4-6 Variation of Parameters

4-6-1 方法的限制

The method can solve the particular solution for **any** linear DE

(1) May not have constant coefficients

(2) $g(x)$ may not be of the special forms

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y=g(x)$$

4-6-2 Case of the 2nd order linear DE

$$a_2(x)y''(x)+a_1(x)y'(x)+a_0(x)y=g(x)$$

associated homogeneous equation: $a_n(x)y''(x)+a_1(x)y'(x)+a_0(x)y=0$

Suppose that the solution of the associated homogeneous equation is

$$c_1y_1(x)+c_2y_2(x)$$

Then the particular solution is assumed as:

$$y_p=u_1(x)y_1(x)+u_2(x)y_2(x)$$

(方法的基本精神)

$$\boxed{y_p = u_1(x)y_1(x) + u_2(x)y_2(x)}$$ 代入原式後，總是可以簡化

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

$$y_p'' = u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''$$

代入 $y''(x) + P(x)y'(x) + Q(x)y = f(x)$

$$P(x) = \frac{a_1(x)}{a_2(x)}, \quad Q(x) = \frac{a_0(x)}{a_2(x)}, \quad f(x) = \frac{g(x)}{a_2(x)}$$

$$y_p'' + P(x)y_p' + Q(x)y_p = u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''$$
$$+P\left(u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'\right) + Q\left(u_1y_1 + u_2y_2\right)$$

$$y_p'' + P(x)y_p' + Q(x)y_p = u_1\underbrace{\left[y_1'' + Py_1' + Qy_1\right]}_{\text{zero}} + u_2\underbrace{\left[y_2'' + Py_2' + Qy_2\right]}_{\text{zero}} + y_1u_1''$$
$$+2u_1'y_1' + y_2u_2'' + 2u_2'y_2' + P\left[y_1u_1' + y_2u_2'\right]$$

$$y_p'' + P(x)y_p' + Q(x)y_p = f(x), \qquad y_p = u_1 y_1 + u_2 y_2$$

簡化

$$y_p'' + P(x)y_p' + Q(x)y_p = y_1 u_1'' + 2u_1' y_1' + y_2 u_2'' + 2u_2' y_2' + P[y_1 u_1' + y_2 u_2']$$

簡化

$$\frac{d}{dx}[y_1 u_1' + y_2 u_2'] + P[y_1 u_1' + y_2 u_2'] + y_1' u_1' + y_2' u_2' = f(x)$$

進一步簡化：

假設 $y_1 u_1' + y_2 u_2' = 0$

$$y_1' u_1' + y_2' u_2' = f(x)$$

聯立方程式

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases}$$

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases} \Rightarrow \boxed{u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W}} \Rightarrow \boxed{u_1 = \int u_1'(x)\,dx}$$

$$\boxed{u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W}} \qquad \boxed{u_2 = \int u_2'(x)\,dx}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $\quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$ $\quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

| |: determinant

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

可以和 1st order case (page 62) 相比較

4-6-3 Process for the 2nd Order Case

Step 2-1 變成 standard form

$$y''(x)+P(x)y'(x)+Q(x)y=f(x)$$

Step 2-2

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1=\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad W_2=\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Step 2-3

$$u_1'=\frac{W_1}{W} \qquad u_2'=\frac{W_2}{W}$$

Step 2-4

$$u_1=\int u_1'(x)\,dx \qquad u_2=\int u_2'(x)\,dx$$

Step 2-5

$$y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)$$

4-6-4 Examples

Example 1 (text page 162)

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Step 1: solution of $y'' - 4y' + 4y = 0$:

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

Step 2-2: $y_p = u_1 y_1 + u_2 y_2$, $y_1 = e^{2x}$, $y_2 = xe^{2x}$

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x} \quad W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x+1)xe^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

Step 2-3: $u_1' = \dfrac{W_1}{W} = -x^2 - x$ $\quad u_2' = \dfrac{W_2}{W} = x + 1$

