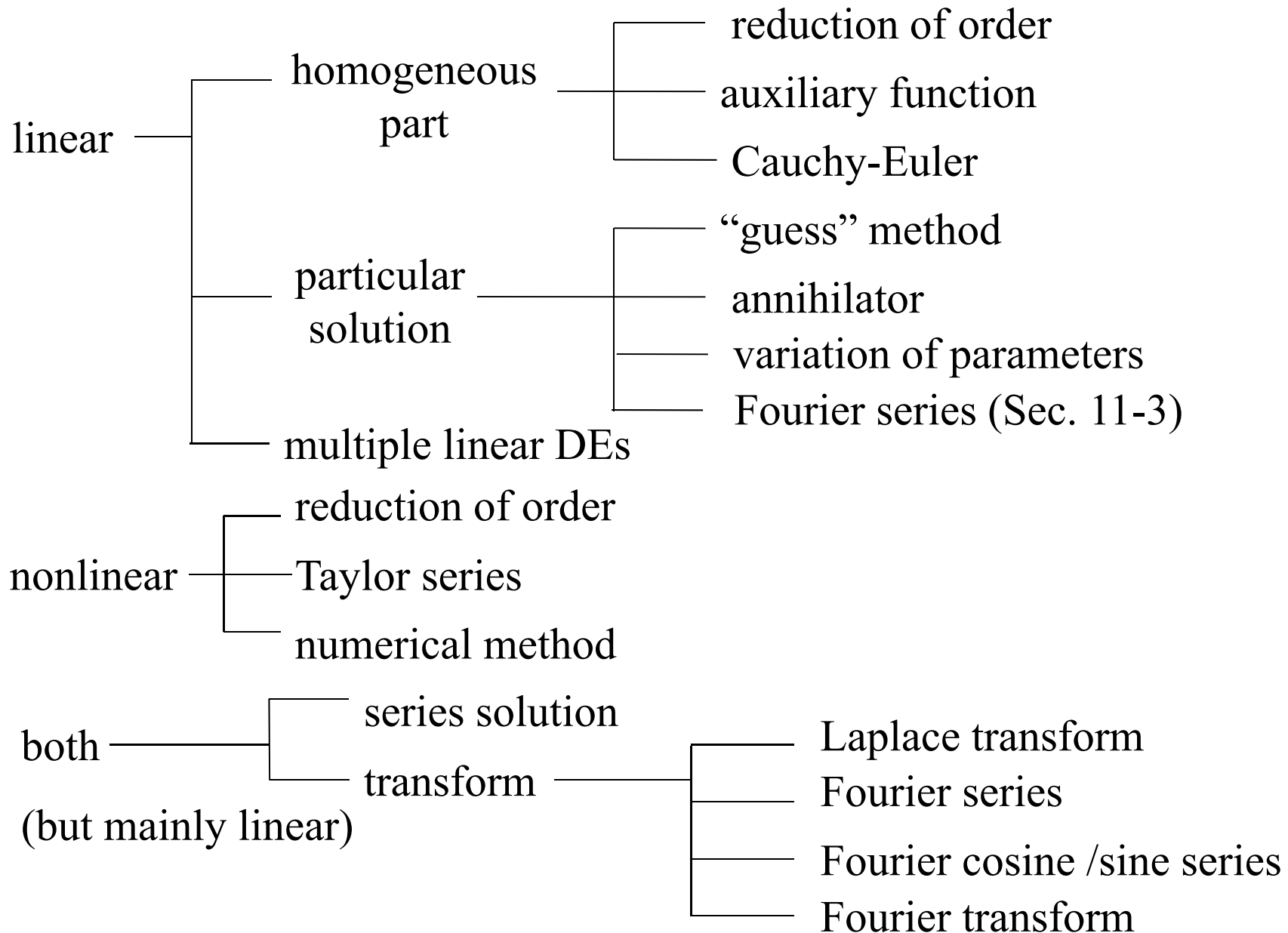
Chapter 4 Higher Order Differential Equations

Highest differentiation: $\frac{d^n y}{dx^n}$, $n > 1$

Most of the methods in Chapter 4 are applied for the **linear** DE.

附錄五 DE 的分類





4-1 Linear Differential Equations: Basic Theory

The n^{th} order linear DE

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

$g(x) = 0 \longrightarrow$ homogeneous

$g(x) \neq 0 \longrightarrow$ nonhomogeneous

4.1.1 Nonhomogeneous Equations (可和 page 60 相比較)

Nonhomogeneous **linear** DE

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y = g(x)$$

Part 1

Associated homogeneous DE

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y = 0$$

find **n linearly independent solutions**

$$y_1(x), y_2(x), \dots, y_n(x)$$

complementary function

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

Part 2

particular solution y_p

(**any** solution of the nonhomogeneous linear DE)

superposition principle (sometimes)

$$g(x) = g_1(x) + g_2(x) + \cdots + g_k(x)$$

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$$

general solution of the nonhomogeneous linear DE

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x)$$

Non-homogeneous Linear DE 解法的步驟 (Also see page 143)

144

Step 1: Find the **general solution** (i.e., the **complementary function**)
of the **associated homogeneous DE**

(Sections 4-2, 4-3, 4-7)

Step 2: Find the **particular solution**

(Sections 4-4, 4-5, 4-6)

Step 3: Combine the complementary function and the particular solution

Extra Step: Consider the initial (or boundary) conditions

4.1.2 Homogeneous Equations and Complementary Function

4.1.2.1 Definition

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

$$g(x) = 0 \longrightarrow \text{homogeneous}$$

$$g(x) \neq 0 \longrightarrow \text{nonhomogeneous}$$

- 重要名詞：Associated homogeneous equation

The associated homogeneous equation of a nonhomogeneous DE:

$$\text{Setting } g(x) = 0$$

- Review: Section 2-3, pages 58, 60

[Example] $y''' - 6y'' + 11y' - 6y = 3x$

Associated homogeneous equation: $y''' - 6y'' + 11y' - 6y = 0$

4.1.2.2 Solution of the Homogeneous Equation

[Important Theory]: An n^{th} order homogeneous linear DE has n linearly independent solutions.

[Theorem 4.1.5]

For an n^{th} order **homogeneous linear** DE, if

① $y_1(t), y_2(t), \dots, y_n(t)$ are the solutions of this DE

② $y_1(t), y_2(t), \dots, y_n(t)$ are **linearly independent**

then any solution of the homogeneous linear DE can be expressed as:

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

可以和矩陣的概念相比較

From Theorem 4.1.5:

An n^{th} order homogeneous linear DE has n linearly independent solutions.

Find n linearly independent solutions

== Find all the solutions of an n^{th} order homogeneous linear DE

$y_1(x), y_2(x), \dots, y_n(x)$: fundamental set of solutions

$y = c_1y_1 + c_2y_2 + \dots + c_ny_n$: general solution of the homogenous linear DE

(又稱做 complementary function)
也是重要名詞

[Definition 4.1] Linear Dependence / Independence

If there is **no solution other than** $c_1 = c_2 = \dots = c_n = 0$ for the following equality

$$c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) = 0 \quad \text{for all } x$$

then $y_1(x), y_2(x), \dots, y_n(x)$ are said to be **linearly independent**.

Otherwise, they are **linearly dependent**.

判斷是否為 linearly independent 的方法: **Wronskian**

[Definition 4.2] Wronskian

$$W(y_1, y_2, \dots, y_n) = \det \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

$$W(y_1, y_2, \dots, y_n) \neq 0 \longrightarrow \text{linearly independent}$$

Note: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

[Example 9] (text page 127)

$$y''' - 6y'' + 11y' - 6y = 0$$

$y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ are three of the solutions

Since

$$\det \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{bmatrix} = e^{x+2x+3x} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = 2e^{6x} \neq 0$$

Therefore, y_1 , y_2 , and y_3 are linear independent for any x

general solution:

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} \quad x \in (-\infty, \infty)$$

4.1.3 Particular Solution

Particular solution:

Any solution of the original nonhomogeneous linear DE.

