

# Chapter 6 Series Solutions of Linear Equations

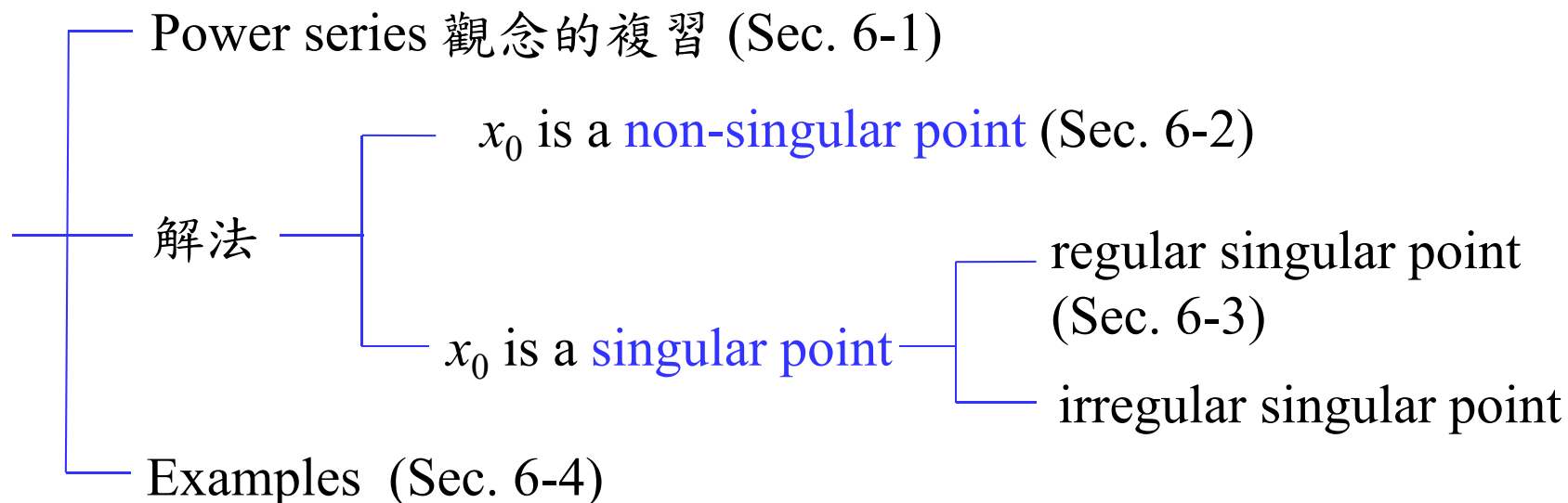
當 DE 為 linear 且 coefficients 為 polynomials

→ 假設 DE 的 solutions 為 polynomial 的型態

(和 Cauchy-Euler Method 以及 Taylor Series 的概念相近)

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

被稱作為 power series centered at  $x_0$



## Section 6-1 Reviews of Power Series

### 6-1-1 定義

#### 1. Power series

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1 (x - x_0) + c_2 (x - x_0)^2 + \dots$$

2. Convergence:  $\lim_{N \rightarrow \infty} \sum_{n=0}^N c_n (x - x_0)^n$  exists

測試方法：Ratio test (test for convergence)

$$\lim_{n \rightarrow \infty} \left| \frac{c_{n+1} (x - x_0)^{n+1}}{c_n (x - x_0)^n} \right| = L \quad L < 1: \text{converge} \quad L > 1: \text{diverge}$$

$$L = 1: \text{不一定}$$

#### 3. Radius of Convergence $R$

$$L < 1 \text{ if } |x - x_0| < R \quad L > 1 \text{ if } |x - x_0| > R$$

Example 1 (text page 238)

For the Power series  $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n n}$

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(x-3)^{n+1}}{2^{n+1}(n+1)}}{\frac{(x-3)^n}{2^n n}} \right| = |x-3| \lim_{n \rightarrow \infty} \frac{n}{2(n+1)} = \frac{|x-3|}{2}$$

$$\frac{|x-3|}{2} < 1 \quad \text{for } 1 < x < 5 \quad \text{Interval of convergence: } (1, 5)$$