Step 2-4: $u_1 = \int u_1' dx = \int (-x^2 - x)dx = -\frac{1}{3}x^3 - \frac{1}{2}x^2 + \cancel{c_1}$

$u_2 = \int u_2' dx = \int (x+1)dx = \frac{1}{2}x^2 + x + \cancel{c_2}$

Step 2-5: $y_p = (-\frac{1}{3}x^3 - \frac{1}{2}x^2)e^{2x} + (\frac{1}{2}x^2 + x)xe^{2x} = (\frac{1}{6}x^3 + \frac{1}{2}x^2)e^{2x}$

Step 3: $\boxed{y = c_1 e^{2x} + c_2 x e^{2x} + (\frac{1}{6}x^3 + \frac{1}{2}x^2)e^{2x}}$

Example 2 (text page 163) $4y'' + 36y = \csc 3x$

Step 1: solution of $4y'' + 36y = 0$: $y_c = c_1 \cos 3x + c_2 \sin 3x$

Step 2-1: standard form: $y'' + 9y = \csc 3x / 4$ $f(x) = \csc 3x / 4$

Step 2-2:

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3 \qquad W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -1/4$$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

Step 2-3: $u_1' = \dfrac{W_1}{W} = -\dfrac{1}{12}$ $u_2' = \dfrac{W_2}{W} = \dfrac{1}{12}\dfrac{\cos 3x}{\sin 3x}$

Step 2-4: $u_1 = -\dfrac{x}{12}$ $u_2 = \dfrac{1}{36}\ln|\sin 3x|$

(未完待續) 注意 $\dfrac{1}{12}\displaystyle\int \frac{\cos 3x}{\sin 3x}dx$ 算法

Step 2-5: $y_p = -\frac{x}{12}\cos 3x + \frac{1}{36}\sin 3x \ln|\sin 3x|$

Step 3: $y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{12}\cos 3x + \frac{1}{36}\sin 3x \ln|\sin 3x|$

Note: 課本 Interval (0, π/6) 應該改為(0, π/3)

Example 3 (text page 164) $y'' - y = 1/x$

$$y_c = c_1 e^{x} + c_2 e^{-x} \qquad f(x) = 1/x$$

$$W = \begin{vmatrix} e^{x} & e^{-x} \\ e^{x} & -e^{-x} \end{vmatrix} = -2$$

Note: $\int \frac{e^{x}}{x} dx$ 沒有 analytic 的解

所以直接表示成 $\int_{x_0}^{x} \frac{e^{t}}{t} dt$ (複習 page 48)

4-6-5 Case of the Higher Order Linear DE

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y=g(x)$$

Solution of the associated homogeneous equation:

$$y_c=c_1y_1(x)+c_2y_2(x)+c_3y_3(x)+\cdots\cdots+c_ny_n(x)$$

The particular solution is assumed as:

$$y_p=u_1(x)y_1(x)+u_2(x)y_2(x)+u_3(x)y_3(x)+\cdots\cdots+u_n(x)y_n(x)$$

$$u_k'(x)=\frac{W_k}{W} \Longrightarrow u_k(x)=\int u_k'(x)dx$$

Process of the Higher Order Case

Step 2-1 變成 standard form

$$y^{(n)}(x)+\frac{a_{n-1}(x)}{a_n(x)}y^{(n-1)}(x)+\cdots+\frac{a_1(x)}{a_n(x)}y'(x)+\frac{a_0(x)}{a_n(x)}y=\frac{g(x)}{a_n(x)}$$

Step 2-2 Calculate W, W_1, W_2, …., W_n (see page 243)

Step 2-3 $u_1'=\frac{W_1}{W}$ $u_2'=\frac{W_2}{W}$ ……… $u_n'=\frac{W_n}{W}$

Step 2-4 $u_1=\int u_1'(x)\,dx$ $u_2=\int u_2'(x)\,dx$ ……. $u_n=\int u_n'(x)\,dx$