[Example 10] (text page 128)

$$y''' - 6y'' + 11y' - 6y = 3x$$

$$y''' - 6y'' + 11y' - 6y = 0$$

$$e^x \quad e^{2x} \quad e^{3x}$$

Complementary function

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

(Example 9)

Particular solution of

$$y''' - 6y'' + 11y' - 6y = 3x$$

$$y_p = -\frac{11}{12} - \frac{1}{2}x$$

General solution:

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x$$

4.1.3.1 Superposition Principle for Particular Solutions

[Theorem 4.1.7] Superposition Principle

If $y_{p_1}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g_1(x)$$

$y_{p_2}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g_2(x)$$

⋮

$y_{p_k}(x)$ is the particular solution of

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g_k(x)$$

then $y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x)$ is the particular solution of

$$\begin{aligned} & a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) \\ &= g_1(x) + g_2(x) + \cdots + g_k(x) \end{aligned}$$

[Example 11] (text page 129)

$y_{p_1}(x) = -4x^2$ is a particular solution of $y'' - 3y' + 4y = -16x^2 + 24x - 8$

(子問題一)

$y_{p_2}(x) = e^{2x}$ is a particular solution of $y'' - 3y' + 4y = 2e^{2x}$

(子問題二)

$y_{p_3}(x) = xe^x$ is a particular solution of $y'' - 3y' + 4y = 2xe^x - e^x$

(子問題三)

$y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^x$ is a particular solution of

$$y'' - 3y' + 4y = -16x^2 + 24x - 8 + 2e^{2x} + 2xe^x - e^x$$

4.1.4 New Notations

Notation: $D^n y = \frac{d^n y}{dx^n}$

$$\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y \xrightarrow{\text{可改寫成}} D^2 y + 5Dy + 6y \xrightarrow{\text{可改寫成}} (D^2 + 5D + 6)y$$

可再改寫成

$$L(y)$$
$$L = D^2 + 5D + 6$$

(Useful for the linear DE with constant coefficients)

4.1.5 Initial-Value and Boundary Value Problems

4.1.5.1 The n^{th} Order Initial Value Problem

i.e., the n^{th} order linear DE with the constraints at the same point

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y''(x_0) = y_2, \quad \dots$$

$$\dots \dots \dots y^{(n-1)}(x_0) = y_{n-1}$$

n initial conditions

(given at the same point)

[Theorem 4.1.1]

For an interval I that contains the point x_0

① If $a_0(x), a_1(x), a_2(x), \dots, a_{n-1}(x), a_n(x)$ are continuous at $x = x_0$

② $a_n(x_0) \neq 0$

(很像Section 2-3 當中 x_0 不是singular point 的條件)

then for the problem on page 155, the solution $y(x)$ exists and is unique on the interval I that contains the point x_0

(Interval I 的範圍，取決於何時 $a_n(x) = 0$ 以及何時 $a_k(x)$ ($k = 0 \sim n$) 不為continuous)

Otherwise, the solution is either **non-unique** or does **not exist**.

(infinite number of solutions) (no solution)



[Example 1] (text page 119)

$$3y''' + 5y'' - y' + 7y = 0 \quad y(1) = 0 \quad y'(1) = 0 \quad y''(1) = 0$$

[Example 2] (text page 120)

$$y'' - 4y' = 12x \quad y(0) = 4 \quad y'(0) = 1$$

- $x^2 y'' - 2xy' + 2y = 6 \quad y(0) = 3 \quad y'(0) = 1$

有無限多組解

$$y = cx^2 + x + 3 \quad c \text{ 為任意之常數}$$

- 比較：

$$x^2 y'' - 2xy' + 2y = 6 \quad y(1) = 3 \quad y'(1) = 1$$

There is only one solution

$$y = x^2 - x + 3$$

$$x \in (0, \infty)$$

- Note:

The initial value can also be the form as:

$$\alpha y(x_0) + \beta y'(x_0) = y_0$$

$$\sum_{n=0}^{N-1} \alpha_n y^{(n)}(x_0) = y_0 \quad (\text{general initial condition})$$

4.1.5.2 n^{th} Order Boundary Value Problem

Boundary conditions are specified at different points

比較：Initial conditions are specified at the same points

例子： $a_2(x)y'' + a_1(x)y' + a_0(x) = g(x)$

subject to $y(a) = y_0, \quad y(b) = y_1 \quad a \neq b$

或 $y'(a) = y_0, \quad y(b) = y_1$

或
$$\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = \gamma_1 \\ \alpha_2 y(b) + \beta_2 y'(b) = \gamma_2 \end{cases}$$

An n^{th} order linear DE with n boundary conditions may have a unique solution, no solution, or infinite number of solutions.

[Example 3] (text page 120)

$$y'' + 16y = 0$$

solution: $y = c_1 \cos(4x) + c_2 \sin(4x)$

(1) $y(0) = 0$ $y(\pi/2) = 0$

$y = c_2 \sin(4x)$ c_2 is any constant (infinite number of solutions)

(2) $y(0) = 0$ $y(\pi/8) = 0$

$y = 0$ (unique solution)

4.1.6 本節要注意的地方

- (1) Most of the theories in Section 4.1 are applied to the linear DE
- (2) 注意 initial conditions 和 boundary conditions 之間的不同
- (3) 快速判斷 linear independent

4.1.6.1 名詞

- general solution of the nonhomogenous linear DE (page 143)

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

- associated homogeneous equation, (page 145)

(重要名詞)

- fundamental set of solutions (page 147)

- complementary function (general solution of the homogenous linear DE)

(重要名詞) (page 147)

- Wronskian (page 149)

- particular solution (page 151)

- initial conditions, boundary conditions (pages 155, 159)

(重要名詞)

(補充 1) Theorem 4.1.1 的解釋

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

$$y(x_0) = y_0 \quad y'(x_0) = y_1 \quad \cdots \quad y^{(n-1)}(x_0) = y_{n-1}$$

When $a_n(x_0) \neq 0$

$$y^{(n)}(x_0) + \frac{a_{n-1}(x_0)}{a_n(x_0)} y^{(n-1)}(x_0) + \cdots + \frac{a_1(x_0)}{a_n(x_0)} y'(x_0) + \frac{a_0(x_0)}{a_n(x_0)} y(x_0) = \frac{g(x_0)}{a_n(x_0)}$$

find $y^{(n)}(x_0)$

$$y^{(n-1)}(x_0 + \Delta) = y^{(n-1)}(x_0) + y^{(n)}(x_0) \Delta \longrightarrow \text{find } y^{(n-1)}(x_0 + \Delta)$$

(根據 $f'(t) = \frac{f(t+\Delta) - f(t)}{\Delta}$, $f(t+\Delta) = f(t) + f'(t)\Delta$)

以此類推

$$y^{(n-2)}(x_0 + \Delta) = y^{(n-2)}(x_0) + y^{(n-1)}(x_0)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0 + \Delta)$$

$$y^{(n-3)}(x_0 + \Delta) = y^{(n-3)}(x_0) + y^{(n-2)}(x_0)\Delta \longrightarrow \text{find } y^{(n-3)}(x_0 + \Delta)$$