However, since when  $x = 1$ , the power series becomes

$$\sum_{n=1}^{\infty} (-1)^n / n,$$

which is also convergent, the interval of convergence is modified as:

$$\text{Interval of convergence: } [1, 5)$$

## 6-1-2 Maclaurin Series (Taylor Series)

$$y(x) = y(x_0) + \frac{y'(x_0)}{1!}(x - x_0) + \frac{y''(x_0)}{2!}(x - x_0)^2 + \frac{y'''(x_0)}{3!}(x - x_0)^3 + \\ + \frac{y^{(4)}(x_0)}{4!}(x - x_0)^4 + \dots$$

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x - x_0)^n$$

Maclaurin Series (Taylor Series)	Interval of Convergence
$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$(-\infty, \infty)$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$	$(-\infty, \infty)$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$(-\infty, \infty)$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	$(-1, 1]$
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$	$(-1, 1)$

$$e^{2x} = ?$$

$$\cos 3x = ?$$

Example 2 (text page 240)

Find a power series representation of  $e^x \sin x$

$$\begin{aligned} e^x \sin x &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right) \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) \\ &= (1)x + (1)x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5 + \dots \\ &= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \dots \end{aligned}$$

## Section 6-2 Solutions about Ordinary Points

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Suppose that the solution is  $\sum_{n=0}^{\infty} c_n (x - x_0)^n$

### 6-2-1 方法適用情形

(1) Linear

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

(2)  $x_0$  is not a singular point

(3) It is better that  $a_0(x), a_1(x), \dots, a_n(x), g(x)$  are all polynomials.

(or can be expressed by Taylor series)

## 6-2-2 解法流程

Step 1 將  $y(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n$  代入 ( $x_0$  必需為 ordinary point)

Step 2 對齊 (一律變成  $(x-x_0)^k$ )

不是 singular point 的  
即為 ordinary point

Step 3 合併

Step 4 比較係數，將  $c_n$  之間的關係找出來

Step 5 Obtained independent solutions and general solution



## 6-2-3 例子

Example 5 (text page 246)

$$y'' - xy = 0$$

Set  $y(x) = \sum_{n=0}^{\infty} c_n x^n$  since  $P(x) = 0$  and  $Q(x) = x$  are analytic at 0

**Step 1**  $y'' + xy = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - x \sum_{n=0}^{\infty} c_n x^n = 0$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

set  $k = n - 2$

set  $k = n + 1$

**Step 2 對齊**  $\sum_{k=0}^{\infty} c_{k+2} (k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 0$

$$\sum_{k=0}^{\infty} c_{k+2} (k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k = 0$$

Step 3  $2c_2 + \sum_{k=1}^{\infty} [c_{k+2} (k+2)(k+1) - c_{k-1}]x^k = 0$

Step 4  $2c_2 = 0$   $c_{k+2} (k+2)(k+1) - c_{k-1} = 0$

$\downarrow$

$c_2 = 0$  或  $c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)}$

recurrence relation

$c_0, c_1$  給定之後

$$k=1 \quad c_3 = \frac{c_0}{2 \cdot 3}$$

$$k=2 \quad c_4 = \frac{c_1}{3 \cdot 4}$$

$$k=3 \quad c_5 = \frac{c_2}{4 \cdot 5} = 0$$

$$c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)}$$

以此類推，所有的  $c_n$  的值都可以算出來 (以  $c_0$  或  $c_1$  表示)

$$k=4 \quad c_6 = \frac{c_3}{5 \cdot 6} = \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6}$$

$$k=5 \quad c_7 = \frac{c_4}{6 \cdot 7} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7}$$

$$k=6 \quad c_8 = \frac{c_5}{7 \cdot 8} = 0$$

$$k=7 \quad c_9 = \frac{c_6}{8 \cdot 9} = \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}$$

$$k=8 \quad c_{10} = \frac{c_7}{9 \cdot 10} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}$$

$$k=9 \quad c_{11} = \frac{c_8}{10 \cdot 11} = 0$$

⋮

⋮

group 1  $c_0, c_3, c_6, c_9 \dots\dots$

group 2  $c_1, c_4, c_7, c_{10} \dots\dots$

group 3  $c_2, c_5, c_8, c_{11} \dots\dots$

Step 5

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} c_{3k} x^{3k} + \sum_{k=0}^{\infty} c_{3k+1} x^{3k+1} + \sum_{k=0}^{\infty} c_{3k+2} x^{3k+2}$$