Step 2-5 $y_p(x)=u_1(x)y_1(x)+u_2(x)y_2(x)+\cdots\cdots+u_n(x)y_n(x)$

$$u_k'(x) = \frac{W_k}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$W_k = \begin{vmatrix} y_1 & y_2 & \cdots & y_{k-1} & 0 & y_{k+1} & \cdots & y_n \\ y_1' & y_2' & \cdots & y_{k-1}' & 0 & y_{k+1}' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{k-1}^{(n-2)} & 0 & y_{k+1}^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{k-1}^{(n-1)} & f(x) & y_{k+1}^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$f(x) = g(x)/a_n(x)$$

W_k: replace the k^{th} column of W by $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$ $\quad f(x) = \dfrac{g(x)}{a_n(x)}$

For example, when $n = 3$,

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix} \qquad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}$$

Exercise 30 $y''' + 4y' = \sec 2x$

Complementary function: $y_c = c_1 + c_2 \cos 2x + c_3 \sin 2x$

$$W = \begin{vmatrix} 1 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ 0 & -4\cos 2x & -4\sin 2x \end{vmatrix} = 8$$

$$W_1 = \begin{vmatrix} 0 & \cos 2x & \sin 2x \\ 0 & -2\sin 2x & 2\cos 2x \\ \sec 2x & -4\cos 2x & -4\sin 2x \end{vmatrix} = 2\sec 2x$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin 2x \\ 0 & 0 & 2\cos 2x \\ 0 & \sec 2x & -4\sin 2x \end{vmatrix} = -2 \qquad W_3 = \begin{vmatrix} 1 & \cos 2x & 0 \\ 0 & -2\sin 2x & 0 \\ 0 & -4\cos 2x & \sec 2x \end{vmatrix} = -2\tan 2x$$

$$u_1' = \frac{W_1}{W} = \frac{\sec 2x}{4} \qquad u_2' = \frac{W_2}{W} = \frac{-1}{4} \qquad u_3' = \frac{W_3}{W} = \frac{-\tan 2x}{4}$$

$$u_1 = \frac{1}{8}\ln\left|\sec 2x + \tan 2x\right| \qquad u_2 = \frac{-x}{4} \qquad u_3 = \frac{1}{8}\ln\left|\cos 2x\right|$$

$$y(x) = c_1 + c_2\cos 2x + c_3\sin 2x$$
$$+\frac{1}{8}\ln\left|\sec 2x + \tan 2x\right| + \frac{-x}{4}\cos 2x + \frac{1}{8}\left(\ln\left|\cos 2x\right|\right)\sin 2x$$

for $-\pi/4 < x < \pi/4$

Note: $-\pi/4$, $\pi/4$ are singular points

4-6-7 本節需注意的地方

(1) 養成先解 associated homogeneous equation 的習慣

(2) 記熟幾個重要公式

(3) 這裡 | | 指的是 determinant

(4) 算出 $u_1'(x)$ 和 $u_2'(x)$ 後別忘了作積分

(5) $f(x) = g(x)/a_n(x)$ (和 1st order 的情形一樣，使用 standard form)

特別要小心

(6) 計算 $u_1'(x)$ 和 $u_2'(x)$ 的積分時，$+\,c$ 可忽略

因為 我們的目的是算particular solution y_p

y_p 是任何一個能滿足原式的解

(7) 這方法解的範圍，不包含 $a_n(x) = 0$ 的地方

4-7 Cauchy-Euler Equation

4-7-1 解法限制條件

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = g(x)$$

not constant coefficients

k

but the coefficients of $y^{(k)}(x)$ have the form of $a_k x^k$

a_k is some constant

associated homogeneous equation

particular solution

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = 0$$

4-7-2 解法

Associated homogeneous equation of the Cauchy-Euler equation

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = 0$$

Guess the solution as $y(x) = x^m$, then

$$a_n x^n m(m-1)(m-2)\cdots\cdots(m-n+1) x^{m-n} +$$
$$a_{n-1} x^{n-1} m(m-1)(m-2)\cdots\cdots(m-n+2) x^{m-n+1} +$$
$$a_{n-2} x^{n-2} m(m-1)(m-2)\cdots\cdots(m-n+3) x^{m-n+2} +$$
$$\vdots$$
$$+a_1 x m x^{m-1}$$
$$+a_0 x^m = 0$$