⋮

$$y(x_0 + \Delta) = y(x_0) + y'(x_0)\Delta \longrightarrow \text{find } y(x_0 + \Delta)$$

$$y^{(n)}(x_0 + \Delta) + \frac{a_{n-1}(x_0 + \Delta)}{a_n(x_0 + \Delta)} y^{(n-1)}(x_0 + \Delta) + \cdots + \frac{a_1(x_0 + \Delta)}{a_n(x_0 + \Delta)} y'(x_0 + \Delta)$$

$$+ \frac{a_0(x_0 + \Delta)}{a_n(x_0 + \Delta)} y(x_0 + \Delta) = \frac{g(x_0 + \Delta)}{a_n(x_0 + \Delta)} \longrightarrow \text{find } y^{(n)}(x_0 + \Delta)$$

$$y^{(n-1)}(x_0 + 2\Delta) = y^{(n-1)}(x_0 + \Delta) + y^{(n)}(x_0 + \Delta)\Delta \longrightarrow \text{find } y^{(n-1)}(x_0 + 2\Delta)$$

$$y^{(n-2)}(x_0 + 2\Delta) = y^{(n-2)}(x_0 + \Delta) + y^{(n-1)}(x_0 + \Delta)\Delta \longrightarrow \text{find } y^{(n-2)}(x_0 + 2\Delta)$$

$$y(x_0 + 2\Delta) = y(x_0 + \Delta) + y'(x_0 + \Delta)\Delta \longrightarrow \boxed{\text{find } y(x_0 + 2\Delta)}$$

$$y^{(n)}(x_0 + 2\Delta) + \frac{a_{n-1}(x_0 + 2\Delta)}{a_n(x_0 + 2\Delta)} y^{(n-1)}(x_0 + 2\Delta) + \cdots + \frac{a_1(x_0 + 2\Delta)}{a_n(x_0 + 2\Delta)} y'(x_0 + 2\Delta) + \frac{a_0(x_0 + 2\Delta)}{a_n(x_0 + 2\Delta)} y(x_0 + 2\Delta) = \frac{g(x_0 + 2\Delta)}{a_n(x_0 + 2\Delta)}$$

以此類推，可將 $y(x_0 + 3\Delta), y(x_0 + 4\Delta), y(x_0 + 5\Delta), \dots$

以至於將 $y(x)$ 所有的值都找出來。

(求 $y(x)$ for $x > x_0$ 時, 用正的 Δ 值,

求 $y(x)$ for $x < x_0$ 時, 用負的 Δ 值)

Requirement 1: $a_0(x), a_1(x), a_2(x), \dots, a_{n-1}(x), a_n(x)$ are continuous
是為了讓 $a_k(x_0+m\Delta)$ 皆可以定義

Requirement 2: $a_n(x) \neq 0$ 是為了讓 $a_k(x_0+m\Delta) / a_n(x_0+m\Delta)$ 不為無限大

4-2 Reduction of Order

4.2.1 適用情形

Suitable for the ⁽¹⁾2nd ⁽²⁾order ⁽³⁾linear homogeneous DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

(4) One of the nontrivial solution $y_1(x)$ has been known.

4.2.2 解法

假設 $y_2(x) = u(x)y_1(x)$

將 DE 變成 Standard form

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

先將 DE 變成 Standard form

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = 0$$

If $y(x) = u(x)y_1(x)$,

(比較 Section 2-3) $P(x) = \frac{a_1(x)}{a_2(x)}$

$$y' = uy_1' + u'y_1 \quad y'' = uy_1'' + 2u'y_1' + u''y_1$$

$$uy_1'' + 2u'y_1' + u''y_1 + P(x)uy_1' + P(x)u'y_1 + Q(x)uy_1 = 0$$

$$u(\underbrace{y_1'' + P(x)y_1' + Q(x)y_1}_{\text{zero}}) + 2u'y_1' + u''y_1 + P(x)u'y_1 = 0$$

zero

$$u''y_1 + u'(2y_1' + P(x)y_1) = 0$$

set $w = u'$

$$\frac{dw}{dx}y_1 + w\left(2\frac{dy_1}{dx} + P(x)y_1\right) = 0$$

multiplied by $dx/(y_1w)$

$$\frac{dw}{w} + 2\frac{dy_1}{y_1} + P(x)dx = 0$$

separable variable
(with 3 variables)

$$\int \frac{dw}{w} + 2\int \frac{dy_1}{y_1} + \int P(x)dx = 0$$

$$\ln|w| + c_3 + 2\ln|y_1| + c_4 = -\int P(x)dx$$

$$\ln|w| + 2\ln|y_1| = \ln|w| + \ln|y_1|^2 = \ln|w||y_1|^2 = \ln|wy_1^2|$$

$$\ln|wy_1^2| = -\int P(x)dx + c$$

$$\ln |wy_1^2| = -\int P(x) dx + c$$

$$wy_1^2 = \pm e^{-\int P(x) dx + c}$$

$$w = c_1 e^{-\int P(x) dx} / y_1^2$$

$$u = \int w dx = c_1 \int \frac{e^{-\int P(x) dx}}{y_1^2} dx + c_2$$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$

We can set $c_1 = 1$ and $c_2 = 0$

(因為我們算 $u(x)$ 的目的，只是為了要算出與 $y_1(x)$ 互相independent的另一個解)

(the coefficient of $y'(x)$ in the standard form)

4.2.3 例子

[Example 1] (text page 132)

$$y'' - y = 0$$

We have known that $y_1 = e^x$ is one of the solution

$$P(x) = 0 \quad y_2(x) = e^x \int ce^{-2x} dx = -\frac{1}{2}ce^{-x}$$

Specially, set $c = -2$, ($y_2(x)$ 只要 independent of $y_1(x)$ 即可
所以 c 的值可以任意設)

$$y_2(x) = e^{-x}$$

General solution: $y(x) = c_1e^x + c_2e^{-x}$

[Example 2] (text page 133)

(將課本 x 的範圍做更改)

$$x^2 y'' - 3xy' + 4y = 0 \quad \text{when } x \in (-\infty, 0)$$

We have known that $y_1 = x^2$ is one of the solution

Note: the interval of x

If $x \in (0, \infty)$ ($x > 0$), $\int dx/x = \ln x$ 如課本

If $x < 0$, $\int dx/x = \ln(-x)$

$$\begin{aligned} y_2(x) &= x^2 \int \frac{e^{3\ln(-x)}}{x^4} dx = x^2 \int \frac{(-x)^3}{(-x)^4} dx \\ &= -x^2 \int \frac{1}{x} dx = -x^2 \ln|x| \end{aligned}$$

$$y(x) = c_1 x^2 + c_2 x^2 \ln|x|$$

4.2.4 本節需注意的地方

(1) 記住公式
$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

(2) 若不背公式 (不建議)，在計算過程中別忘了對 $w(x)$ 做積分

(3) 別忘了 $P(x)$ 是 “standard form” 一次微分項的 coefficient term

(4) 同樣有 singular point 的問題

(5) 因為 $y_2(x)$ 是 homogeneous linear DE 的 “任意解”，所以計算時，常數的選擇以方便為原則

(6) 由於 $\int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$ 的計算較複雜且花時間，所以要多加練習