$$= c_0 \left[ 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \dots \right] \quad y_1$$

$$+ c_1 \left[ x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \frac{x^{10}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} + \dots \right] \quad y_2$$

$$y(x) = c_0 y_1(x) + c_1 y_2(x)$$

不寫在  
 $\Sigma$  裡面

$$y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{1}{2 \cdot 3 \cdots (3k-1)(3k)} x^{3k}$$

$$y_2(x) = x + \sum_{k=1}^{\infty} \frac{1}{3 \cdot 4 \cdots (3k)(3k+1)} x^{3k+1}$$

ratio test:

Example 6 (text page 248)

$$(x^2 + 1)y'' + xy' - y = 0 \quad y(x) = \sum_{n=0}^{\infty} c_n x^n \quad (\text{analytic at } x = 0)$$

Radius of convergence?

Step 1

$$(x^2 + 1) \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

Step 2

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

$\downarrow k=n$                        $\downarrow k=n-2$                        $\downarrow k=n$                        $\swarrow k=n$

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

Step 3

$$\frac{2c_2 - c_0}{k=0} + \frac{(6c_3 + c_1 - c_1)x}{k=1} + \sum_{k=2}^{\infty} [k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k] x^k = 0$$

$$2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty} [(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}] x^k = 0$$

Step 4

$$2c_2 - c_0 = 0$$

$$6c_3 = 0$$

$$c_{k+2} = -\frac{k-1}{k+2} c_k$$

$$2c_2 - c_0 = 0$$

$$c_2 = c_0/2$$

$$6c_3 = 0$$

$$c_3 = 0$$

$$c_{k+2} = \frac{1-k}{k+2} c_k$$

$c_0, c_1$  給定之後

$$c_2 = c_0/2$$

$$c_3 = 0$$

$$c_4 = -\frac{1}{4}c_2 = -\frac{1}{2 \cdot 4}c_0 = -\frac{1}{2^2 \cdot 2!}c_0$$

$$c_5 = -\frac{2}{5}c_3 = 0$$

$$c_6 = -\frac{3}{6}c_4 = \frac{3}{2 \cdot 4 \cdot 6}c_0 = \frac{3}{2^3 \cdot 3!}c_0$$

$$c_7 = -\frac{4}{7}c_5 = 0$$

$$c_8 = -\frac{5}{8}c_6 = -\frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}c_0 = -\frac{3 \cdot 5}{2^4 \cdot 4!}c_0$$

$$c_9 = -\frac{6}{9}c_7 = 0$$

:

:

**Step 5**  $y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1}$

$$= c_0 \left[ 1 + \frac{x^2}{2} - \frac{x^4}{2^2 \cdot 2!} + \frac{3x^6}{2^3 \cdot 3!} - \frac{3 \cdot 5 x^8}{2^4 \cdot 4!} + \frac{3 \cdot 5 \cdot 7 x^{10}}{2^5 \cdot 5!} \dots \right] + c_1 x$$


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$y(x) = c_0 y_1(x) + c_1 y_2(x)$

不寫在  $\Sigma$  裡面

$$y_1(x) = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^{2n}$$

$|x| < 1$  (Why?)

$$y_2(x) = x$$

ratio test:



Example 8 (text page 250)

$$y'' + (\cos x)y = 0$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots\dots\dots$$

$$y_1 = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \dots\dots\dots$$

$$y_2 = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \dots\dots\dots$$

## 6-2-4 定義

1. Analytic at  $x_0$ : If a function can be expressed as a power series and the **radius of convergence** of the power series is **nonzero**

簡單的判斷  $f(x)$  在  $x_0$  是否為 analytic 方法

(1)  $f(x_0)$  should be neither  $\infty$  nor  $-\infty$

(2)  $f^{(m)}(x_0)$  should be neither  $\infty$  nor  $-\infty$

$m = 1, 2, 3, \dots$

## 2. Ordinary Point and Singular Point:

- For the 2<sup>nd</sup> order linear DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \longrightarrow y'' + P(x)y' + Q(x)y = 0$$

### Definition 6.1

$x_0$  is an **ordinary point** of the 2<sup>nd</sup> order linear DE if both  $P(x)$  and  $Q(x)$  are **analytic** at  $x_0$

Otherwise,  $x_0$  is a **singular point**.