Delete x^m on the previous page

$$\begin{aligned}
&a_n m(m-1)(m-2)\cdots\cdots(m-n+1) \\
&+a_{n-1} m(m-1)(m-2)\cdots\cdots(m-n+2) \\
&+a_{n-2} m(m-1)(m-2)\cdots\cdots(m-n+3) \\
&\quad\vdots \\
&+a_1 m \\
&+a_0 = 0
\end{aligned}$$

⇨ auxiliary function

比較: 和 constant coefficient 時有何不同？

規則：把 $x^k \frac{d^k}{dx^k}$ 變成 $\frac{m!}{(m-k)!} = m(m-1)\cdots(m-k+1)$

4-7-3 For the 2nd Order Case

$$a_2 x^2 y''(x) + a_1 x y'(x) + a_0 y = 0$$

auxiliary function:

$$a_2 m(m-1) + a_1 m + a_0 = 0 \qquad a_2 m^2 + (a_1 - a_2) m + a_0 = 0$$

roots

$$m_1 = \frac{a_2 - a_1 + \sqrt{(a_1 - a_2)^2 - 4a_2 a_0}}{2a_2} \qquad m_2 = \frac{a_2 - a_1 - \sqrt{(a_1 - a_2)^2 - 4a_2 a_0}}{2a_2}$$

[Case 1]: $m_1 \neq m_2$ and m_1, m_2 are real

two independent solution of the homogeneous part:

x^{m_1} and x^{m_2}

$$y_c = c_1 x^{m_1} + c_2 x^{m_2}$$

[Case 2]: $m_1 = m_2$

Use the method of reduction of order

$$y_1 = x^{m_1}$$

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx = x^{m_1}\int \frac{e^{-\int \frac{a_1}{a_2 x}dx}}{x^{2m_1}}dx$$

Note 1: 原式 $\longrightarrow y''(x) + \frac{a_1}{a_2 x}y'(x) + \frac{a_0}{a_2 x^2}y = 0,\qquad P(x) = \frac{a_1}{a_2 x}$

Note 2: 此時 $\quad m_1 = m_2 = \frac{a_2 - a_1}{2a_2}$

$$y_2(x) = x^{m_1}\int \frac{e^{-\int \frac{a_1}{a_2 x}dx}}{x^{2m_1}}dx = x^{m_1}\int \frac{e^{-\frac{a_1}{a_2}\ln|x|}}{x^{2m_1}}dx = x^{m_1}\int \frac{|x|^{-\frac{a_1}{a_2}}}{x^{2m_1}}dx \qquad m_1 = \frac{a_2 - a_1}{2a_2}$$

$$= (-1)^{-\frac{a_1}{a_2}} x^{m_1}\int x^{-\frac{a_1}{a_2}} x^{\frac{a_1 - a_2}{a_2}} dx = x^{m_1}\int x^{-1}dx = x^{m_1}\ln|x|$$