多算習題

附錄七： Hyperbolic Function

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

比較： $\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

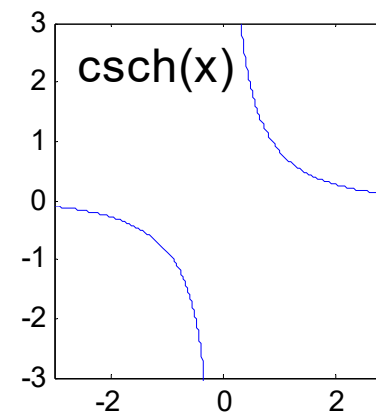
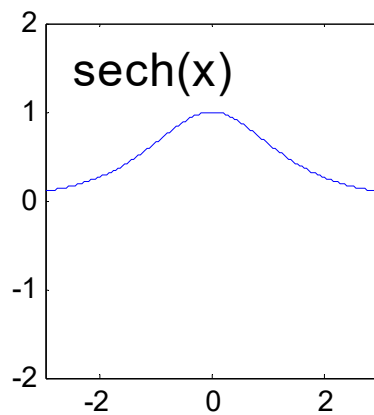
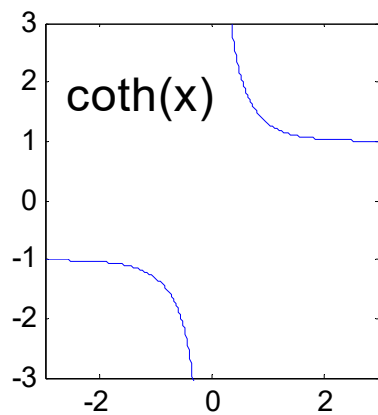
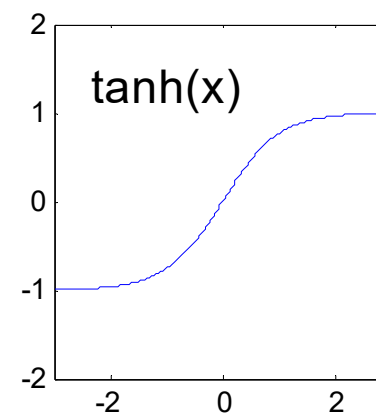
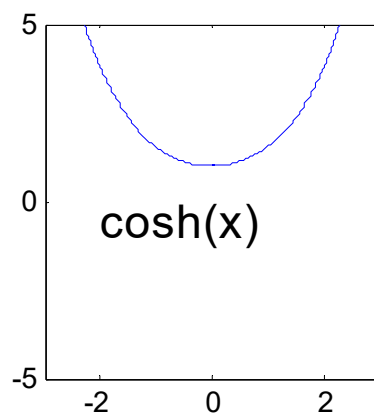
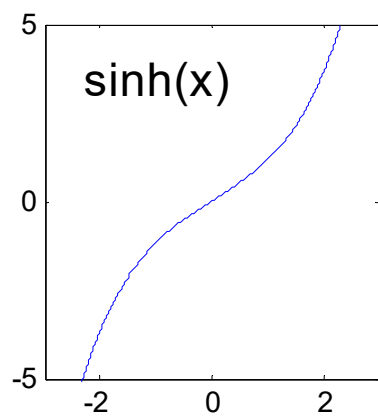
$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$$

Note: $\sinh(x) = -\sinh(-x)$

$$\cosh(x) = \cosh(-x)$$



$$\frac{d}{dx} \sinh(ax) = a \cosh(ax)$$

$$\sinh(0) = 0$$

$$\frac{d}{dx} \cosh(ax) = a \sinh(ax)$$

$$\cosh(0) = 1$$

$$\frac{d}{dx} \tanh(ax) = a \operatorname{sech}^2(ax)$$

$$\sinh'(0) = 1$$

$$\frac{d}{dx} \coth(ax) = -a \operatorname{csch}^2(ax)$$

$$\cosh'(0) = 0$$

$$\frac{d}{dx} \operatorname{sech}(ax) = -a \operatorname{sech}(ax) \tanh(ax)$$

$$\sin(ix) = i \sinh(x)$$

$$\frac{d}{dx} \operatorname{csch}(ax) = -a \operatorname{csch}(ax) \coth(ax)$$

$$\cos(ix) = \cosh(x)$$

$$\int \sinh(ax) dx = \frac{\cosh(ax)}{a} + c$$

$$\int \cosh(ax) dx = \frac{\sinh(ax)}{a} + c$$

$$\int \tanh(ax) dx = \frac{\ln|\cosh(ax)|}{a} + c$$

$$\int \coth(ax) dx = \frac{\ln|\sinh(ax)|}{a} + c$$

$$\int \operatorname{sech}(ax) dx = \frac{2 \tan^{-1}\left(\tanh\left(\frac{a}{2}x\right)\right)}{a} + c$$

$$\int \operatorname{csch}(ax) dx = \frac{\ln\left|\tanh\left(\frac{a}{2}x\right)\right|}{a} + c$$

4-3 Homogeneous Linear Equations with Constant Coefficients

本節使用 **auxiliary equation** 的方法來解 homogeneous DE

KK: [əg`zɪ l j ə r ɪ]

4-3-1 限制條件

限制條件: (1) homogeneous

(2) linear

(3) **constant coefficients**

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$$

$a_0, a_1, a_2, \dots, a_n$ are constants

(the simplest case of the higher order DEs)

4-3-2 解法

解法核心：

Suppose that the solutions has the form of e^{mx}

Example: $y''(x) - 3y'(x) + 2y(x) = 0$

Set $y(x) = e^{mx}$, $m^2 e^{mx} - 3m e^{mx} + 2 e^{mx} = 0$

$$m^2 - 3m + 2 = 0 \longrightarrow \text{solve } m$$

可以直接把 n 次微分 用 m^n 取代，變成一個多項式

這個多項式被稱為 **auxiliary equation**

• 解法流程

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$$

Step 1-1

auxiliary function

$$y^{(k)}(x) \rightarrow m^k$$

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

Step 1-1

Find n roots, $m_1, m_2, m_3, \dots, m_n$

(If $m_1, m_2, m_3, \dots, m_n$ are distinct)

Step 1-2 n linearly independent solutions $e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \dots, e^{m_n x}$

(有三個 Cases)

Step 1-3 Complementary function

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

4-3-3 Three Cases for Roots (2nd Order DE)

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0$$

$$a_2 m^2 + a_1 m + a_0 = 0$$

$$\text{roots } m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \quad m_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

solutions

Case 1 $m_1 \neq m_2$, m_1, m_2 are real

(其實 m_1, m_2 不必限制為 real)

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Case 2 $m_1 = m_2$ (m_1 and m_2 are of course real)

First solution: $y_1 = e^{m_1 x}$

Second solution: using the method of “Reduction of Order”

$$\begin{aligned}
 y_2(x) &= y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx \\
 &= e^{m_1 x} \int e^{-2m_1 x} e^{-\int a_1/a_2 dx} dx \\
 &= e^{m_1 x} \int e^{(-2m_1 - \frac{a_1}{a_2})x} dx \\
 &= e^{m_1 x} \int dx = e^{m_1 x} (x + c)
 \end{aligned}
 \qquad m_1 = \frac{-a_1}{2a_2}$$

$$y_2(x) = x e^{m_1 x}$$

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

Case 3 $m_1 \neq m_2$, m_1 and m_2 are conjugate and complex

$$m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2} = \alpha + j\beta \quad m_2 = \alpha - j\beta$$

$$\alpha = -a_1 / 2a_2, \quad \beta = \sqrt{4a_2a_0 - a_1^2} / 2a_2$$

Solution: $y = C_1 e^{\alpha x + j\beta x} + C_2 e^{\alpha x - j\beta x}$

Another form:

$$\begin{aligned} y &= e^{\alpha x} (C_1 e^{j\beta x} + C_2 e^{-j\beta x}) \\ &= e^{\alpha x} (C_1 \cos \beta x + jC_1 \sin \beta x + C_2 \cos \beta x - jC_2 \sin \beta x) \end{aligned}$$

$$\text{set } c_1 = C_1 + C_2 \text{ and } c_2 = jC_1 - jC_2$$

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) \quad c_1 \text{ and } c_2 \text{ are some constant}$$

[Example 1] (text page 137)

(a) $2y'' - 5y' - 3y = 0$

$$2m^2 - 5m - 3 = 0, \quad m_1 = -1/2, \quad m_2 = 3$$

$$y = c_1 e^{-x/2} + c_2 e^{3x}$$

(b) $y'' - 10y' + 25y = 0$

$$m^2 - 10m + 25 = 0, \quad m_1 = 5, \quad m_2 = 5$$

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

(c) $y'' + 4y' + 7y = 0$

$$m^2 + 4m + 7 = 0, \quad m_1 = -2 + i\sqrt{3}, \quad m_2 = -2 - i\sqrt{3}$$

$$y = e^{-2x} (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

4-3-4 Three + 1 Cases for Roots (Higher Order DE)

For higher order case $a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$

auxiliary function: $a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$

roots: $m_1, m_2, m_3, \dots, m_n$

(1) If $m_p \neq m_q$ for $p = 1, 2, \dots, n$ and $p \neq q$

(也就是這個多項式在 m_q 的地方只有一個根)

then $e^{m_q x}$ is a solution of the DE.