### Theorem 6.1

If  $x_0$  is an **ordinary point** of the 2<sup>nd</sup> order linear DE, then we can find **two linearly independent solutions** in the form of a power series centered at  $x_0$ , i.e.,

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

- For the  $k^{\text{th}}$  order linear DE

$$a_k(x)y^{(k)} + a_{k-1}(x)y^{(k-1)} + \dots + a_1(x)y' + a_0(x)y = 0$$

$$\longrightarrow y^{(k)} + P_{k-1}(x)y^{(k-1)} + \dots + P_1(x)y' + P_0(x)y = 0 \quad P_m(x) = \frac{a_m(x)}{a_n(x)}$$

### Extension of Definition 6.2.1

$x_0$  is an **ordinary point** of the  $k^{\text{th}}$  order linear DE if  $P_0(x)$ ,  $P_1(x)$ ,  $P_2(x)$ ,  $\dots$ ,  $P_{k-1}(x)$ , are **analytic** at  $x_0$

Otherwise,  $x_0$  is a **singular point**.

### Extension of Theorem 6.2.1

If  $x_0$  is an **ordinary point** of the  $k^{\text{th}}$  order linear DE, then we can find  $k$  linearly independent solutions in the form of a power series centered at  $x_0$ , i.e.,

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

## 6-2-5 Interval of Convergence 的判斷方法

判斷方法一：找出  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n} \right| < 1$  的條件

判斷方法二

(較快速，但較不精準

找出的收斂的範圍有時會比實際的收斂範圍小)

$$|x - x_0| < R$$

其中  $R$  是  $x_0$  和最近的 singular point 的距離

Singular point can be a complex number, see Example 6

超過這個範圍未必不為 convergence

### 6-2-6 思考

(1) 對於 nonhomogeneous 的情形.....

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x)$$

(2) 這方法還可以用在什麼情形？

### 6-2-7 本節需注意的地方

- (1) 要了解幾個重要定義：(a) convergence, (b) radius of convergence, (c) analytic at  $x_0$ , (d) singular point, (e) ordinary point
- (2) 複習一下 Taylor (Maclaurin) series (如 page 275)
- (3) Index 的地方計算要小心
  - (a) 先都化成  $x^k$  再合併，(b) 頭幾項可能要獨立出來
  - (c) Index 對齊計算要小心
- (4)  $n^{\text{th}}$  order linear DE 要有  $n$  個 linearly independent 解
- (5) 有時要考慮 interval of convergence

## Section 6-3 Solutions about Singular Points

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = 0$$

假設解為  $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$

### 6-3-1 方法適用情形

(1) Linear

(2)  $(x - x_0)P_{n-1}(x), (x - x_0)^2P_{n-2}(x), \dots, (x - x_0)^{n-1}P_1(x),$

$(x - x_0)^nP_0(x)$  are **analytic** at  $x_0$

(比較： Section 6-2 要求  $P_{n-1}(x), P_{n-2}(x), \dots,$

$P_1(x), P_0(x)$  are **analytic** at  $x_0$ )

(3) It is better that  $P_0(x), P_1(x), \dots, P_{n-1}(x)$  are all polynomials.



## 6-3-2 定義

Singular Points 分成二種

- If  $x_0$  is a singular point but  $(x - x_0)P_{n-1}(x)$ ,  $(x - x_0)^2P_{n-2}(x)$ ,  
.....,  $(x - x_0)^{n-1}P_1(x)$ ,  $(x - x_0)^nP_0(x)$  are analytic at  $x_0$

$x_0$  : regular singular point

- If  $(x - x_0)P_{n-1}(x)$ ,  $(x - x_0)^2P_{n-2}(x)$ , ..... ,  $(x - x_0)^{n-1}P_1(x)$ ,  
 $(x - x_0)^nP_0(x)$  are not analytic at  $x_0$