If $y_2(x)$ is a solution of a homogeneous DE

then $c\, y_2(x)$ is also a solution of the homogeneous DE

If we constrain that $x > 0$, then $y_2 = x^{m_1}\ln x$

$$y_c = c_1 x^{m_1} + c_2 x^{m_1}\ln x$$

[Case 3]: $m_1 \neq m_2$ and m_1, m_2 are the form of

$$m_1 = \alpha + j\beta \qquad m_2 = \alpha - j\beta$$

two independent solution of the homogeneous part:

$$x^{\alpha+j\beta} \quad \text{and} \quad x^{\alpha-j\beta}$$

$$y_c = C_1 x^{\alpha+j\beta} + C_2 x^{\alpha-j\beta}$$

$$x^{\alpha+j\beta} = (e^{\ln x})^{\alpha+j\beta} = e^{(\alpha+j\beta)\ln x} = e^{\alpha \ln x} e^{j\beta \ln x}$$

$$= x^{\alpha}\left(\cos(\beta \ln x) + j\sin(\beta \ln x)\right)$$

同理 $x^{\alpha-j\beta} = x^{\alpha}\left(\cos(\beta \ln x) - j\sin(\beta \ln x)\right)$

$$y_c = x^{\alpha}[(C_1 + C_2)\cos(\beta \ln x) + j(C_1 - C_2)\sin(\beta \ln x)]$$

$$\boxed{y_c = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]}$$

Example 1 (text page 167)

$$x^2 y''(x) - 2xy'(x) - 4y = 0$$

Example 2 (text page 168)

$$4x^2 y''(x) + 8xy'(x) + y = 0$$

Example 3 (text page 169)

$$\boxed{4x^2 y''(x) + 17y = 0} \qquad y(1) = -1 \quad y'(1) = -\frac{1}{2}$$

4-7-4 For the Higher Order Case

Process: auxiliary function

↓ Step 1-1

roots

↓ Step 1-2

n independent solutions

↓ Step 1-3

solution of the n^{th} order associated homogeneous equation

(1) 若 auxiliary function 在 m_0 的地方只有一個根

$$x^{m_0}$$

是 associated homogeneous equation 的其中一個解

(2) 若 auxiliary function 在 m_0 的地方有 k 個重根

$$x^{m_0}, \quad x^{m_0}\ln x, \quad x^{m_0}(\ln x)^2, \quad \cdots\cdots, \quad x^{m_0}(\ln x)^{k-1}$$

皆為 associated homogeneous equation 的解

(3) 若 auxiliary function 在 $\alpha+j\beta$ 和 $\alpha-j\beta$ 的地方各有一個根
(未出現重根)

$$x^{\alpha}\cos(\beta\ln x),\quad x^{\alpha}\sin(\beta\ln x)$$

是 associated homogeneous equation 的其中二個解

(4) 若 auxiliary function 在 $\alpha+j\beta$ 和 $\alpha-j\beta$ 的地方皆有 k 個重根

$$x^{\alpha}\cos(\beta\ln x),\quad x^{\alpha}\cos(\beta\ln x)\ln x,\quad x^{\alpha}\cos(\beta\ln x)(\ln x)^{2},\quad \cdots\cdots,$$

$$x^{\alpha}\cos(\beta\ln x)(\ln x)^{k-1}$$

$$x^{\alpha}\sin(\beta\ln x),\quad x^{\alpha}\sin(\beta\ln x)\ln x,\quad x^{\alpha}\sin(\beta\ln x)(\ln x)^{2},\quad \cdots\cdots,$$

$$x^{\alpha}\sin(\beta\ln x)(\ln x)^{k-1}$$

是 associated homogeneous equation 的其中 $2k$ 個解

Example 4 (text page 169)

$$x^3 y'''(x) + 5x^2 y''(x) + 7xy'(x) + 8y = 0$$

auxiliary function

$$m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0$$

$$m^3 - 3m^2 + 2m + 5m^2 - 5m + 7m + 8 = 0$$

$$m^3 + 2m^2 + 4m + 8 = 0$$

$$(m+2)(m^2+4) = 0$$

4-7-5 Nonhomogeneous Case

To solve the nonhomogeneous Cauchy-Euler equation:

Method 1: (See Example 5)

(1) Find the complementary function (general solutions of the associated homogeneous equation) from the rules on pages 251-254, 258-259.

(2) Use the method in Sec. 4-6 (Variation of Parameters) to find the particular solution.