重覆次數

(2) If the multiplicities of m_q is k (當這個多項式在 m_q 的地方有 k 個根),

$$e^{m_q x}, x e^{m_q x}, x^2 e^{m_q x}, \dots, x^{k-1} e^{m_q x}$$

are the solutions of the DE.

(3) If both $\alpha + j\beta$ and $\alpha - j\beta$ are the roots of the auxiliary function,
then

$$e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x)$$

are the solutions of the DE.

(4) If the multiplicities of $\alpha + j\beta$ is k and the multiplicities of $\alpha - j\beta$ is also k , then

$$e^{\alpha x} \cos(\beta x), xe^{\alpha x} \cos(\beta x), x^2 e^{\alpha x} \cos(\beta x), \dots, x^{k-1} e^{\alpha x} \cos(\beta x)$$

$$e^{\alpha x} \sin(\beta x), xe^{\alpha x} \sin(\beta x), x^2 e^{\alpha x} \sin(\beta x), \dots, x^{k-1} e^{\alpha x} \sin(\beta x)$$

are the solutions of the DE.

Note: If $\alpha + j\beta$ is a root of a **real coefficient** polynomial,
then $\alpha - j\beta$ is also a root of the polynomial.

$$a_n(\alpha + j\beta)^n + a_{n-1}(\alpha + j\beta)^{n-1} + \cdots + a_1(\alpha + j\beta) + a_0 = 0$$

$a_0, a_1, a_2, \dots, a_n$ are real

[Example 3] (text page 138)

Solve $y''' + 3y'' - 4y = 0$



Step 1-1 $m^3 + 3m^2 - 4 = 0$

$$(m - 1)(m^2 + 4m + 4) = 0$$

$$m_1 = 1, \quad m_2 = m_3 = -2$$

Step 1-2 3 independent solutions: e^x, e^{-2x}, xe^{-2x}

Step 1-3 general solution: $y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}$

[Example 4] (text page 138)

Solve $y^{(4)}(x) + 2y''(x) + y(x) = 0$

Step 1-1 $m^4 + 2m^2 + 1 = 0$

$$(m^2 + 1)^2 = 0 \quad \text{four roots: } i, i, -i, -i$$

Step 1-2 4 independent solutions: $\cos x$, $x \cos x$, $\sin x$, $x \sin x$

Step 1-3 general solution: $y = c_1 \cos x + c_2 x \cos x + c_3 \sin x + c_4 x \sin x$

4-3-5 How to Find the Roots

190

(1) Formulas

$$\boxed{a_2 m^2 + a_1 m + a_0 = 0} \implies m_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2 a_0}}{2a_2} \quad m_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}$$

$$\boxed{a_3 m^3 + a_2 m^2 + a_1 m + a_0}$$

$$\implies m_1 = S + T - \frac{a_2}{3a_3}$$

Solutions:

$$m_2 = -\frac{1}{2}(S + T) - \frac{a_2}{3a_3} + i\frac{1}{2}\sqrt{3}(S - T)$$

$$m_3 = -\frac{1}{2}(S + T) - \frac{a_2}{3a_3} - i\frac{1}{2}\sqrt{3}(S - T)$$

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^3}} \quad , \quad T = \sqrt[3]{R - \sqrt{Q^3 + R^3}}$$

$$Q = \frac{1a_1}{3a_3} - \frac{a_2^2}{9a_3^2} \quad , \quad R = \frac{9a_1 a_2 - 27a_3 a_0 - 2a_2^3}{54a_3^2} \quad (\text{太複雜了})$$

(2) Observing

例如：1 是否為 root \longrightarrow 看係數和是否為 0

又如：

$$3m^3 + 5m^2 + 10m - 4$$

factor: 1,3 factor: 1,2,4

possible roots: $\pm 1, \pm 2, \pm 4, \pm 1/3, \pm 2/3, \pm 4/3$

test for each possible root \longrightarrow find that $1/3$ is indeed a root

$$3m^3 + 5m^2 + 10m - 4 = \left(m - \frac{1}{3}\right)(3m^2 + 6m + 12)$$

(3) Solving the roots of a polynomial by software

Maple

Mathematica (by the commands of **Nsolve** and **FindRoot**)

Matlab (by the command of **roots**)

4-3-6 本節需注意的地方

- (1) 注意重根和 conjugate complex roots 的情形
- (2) 寫解答時，要將 “General solution” 寫出來

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

- (3) 因式分解要熟練
- (4) 本節的方法，也適用於 1st order 的情形

4-4 Undetermined Coefficients – Superposition Approach

This section introduces some method of “guessing” the particular solution.

4-4-1 方法適用條件

(1) (2)
Suitable for linear and constant coefficient DE.

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = g(x)$$

(3) $g(x), g'(x), g''(x), g'''(x), g^{(4)}(x), g^{(5)}(x), \dots$ contain **finite** number of terms.

4-4-2 方法

把握一個原則：

$g(x)$ 長什麼樣子，particular solution 就應該是什麼樣子。

記熟下一頁的規則

(計算時要把 A, B, C, \dots 這些 unknowns 解出來)

Trial Particular Solutions (from text page 146)

$g(x)$	Form of y_p
1 (any constant)	A
$5x + 7$	$Ax + B$
$3x^2 - 2$	$Ax^2 + Bx + C$
$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
$\sin 4x$	$A\cos 4x + B\sin 4x$
$\cos 4x$	$A\cos 4x + B\sin 4x$
e^{5x}	Ae^{5x}
$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x + Be^{3x}\sin 4x$
$5x^2\sin 4x$	$(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$
$xe^{3x}\cos 4x$	$(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$



It comes from the “**form rule**”. See page 201.

$$g(x) = e^{2x} + xe^{3x} \quad y_p = ?$$

$$g(x) = \cos(x) + x^2 \sin(2x) \quad y_p = ?$$

$$g(x) = \cosh(2x) \quad y_p = ?$$

4-4-3 Examples

[Example 2] $y'' - y' + y = 2 \sin 3x$ (text page 144)

Step 1: find the solution of the associated homogeneous equation

$$y_c = e^{x/2} \left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right) \quad \text{Guess}$$

Step 2: particular solution

$$y_p = A \cos 3x + B \sin 3x$$

$$y_p' = -3A \sin 3x + 3B \cos 3x$$

$$y_p'' = -9A \cos 3x - 9B \sin 3x$$

$$y_p'' - y_p' + y_p = (-8A - 3B) \cos 3x + (3A - 8B) \sin 3x = 2 \sin 3x$$

$$\begin{cases} -8A - 3B = 0 \\ 3A - 8B = 2 \end{cases} \quad \Longrightarrow \quad A = 6/73, \quad B = -16/73$$

$$y_p = \frac{6}{73} \cos 3x - \frac{16}{73} \sin 3x$$

Step 3: General solution:

$$y = e^{x/2} \left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right) + \frac{6}{73} \cos 3x - \frac{16}{73} \sin 3x$$

[Example 3] $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$ (text page 145)