$x_0$  : irregular singular point

Example 1 (text page 253)

$$(x^2 - 4)^2 y'' + 3(x - 2)y' + 5y = 0$$

$$P(x) = \frac{3}{(x - 2)(x + 2)^2} \quad Q(x) = \frac{5}{(x - 2)^2(x + 2)^2}$$

$x = 2$  is a point

$x = -2$  is a point

### 6-3-3 解法

解法的關鍵：

假設解為 
$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

#### Theorem 6.3.1 Frobenius' Theorem

若  $x_0$  是 linear DE 當中的一個 regular singular point

則這個 linear DE 至少有一個解是  $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$  的型態

## Process

Step 1 將  $y(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$  代入

Step 2 Power 對齊(以指數較小的為準)

Step 3 合併

Step 4 算出  $r$

Step 5 比較係數，將  $c_n$  之間的關係找出來

Step 6 將 Step 4 得出的  $r$  代入 Step 5

得出所有的 independent solutions 及 general solution

Step 7 (見後頁)

**(Step 7)**

當 (1)  $r$  有重根

或 (2)  $r$  的根之間的差為整數，且從 Step 6 得出來的解不為 independent 時

$$\text{用 } y_2(x) = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx \text{ 和長除法找出 } y_2(x)$$

(參考 Section 6-3 的 Examples 4, 5)

當  $r$  的根之間的差為整數，但從 Step 6 得出來的解為 independent 時，不需進行這個步驟

### 6-3-4 範例

Example 2 (text page 255)

$$3xy'' + y' - y = 0$$

Step 1 將  $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$  代入

$$3 \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Step 2 Power 對齊

$$\begin{array}{ccc}
 \downarrow n=k & \downarrow n=k & \downarrow \begin{array}{l} n=k-1 \\ k=n+1 \end{array} \\
 3 \sum_{k=0}^{\infty} c_k (k+r)(k+r-1) x^{k+r-1} + \sum_{k=0}^{\infty} c_k (k+r) x^{k+r-1} - \sum_{k=1}^{\infty} c_{k-1} x^{k+r-1} = 0
 \end{array}$$

$$3 \sum_{k=0}^{\infty} c_k (k+r)(k+r-1)x^{k+r-1} + \sum_{k=0}^{\infty} c_k (k+r)x^{k+r-1} - \sum_{k=1}^{\infty} c_{k-1} x^{k+r-1} = 0$$

Step 3 合并

$$[3c_0 r(r-1) + c_0 r] x^{r-1} + \sum_{k=1}^{\infty} [3c_k (k+r)(k+r-1) + c_k (k+r) - c_{k-1}] x^{k+r-1} = 0$$

Step 4 算出  $r$

$$3r(r-1) + r = 0 \quad 3r^2 - 2r = 0 \quad r(3r-2) = 0$$

$$r = 0 \quad \text{or} \quad 2/3$$

Step 5

$$3c_k (k+r)(k+r-1) + c_k (k+r) - c_{k-1} = 0$$

$$c_k = \frac{1}{(k+r)(3k+3r-2)} c_{k-1}$$

$$c_k = \frac{1}{(k+r)(3k+3r-2)} c_{k-1}$$

Step 6

當  $r = 0$   $c_k = \frac{c_{k-1}}{k(3k-2)}$

$$c_1 = \frac{c_0}{1 \cdot 1}$$

$$c_2 = \frac{c_1}{2 \cdot 4} = \frac{c_0}{2! \cdot 4}$$

$$c_3 = \frac{c_2}{3 \cdot 7} = \frac{c_0}{3! \cdot 4 \cdot 7}$$

$$c_4 = \frac{c_3}{4 \cdot 10} = \frac{c_0}{4! \cdot 4 \cdot 7 \cdot 10}$$

⋮

$$c_n = \frac{c_0}{n! \cdot 4 \cdot 7 \cdots (3n-2)}$$

當  $r = 2/3$   $c_k = \frac{1}{(3k+2)k} c_{k-1}$

$$c_1 = \frac{c_0}{5 \cdot 1}$$

$$c_2 = \frac{c_1}{8 \cdot 2} = \frac{c_0}{2! \cdot 5 \cdot 8}$$

$$c_3 = \frac{c_2}{11 \cdot 3} = \frac{c_0}{3! \cdot 5 \cdot 8 \cdot 11}$$

$$c_4 = \frac{c_3}{14 \cdot 4} = \frac{c_0}{4! \cdot 5 \cdot 8 \cdot 11 \cdot 14}$$