(3) Solution = complementary function + particular solution

Method 2: See Example 6，很重要

Set $x = e^t$, $t = \ln x$

Example 5 (text page 169, illustration for method 1)

$$x^2y''(x)-3xy'(x)+3y=2x^4e^x$$

Step 1 solution of the associated homogeneous equation

auxiliary function

$$m(m-1)-3m+3=0 \qquad m^2-4m+3=0 \qquad m_1=1$$

$$m_2=3$$

$$y_c=c_1x+c_2x^3$$

Step 2-2 Particular solution $W=\begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}=2x^3$

$$W_1=\begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix}=-2x^5e^x \qquad W_2=\begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix}=2x^3e^x$$

Step 2-3 $u_1'=\dfrac{W_1}{W}=-x^2e^x \qquad u_2'=\dfrac{W_2}{W}=e^x$

Step 2-4 $u_1 = \int u_1' dx = -x^2 e^x + 2xe^x - 2e^x$

$$u_2 = \int u_2' dx = e^x$$

Step 2-5 $\boxed{y_p = u_1 y_1 + u_2 y_2 = 2x^2 e^x - 2xe^x}$

Step 3 $\boxed{y = c_1 x + c_2 x^3 + 2x^2 e^x - 2xe^x}$

Example 6 (text page 170, illustration for method 2)

$$x^2 y''(x) - xy'(x) + y = \ln x$$

Set $x = e^t$, $t = \ln x$ (Step 1)

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dt} \qquad \text{(chain rule)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{1}{x}\frac{d}{dt}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{1}{x^2}\frac{d^2y}{dt^2} + \frac{1}{x}\left(\frac{d}{dt}\frac{1}{x}\right)\left(\frac{dy}{dt}\right) = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Therefore, the original equation is changed into

$$\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + y(t) = t$$

(Step 2)

$$\frac{d^2}{dt^2}y(t)-2\frac{d}{dt}y(t)+y(t)=t$$

$\Longrightarrow$ $y(t)=c_1e^t+c_2te^t+t+2$ (Step 3)

$\Longrightarrow$ $y(x)=c_1x+c_2x\ln x+\ln x+2$ (別忘了 $t=\ln x$ 要代回來)

(Step 4)

Note 1: 以此類推

$$x^k\frac{d^ky}{dx^k}=(D_t-k+1)\cdots(D_t-1)D_ty$$

(Step 2) D_t means $\frac{d}{dt}$

(Step 2) Determine the auxiliary function, then replace m by D_t

Note 2: 簡化計算的小技巧: 使用Cauchy-Euler equation 的 auxiliary function

4-7-6 本節要注意的地方

(1) 本節公式記憶的方法：

把 Section 4-3 的 e^x 改成 x，x 改成 $\ln(x)$

把 auxiliary function 的 m^n 改成 $m(m-1)(m-2)\cdots\cdots(m-n+1)$

(2) 如何解 particular solution?

Variation of Parameters 的方法

(3) 解的範圍將不包括 $x=0$ 的地方 (Why?)

Extra Problems:

How do we solve

(1) $xy''(x) + y'(x) = 0$

(2) $(x-1)^2 y''(x) + y(x) = 0$

還有很多 linear DE 沒有辦法解，怎麼辦

(1) numerical approach (Section 4-9-3)

(2) using special function (Chap. 6)

(3) Laplace transform and Fourier transform (Chaps. 7, 11, 14)

(4) 查表 (table lookup)

(1) 即使用了 Section 4-7 的方法，大部分的 DE還是沒有辦法解

(2) 所幸，自然界真的有不少的例子是 linear DE

甚至是 constant coefficient linear DE

Exercises for practicing

Section 4-6	4, 5, 8, 13, 14, 17, 18, 21, 25, 28, 29, 34
Section 4-7	11, 17, 18, 20, 21, 24, 32, 35, 36, 37, 40, 42
Review 4	27, 28, 29, 30, 32, 42