Step 1: Find the solution of

$$y'' - 2y' - 3y = 0.$$

$$y_c = c_1 e^{3x} + c_2 e^{-x}$$

Step 2: Particular solution

$$y'' - 2y' - 3y = 4x - 5$$

guess

$$y_{p_1} = Ax + B$$

$$y'_{p_1} = A$$

$$y''_{p_1} = 0$$

$$-3Ax - 2A - 3B = 4x - 5$$

$$A = -\frac{4}{3}, \quad B = \frac{23}{9}$$

$$y_{p_1} = -\frac{4}{3}x + \frac{23}{9}$$

$$y'' - 2y' - 3y = 6xe^{2x}$$

guess

$$y_{p_2} = Cxe^{2x} + Ee^{2x}$$

$$y'_{p_2} = 2Cxe^{2x} + Ce^{2x} + 2Ee^{2x}$$

$$y''_{p_2} = 4Cxe^{2x} + 4Ce^{2x} + 4Ee^{2x}$$

$$-3Cxe^{2x} + (2C - 3E)e^{2x} = 6xe^{2x}$$

$$C = -2, \quad E = -\frac{4}{3}$$

$$y_{p_2} = -(2x + \frac{4}{3})e^{2x}$$

Particular solution

$$y_p = y_{p_1} + y_{p_2} = -\frac{4}{3}x + \frac{23}{9} - (2x + \frac{4}{3})e^x$$

Step 3: General solution

$$y = y_c + y_p$$

$$y = c_1e^{3x} + c_2e^{-x} - \frac{4}{3}x + \frac{23}{9} - (2x + \frac{4}{3})e^{2x}$$

4-4-4 方法的解釋：Form Rule

Form Rule: y_p should be a linear combination of $g(x)$, $g'(x)$,
 $g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$,


Why? 如此一來，在比較係數時才不會出現多餘的項

When $g(x) = x^n$

$$x^n \rightarrow x^{n-1} \rightarrow x^{n-2} \rightarrow x^{n-3} \rightarrow \dots \rightarrow 1 \rightarrow 0$$


$$y_p = A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \dots + A_0$$

When $g(x) = \cos kx$

$$\cos kx \rightarrow \sin kx$$


$$y_p = A_1 \cos kx + A_2 \sin kx$$

When $g(x) = \exp(kx)$

$$e^{kx}$$


$$y_p = A \exp(kx)$$

When $g(x) = x^n \exp(kx)$

$$g'(x) = nx^{n-1}e^{kx} + kx^n e^{kx}$$

$$g''(x) = n(n-1)x^{n-2}e^{kx} + 2nkx^{n-1}e^{kx} + k^2x^n e^{kx}$$

$$g'''(x) = n(n-1)(n-2)x^{n-3}e^{kx} + 3kn(n-1)x^{n-2}e^{kx} \\ + 3k^2nx^{n-1}e^{kx} + k^3x^n e^{kx}$$

⋮

⋮

會發現 $g(x)$ 不管多少次微分，永遠只出現

$$x^n e^{kx}, x^{n-1} e^{kx}, x^{n-2} e^{kx}, x^{n-3} e^{kx}, \dots, e^{kx}$$

$$y_p = c_n x^n e^{kx} + c_{n-1} x^{n-1} e^{kx} + c_{n-2} x^{n-2} e^{kx} + \dots + c_0 e^{kx}$$

4-4-5 Glitch of the method:

[Example 4] $y'' - 5y' + 4y = 8e^x$ (text page 146)

Particular solution guessed by Form Rule:

$$y_p = Ae^x$$

$$y_p'' - 5y_p' + 4y_p = Ae^x - 5Ae^x + 4Ae^x = 8e^x$$

$$0 = 8e^x \quad (\text{no solution})$$

Why?

Glitch condition 1: The **particular solution** we guess belongs to the complementary function.

For Example 4 $y'' - 5y' + 4y = 8e^x$

Complementary function $y_c = c_1e^x + c_2e^{4x}$ $Ae^x \in y_c$

解決方法：再乘一個 x

$$y_p = Axe^x \quad y'_p = Axe^x + Ae^x$$

$$y''_p = Axe^x + 2Ae^x$$

$$y''_p - 5y'_p + 4y_p = -3Ae^x = 8e^x \implies A = -8/3$$

$$y_p = -\frac{8}{3}xe^x$$

$$y = c_1e^x + c_2e^{4x} - \frac{8}{3}xe^x$$

[Example 7] $y'' - 2y' + y = e^x$ (text page 148)

$$y_c = c_1 e^x + c_2 x e^x$$

From Form Rule, the particular solution is Ae^x

$$Ae^x \in y_c$$

$$Axe^x \in y_c$$

如果乘一個 x 不夠，則再乘一個 x

$$y_p = Ax^2 e^x$$

$$y'_p = (Ax^2 + 2Ax)e^x$$

$$y''_p = (Ax^2 + 4Ax + 2A)e^x$$

$$y''_p - 2y'_p + y_p = 2Ae^x = e^x \implies A = 1/2$$

$$y_p = x^2 e^x / 2$$

$$y = c_1 e^x + c_2 x e^x + x^2 e^x / 2$$

[Example 8] (text page 148)

$$y'' + y = 4x + 10\sin x \quad y(\pi) = 0 \quad y'(\pi) = 2$$

子問題1

子問題2

Step 1

$$y_c = c_1 \cos x + c_2 \sin x$$

Step 2

$$y_p = Ax + B + C \sin x + E \cos x$$

修正

$$y_p = Ax + B + Cx \sin x + Ex \cos x$$

注意： $\sin x, \cos x$ 都要
乘上 x

$$y_p = 4x - 5x \cos x$$

Step 3

$$y = c_1 \cos x + c_2 \sin x + 4x - 5x \cos x$$

Step 4

Solving c_1 and c_2 by initial conditions (最後才解 IVP)

$$y(\pi) = -c_1 + 4\pi + 5\pi = 0 \implies c_1 = 9\pi$$

$$y' = -c_1 \sin x + c_2 \cos x + 4 - 5 \cos x + 5x \sin x$$

$$y'(\pi) = -c_2 + 9 = 2 \implies c_2 = 7$$

$$y = 9\pi \cos x + 7 \sin x + 4x - 5x \cos x$$

[Example 11] (text page 149) 子問題1 $y_{p,1} = A$

$$y^{(4)} + y''' = 1 - x^2 e^{-x} \quad \text{子問題2} \quad y_{p,2} = Bx^2 e^{-x} + Cx e^{-x} + Ee^{-x}$$

$$y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$$

From Form Rule

$$y_p = A + Bx^2 e^{-x} + Cx e^{-x} + Ee^{-x}$$

y_p 只要有一部分和 y_c 相同就作修正

乘上 x^3

修正

乘上 x

$$y_p = Ax^3 + Bx^3 e^{-x} + Cx^2 e^{-x} + Exe^{-x}$$

來自同一個子問題的項，乘的要相同

If we choose $y_p = A + Bx^2 e^{-x} + Cx e^{-x} + Ee^{-x}$

X

$$y_p^{(4)} + y_{(p)}''' = \underline{-2Bxe^{-x} + (6B - C)e^{-x}} = 1 - x^2 e^{-x}$$

沒有 $1, x^2 e^{-x}$ 兩項，不能比較係數，無解

If we choose $y_p = Ax^3 + Bx^2e^{-x} + Cxe^{-x} + Ee^{-x}$

X

$$y_p^{(4)} + y_{(p)}''' = \underline{6A - 2Bxe^{-x} + (6B - C)e^{-x}} = 1 - x^2e^{-x}$$

沒有 x^2e^{-x} 這一項，不能比較係數，無解

If we choose $y_p = Ax^3 + Bx^3e^{-x} + Cx^2e^{-x} + Exe^{-x}$

O

$$y_p^{(4)} + y_{(p)}'''$$

$$= 6A - 3Bx^2e^{-x} + (18B - 2C)xe^{-x} + (-18B + 6C - E)e^{-x}$$

$$= 1 - x^2e^{-x}$$

$$A = 1/6, B = 1/3, C = 3, E = 12$$

$$y_p = \frac{1}{6}x^3 + \frac{1}{3}x^3e^{-x} + 3x^2e^{-x} + 12xe^{-x}$$

$$y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + \frac{1}{6}x^3 + \frac{1}{3}x^3e^{-x} + 3x^2e^{-x} + 12xe^{-x}$$

Glitch condition 2: $g(x), g'(x), g''(x), g'''(x), g^{(4)}(x), g^{(5)}(x), \dots$
 contain **infinite number of terms**.