⋮

$$c_n = \frac{c_0}{n! \cdot 5 \cdot 8 \cdot 11 \cdots (3n+2)}$$



Solution of Example 2 (別忘了將最後的解寫出)

$$y = C_1 y_1 + C_2 y_2$$

$$\sum_{n=0}^{\infty} c_n x^{n+r} = x^r \sum_{n=0}^{\infty} c_n x^n$$

$$y_1(x) = x^0 \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{n! \cdot 1 \cdot 4 \cdot 7 \cdots (3n-2)} x^n \right] \quad (\text{Set } c_0 = 1)$$

$$y_2(x) = x^{2/3} \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{n! \cdot 5 \cdot 8 \cdot 11 \cdots (3n+2)} x^n \right]$$

$$x \in (0, \infty)$$

(別忘了寫出  $x$  的範圍)

Examples 4, 5 (text pages 258, 259)

$$xy'' + y = 0$$

Step 1 將  $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$  代入

$$\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Step 2 對齊

$$n = k$$

$$n = k-1$$

$$\sum_{k=0}^{\infty} c_k (k+r)(k+r-1)x^{k+r-1} + \sum_{k=1}^{\infty} c_{k-1} x^{k+r-1} = 0$$

Step 3 合併

$$c_0 r(r-1)x^{r-1} + \sum_{k=1}^{\infty} [c_k (k+r)(k+r-1) + c_{k-1}] x^{k+r-1} = 0$$

$$c_0 r(r-1)x^{r-1} + \sum_{k=1}^{\infty} [c_k (k+r)(k+r-1) + c_{k-1}] x^{k+r-1} = 0$$

**Step 4**  $r(r-1) = 0$

$$r = 0 \quad \text{or} \quad 1$$

**Step 5**  $c_k = -\frac{c_{k-1}}{(k+r)(k+r-1)}$

**Step 6** 當  $r = 1$   $c_k = -\frac{c_{k-1}}{(k+1)k}$

$(-1)^n$ : from -1

$(n+1)!$ : from  $k+1, k = 1 \sim n$

$n!$ : from  $k, k = 1 \sim n$

$$c_n = (-1)^n \frac{c_0}{(n+1)!n!}$$

$$y_1(x) = x \left[ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^n \right] = x \left[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^n \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1}$$

Step 6 當  $r=0$   $c_k = -\frac{c_{k-1}}{(k-1)k}$

$k=1$  時不能算

此時，應該根據 Step 3，由

$$c_k(k+r)(k+r-1) + c_{k-1} = c_1 \cdot 0 + c_0 = 0$$

( $k=1, r=0$  代入)

$c_0$  必需等於 0,  $c_1$  可為任意值

$$c_2 = -\frac{c_1}{1 \cdot 2} \quad c_3 = -\frac{c_2}{2 \cdot 3} = \frac{c_1}{1 \cdot 2 \cdot 2 \cdot 3} \quad \dots\dots\dots$$

$$c_n = \frac{(-1)^{n-1} c_1}{(n-1)! \cdot n!}$$

這地方容易犯錯，  
要小心

$$y_2(x) = x^0 \left[ x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(n-1)! \cdot n!} x^n \right] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)! \cdot n!} x^n = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} x^{m+1}$$

$m = n - 1$

$$y_2(x) = y_1(x)$$

因為前頁算出來的  $y_2(x)$  等於  $y_1(x)$ ，

只好另外用 Sec. 4-2 “reduction of order” 的方法求解

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx$$

註：若算出的  $y_2(x)$  等於  $y_1(x)$  乘上一個 constant, 則  $y_2(x)$  和  $y_1(x)$  仍視同 linearly dependent, 仍需要用上方 “reduction of order” 的方法來求另外一個解

$$y_2(x) = y_1(x) \int \frac{e^{-\int 0 dx}}{y_1^2(x)} dx = y_1(x) \int \frac{dx}{\left[x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots\right]^2}$$

$$= y_1(x) \int \frac{dx}{\left[x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \dots\right]}$$

long division

$$= y_1(x) \int \left[ \frac{1}{x^2} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x + \dots \right] dx$$