If $g(x) = \ln x$

$$\ln x \rightarrow \frac{1}{x} \rightarrow \frac{1}{x^2} \rightarrow \frac{1}{x^3} \rightarrow \dots$$

If $g(x) = \exp(x^2)$

$$g'(x) \rightarrow 2xe^{x^2}$$

$$g''(x) \rightarrow (4x^2 + 2)e^{x^2}$$

$$g'''(x) \rightarrow (8x^3 + 12x)e^{x^2}$$

:

:

4-4-6 本節需要注意的地方

(1) 記住 Table 4.1 的 particular solution 的假設方法
(其實和 “form rule” 有相密切的關聯)

(2) 注意 “glitch condition”

另外，“同一類”的 term 要乘上相同的東西 (參考 Example 11)

(3) 所以要先算 complementary function，再算 particular solution

(4) 同樣的方法，也可以用在 1st order 的情形

(5) 本方法只適用於 linear, constant coefficient DE

4-5 Undetermined Coefficients – Annihilator Approach

For a linear DE:

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$$

Annihilator Operator:

能夠「殲滅」 $g(x)$ 的 operator

4-5-1 方法適用條件

- (1) Linear , (2) Constant coefficients
- (3) $g(x), g'(x), g''(x), g'''(x), g^{(4)}(x), g^{(5)}(x), \dots$ contain **finite** number of terms.

4-5-2 Find the Annihilator

[Example 1] (text page 153)

$$g(x) = 1 - 5x^2 + 8x^3 \longrightarrow \text{annihilator: } D^4 \quad D^k g(x) = \frac{d^k}{dx^k} g(x)$$

$$g(x) = e^{-3x} \longrightarrow \text{annihilator: } D + 3$$
$$\frac{d}{dx} g(x) + 3g(x) = 0$$

$$g(x) = 4e^{2x} - 10xe^{2x} \longrightarrow \text{annihilator: } (D - 2)^2$$

$$(D - 2)^2 = D^2 - 4D + 4$$

$$\frac{d^2}{dx^2} g(x) - 4 \frac{d}{dx} g(x) + 4g(x) = 0$$

$$(D - 2)g(x) = 8e^{2x} - (20x + 10)e^{2x} - 8e^{2x} + 20xe^{2x} = -10e^{2x}$$

$$(D - 2)^2 g(x) = 0$$

註：當各個微分項的 coefficients 皆為 constants 時，function of D 的計算方式和 function of x 的計算方式相同

$$(x - 2)^2 = x^2 - 4x + 4$$

$$\Rightarrow (D - 2)^2 = D^2 - 4D + 4$$

General rule 1:

$$\text{If } g(x) = (a_n x^n + a_{n-1} x^{n-1} + \dots + a_0) e^{\alpha x}$$

then the annihilator is $[D - \alpha]^{n+1}$

注意：annihilator 和 a_0, a_1, \dots, a_n 無關

只和 α, n 有關

The annihilator is independent of the constant multiplied in the front of each term.

General rule 2:

If $g(x) = (a_n x^n + a_{n-1} x^{n-1} + \dots + a_0) e^{\alpha x} (b_1 \cos \beta x + b_2 \sin \beta x)$

$$b_1 \neq 0 \text{ or } b_2 \neq 0$$

then the annihilator is $\left[D^2 - 2\alpha D + (\alpha^2 + \beta^2) \right]^{n+1}$

Example 2: (text page 154) $g(x) = 5e^{-x} \cos 2x - 9e^{-x} \sin 2x$

annihilator $D^2 + 2D + 5$

Example 5: (text page 156) $g(x) = x \cos x - \cos x$

annihilator $\left[D^2 + 1 \right]^2$

Example 6: (text page 157) $g(x) = 10e^{-2x} \cos x$

annihilator $D^2 + 4D + 5$

General rule 3:

If $g(x) = g_1(x) + g_2(x) + \dots + g_k(x)$

$$L_h[g_h(x)] = 0 \text{ but } L_h[g_m(x)] \neq 0 \text{ if } m \neq h,$$

then the annihilator of $g(x)$ is the product of L_h ($h = 1 \sim k$)

$$L_k L_{k-1} \cdots L_2 L_1$$

Proof:

$$\begin{aligned} & L_k L_{k-1} \cdots L_3 L_2 L_1 [g_1 + g_2 + g_3 + \cdots + g_k] \\ &= L_k L_{k-1} \cdots L_3 L_2 L_1 g_1 + L_k L_{k-1} \cdots L_3 L_2 L_1 g_2 + \\ & \quad L_k L_{k-1} \cdots L_3 L_2 L_1 g_3 + \cdots + L_k L_{k-1} \cdots L_3 L_2 L_1 g_k \end{aligned}$$

$$L_k L_{k-1} \cdots L_3 L_2 L_1 g_1 = L_k L_{k-1} \cdots L_3 L_2 [L_1 g_1] = 0$$

$$L_k L_{k-1} \cdots L_3 L_2 L_1 g_2 = L_k L_{k-1} \cdots L_3 L_1 [L_2 g_2] = 0$$

(因為 L_1, L_2 為 linear DE with constant coefficient,

$$L_1 L_2 = L_2 L_1)$$

Similarly,

$$L_k L_{k-1} \cdots L_4 L_3 L_2 L_1 g_3 = L_k L_{k-1} \cdots L_4 L_2 L_1 [L_3 g_3] = 0$$

$$\vdots$$

$$\vdots$$

$$L_k L_{k-1} \cdots L_4 L_3 L_2 L_1 g_k = L_{k-1} \cdots L_4 L_3 L_2 L_1 [L_k g_k] = 0$$

Therefore,

$$\begin{aligned} & L_k L_{k-1} \cdots L_3 L_2 L_1 [g_1 + g_2 + g_3 + \cdots + g_k] \\ &= 0 + 0 + 0 + \cdots + 0 \\ &= 0 \end{aligned}$$

[Example 7] (text page 157)

$$g(x) = \underline{5x^2 - 6x} + \underline{4x^2 e^{2x}} + \underline{3e^{5x}}$$

annihilator: D^3

annihilator: $(D - 2)^3$

annihilator: $D - 5$

annihilator of $g(x)$: $D^3 (D - 2)^3 (D - 5)$

4-5-3 Using the Annihilator to Find the Particular Solution

Step 2-1 Find the annihilator L_1 of $g(x)$

Step 2-2 如果原來的 linear & constant coefficient DE 是

$$L(y) = g(x)$$

那麼將 DE 變成如下的型態：

$$L_1[L(y)] = L_1[g(x)] = 0$$

(homogeneous linear & constant coefficient DE)

註： If $a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = g(x)$

then $L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$

Step 2-3 Use the method in Section 4-3 to find the solution of

$$L_1[L(y)] = 0$$

Step 2-4 Find the particular solution.

The particular solution y_p is a solution of

$$L_1[L(y)] = 0$$

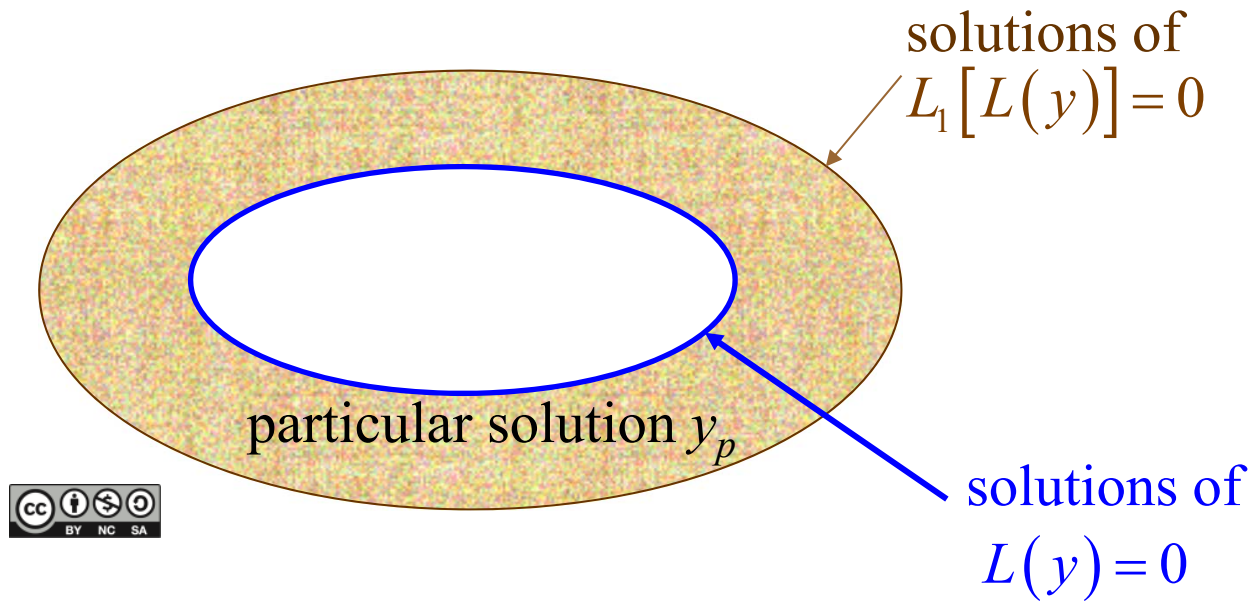
but not a solution of

$$L(y) = 0$$

(Proof): Since $L(y_p) = g(x)$, if $g(x) \neq 0$, $L(y_p)$ should be nonzero.

Moreover, $L_1[L(y_p)] = L_1[g(x)] = 0$.

Step 2-5 Solve the unknowns



particular solution $y_p \in$ solutions of $L_1[L(y)] = 0$

\notin solutions of $L(y) = 0$

本節核心概念

4-5-4 Examples

[Example 3] (text page 155)

$$y'' + 3y' + 2y = 4x^2 \quad (L = D^2 + 3D + 2)$$

Step 1: Complementary function

(solution of the associated homogeneous function)

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

Step 2-1: Annihilation: D^3 $(L_1 = D^3)$

$$L_1[L(y)] = L_1[g(x)] = 0$$

Step 2-2: $D^3(D^2 + 3D + 2)y = 0$

Step 2-3: auxiliary function $m^3(m^2 + 3m + 2) = 0$

roots: $m_1 = m_2 = m_3 = 0, m_4 = -1, m_5 = -2$

Solution for $L_1[L(\tilde{y})] = 0$:

$$\tilde{y} = \boxed{d_1 + d_2 x + d_3 x^2} + d_4 e^{-x} + d_5 e^{-2x}$$

移除和 complementary
function 相同的部分

Step 2-4: particular solution $y_p = A + Bx + Cx^2$ $y'_p = B + 2Cx$
 $y''_p = 2C$

Step 2-5: $y''_p + 3y'_p + 2y_p = 2Cx^2 + (2B + 6C)x + (2A + 3B + 2C) = 4x^2$

$$\begin{cases} 2C = 4 \\ 2B + 6C = 0 \\ 2A + 3B + 2C = 0 \end{cases} \quad \longrightarrow \quad \begin{cases} C = 2 \\ B = -6 \\ A = 7 \end{cases}$$

$$y_p = 7 - 6x + 2x^2$$

Step 3: $y = y_c + y_p = c_1e^{-x} + c_2e^{-2x} + 7 - 6x + 2x^2$

[Example 4] (text page 156)

$$y'' - 3y' = 8e^{3x} + 4\sin x \quad (L = D^2 - 3D)$$

Step 1: Complementary function

From auxiliary function, $m^2 - 3m = 0$, roots: 0, 3

$$y_c = c_1 + c_2 e^{3x}$$

Step 2-1: Find the annihilator

$D - 3$ annihilate $8e^{3x}$ but cannot annihilate $4\sin x$

$(D^2 + 1)$ annihilate $4\sin x$ but cannot annihilate $8e^{3x}$



$L_1 = (D - 3)(D^2 + 1)$ is the annihilator of $8e^{3x} + 4\sin x$

Step 2-2: $(D - 3)(D^2 + 1)(D^2 - 3D)y = 0$

Step 2-3: auxiliary function: $(m-3)(m^2+1)(m^2-3m)$ } 易犯錯的地方
 $= m(m-3)^2(m^2+1) = 0$ ←

solution of $(D-3)(D^2+1)(D^2-3D)\tilde{y} = 0$:

$$\tilde{y} = \cancel{d_1} + \cancel{d_2 e^{3x}} + \boxed{d_3 x e^{3x} + d_4 \cos x + d_5 \sin x}$$

Step 2-4: particular solution

$$y_p = d_3 x e^{3x} + d_4 \cos x + d_5 \sin x$$

↓ 代回原式
 並比較係數

Step 2-5: $y_p = \frac{8}{3} x e^{3x} + \frac{6}{5} \cos x - \frac{2}{5} \sin x$

Step 3: general solution $\boxed{y = c_1 + c_2 e^{3x} + \frac{8}{3} x e^{3x} + \frac{6}{5} \cos x - \frac{2}{5} \sin x}$

4-5-5 本節要注意的地方

- (1) 所以要先算 complementary function，再算 particular solution
- (2) 若有兩個以上的 annihilator，選其中較簡單的即可
- (3) 計算 auxiliary function 時有時容易犯錯
- (4) $L_1[L(\tilde{y})]=0$ 的解和 $L(y)=0$ 的解不一樣。
- (5) 這方法，只適用於 constant coefficient linear DE
(因為，還需借助 auxiliary function)

The thing that can be done by the **annihilator approach** can always be done by the “**guessing**” method in Section 4-4, too.

附錄九：Reviews for Higher Order DE

(A) Linear DE Complementary Function 3 大解法的前 2 個

(1) Reduction of Order (Section 4-2)

適用情形：

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$

(2) Auxiliary Function (Section 4-3)

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y = 0$$

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

4 Cases (See pages 181-183, 185-186)

適用情形：

(B) Linear DE Particular solution 3 大解法的前 2個**(1) Guess (Section 4-4) (熟悉講義 page 196 的表)**

要訣： y_p should be a linear combination of $g(x)$, $g'(x)$,

$g''(x)$, $g'''(x)$, $g^{(4)}(x)$, $g^{(5)}(x)$,

適用情形：

遇到重覆，乘 x (Sec. 4-4, constant coefficients)

或乘 $\ln x$ (Sec 4-7, Cauchy-Euler)

(2) Annihilator (Section 4-5)

若原本的 DE 為 $L[y(x)] = g(x)$ Annihilator: $L_1[g(x)] = 0$

Particular solution 為 $L_1\{L[y(x)]\} = 0$ 的解

(扣去和 $L[y(x)] = 0$ 的解重複的部分)

$$y = y_c + y_p$$

適用情形：

Annihilator 算法三大規則：Pages 215-217

練習題

Section 4-1: 3, 7, 8, 10, 13, 20, 24, 29, 33, 36

Section 4-2: 2, 4, 9, 13, 14, 16, 18, 19, 22

Section 4-3: 7, 16, 20, 22, 24, 28, 33, 39, 41, 52, 54, 56, 59, 61, 63

Section 4-4: 5, 6, 14, 17, 18, 24, 26, 32, 33, 39, 42

Section 4-5: 2, 7, 8, 13, 18, 31, 45, 60, 62, 69, 70

Review 4: 2, 21, 22, 25, 33, 34, 37