長除法

$$= y_1(x) \left[ -\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^2 + \dots \right]$$

思考：為何不是  $\ln|x|$ ？

$$= y_1(x) \ln x + y_1(x) \left[ -\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^2 + \dots \right]$$

$$= y_1(x) \ln x + \left[ -1 - \frac{1}{2}x + \frac{1}{2}x^2 + \dots \right]$$



## 6-3-6 Indicial Equation

2<sup>nd</sup> order case  $y'' + P(x)y' + Q(x)y = 0$

If  $x_0$  is a regular singular point

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0$$

where  $p(x) = (x - x_0)P(x)$      $q(x) = (x - x_0)^2 Q(x)$

由於  $p(x)$  和  $q(x)$  皆為 analytic

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

$$q(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \dots$$

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r},$$

$$y'(x) = \sum_{n=0}^{\infty} c_n (n+r)(x - x_0)^{n+r-1},$$

$$y''(x) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)(x - x_0)^{n+r-2},$$



將  $y(x)$ ,  $y'(x)$ ,  $y''(x)$ ,  $p(x)$ ,  $q(x)$  代入

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0$$

$$\begin{aligned} & \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)(x-x_0)^{n+r} \\ & + \left( a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots \right) \sum_{n=0}^{\infty} c_n (n+r)(x-x_0)^{n+r} \\ & + \left( b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \dots \right) \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r} = 0 \end{aligned}$$

其中  $(x-x_0)^r$  的 coefficient 為

$$c_0 r(r-1) + c_0 a_0 r + c_0 b_0$$

$$\boxed{r(r-1) + a_0 r + b_0 = 0} \longrightarrow \text{indicial equation}$$

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

當 linear DE 為 2<sup>nd</sup> order 時， $r$  可以由  $r(r-1) + a_0r + b_0 = 0$  求出  
其中

$$\begin{array}{lll} \boxed{a_0 = p(x_0)} & p(x) = (x - x_0)P(x) & y'' + P(x)y' + Q(x)y = 0 \\ \boxed{b_0 = q(x_0)} & q(x) = (x - x_0)^2 Q(x) & \end{array}$$

For the 2<sup>nd</sup> order case

$$r(r-1) + a_0r + b_0 = 0 \quad \text{two roots: } r_1, r_2$$

(Case 1)  $r_1 \neq r_2$  and  $r_1, r_2$  are real,  $r_2 - r_1 \neq \text{integer}$

可以找出兩組  $y(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$  的解

(Case 2)  $r_1 \neq r_2$  and  $r_1, r_2$  are real,  $r_2 - r_1 = \text{integer}$

有時可以找出兩組  $y(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$  的解

有時一個解是  $y_1(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1}$

另一個解是  $y_2(x) = Cy_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$

(Case 3)  $r_1 = r_2$  時

$$y_1(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1}$$

$$y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2} \quad C \text{ 一定不為 } 0$$

或寫成  $y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} \tilde{b}_n (x - x_0)^{n+r_2} \quad \tilde{b}_n = b_n / C$

(Case 4)  $r_1 \neq r_2$  and  $r_1, r_2$  are complex

在此不予討論

### 6-3-7 Indicial Equation for Higher Order Case (補充)

當 linear DE 為  $n^{\text{th}}$  order 時

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \dots + P_1(x)y' + P_0(x)y = 0$$

$$y(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r} \quad \text{當中的 } r \text{ 可以由}$$

$$\frac{r!}{(r-n)!} + a_{n-1,0} \frac{r!}{(r-n+1)!} + a_{n-2,0} \frac{r!}{(r-n+2)!} + \dots + a_{1,0} \frac{r!}{(r-1)!} + a_{0,0} = 0$$

求出

$$\text{其中 } a_{k,0} = p_k(x_0), \quad p_k(x_0) = (x-x_0)^{n-k} P_k(x_0)$$

$$k = 0, 1, 2, \dots, n-1$$

### 6-3-8 本節需要注意的地方

(1) Index 「對齊」的計算要小心

(建議可以向 power 較小的對齊，否則會出現負的  $k$ )

例如在 page 300 的 Step 2，一律對齊為  $x^{k+r-1}$  而非  $x^{k+r}$

(2) 若  $x = 0$  為 regular singular point, 設  $x_0 = 0$  即可

(3) 如果是  $c_k$  和  $c_{k-1}$  (或  $c_{k-1}$  和  $c_k$ ) 的 recursive relation

其實有時可以立刻將  $c_n$  的式子觀察出來

(但是分母不可變為 0)

(4) Recursive relation 當中小心分母為 0 的情形 (如 page 306)

(5) 小心算出來的  $y_2(x)$  和  $y_1(x)$  相同的情形 (如 pages 306, 307)

(6) 別忘了將最後的解寫出

最後的解易出錯的地方：別忘了乘上  $x^r$

(7) Interval of solution 依然要考慮，  
且 interval 不包括任何singular point，  
即使是 regular singular point

(8) 複習長除法

## Section 6-4 Special Functions

(本節這學期不在授課範圍內)

Special cases of Sections 6-2 and 6-3

- **Bessel's equation of order  $\nu$**

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \quad \text{Solution: } c_1 J_\nu(x) + c_2 Y_\nu(x)$$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu} \quad : \text{1st kind Bessel function}$$

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} \quad : \text{2nd kind Bessel function}$$

- **Legendre's equation of order  $n$**

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

One of the solution: Legendre polynomials

(See page 336)



## 其他名詞

- Gamma function  $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$

- modified Bessel equation of order  $\nu$

$$x^2 y'' + xy' - (x^2 + \nu^2)y = 0 \quad \text{解： } c_1 I_\nu(x) + c_2 K_\nu(x)$$

- modified Bessel equation of the 1<sup>st</sup> kind  $I_\nu(x) = i^{-\nu} J_\nu(ix)$

- modified Bessel equation of the 2<sup>nd</sup> kind  $K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}$

- Bessel 的另一種變型

$$x^2 y'' + (1 - 2a)xy' + (b^2 c^2 x^{2c} + a^2 - p^2 c^2)y = 0$$

$$\text{解 } y = x^a \left[ c_1 J_p(bx^c) + c_2 Y_p(bx^c) \right]$$

## 6.4.1 Bessel's Equation

### 6.4.1.1 Solving for Bessel's equation of order $\nu$

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

**Steps 1~3** 將  $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$  代入

經過一些計算 (See text pages 262, 263) 得出

$$c_0(r^2 - \nu^2)x^r + c_1((1+r)^2 - \nu^2)x^{r+1} + \sum_{k=2}^{\infty} [c_k((k+r)^2 - \nu^2) + c_{k-2}]x^{r+k} = 0$$

**Step 4**  $r^2 - \nu^2 = 0$       two roots:  $\nu$  and  $-\nu$

**Step 5**  $c_1((1+r)^2 - \nu^2) = 0$        $c_k = \frac{c_{k-2}}{\nu^2 - (k+r)^2}$   
 $c_1 = 0$

$$c_k = \frac{c_{k-2}}{v^2 - (k+r)^2}$$

**Step 6** 當  $r = v$   $c_k = -\frac{c_{k-2}}{k(k+2v)}$       當  $r = -v$   $c_k = -\frac{c_{k-2}}{k(k-2v)}$

由於  $c_1 = 0, c_3 = c_5 = c_7 = c_9 = \dots = 0$

$$\begin{aligned} c_{2n} &= (-1)^n \frac{c_0}{2 \cdot 4 \cdot 6 \cdots 2n \cdot (2+2v)(4+2v)(6+2v) \cdots (2n+2v)} \\ &= \frac{(-1)^n c_0}{2^{2n} n! (1+v)(2+v)(3+v) \cdots (n+v)} \quad \text{when } r = v \end{aligned}$$

$$\begin{aligned} c_{2n} &= (-1)^n \frac{c_0}{2 \cdot 4 \cdot 6 \cdots 2n \cdot (2-2v)(4-2v)(6-2v) \cdots (2n-2v)} \\ &= \frac{(-1)^n c_0}{2^{2n} n! (1-v)(2-v)(3-v) \cdots (n-v)} \quad \text{when } r = -v \end{aligned}$$

### 6.4.1.2 Gamma function: a generalization of $n!$

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

(Note 1)

properties of Gamma function

(1)  $\Gamma(n+1) = n!$  when  $n$  is a positive integer

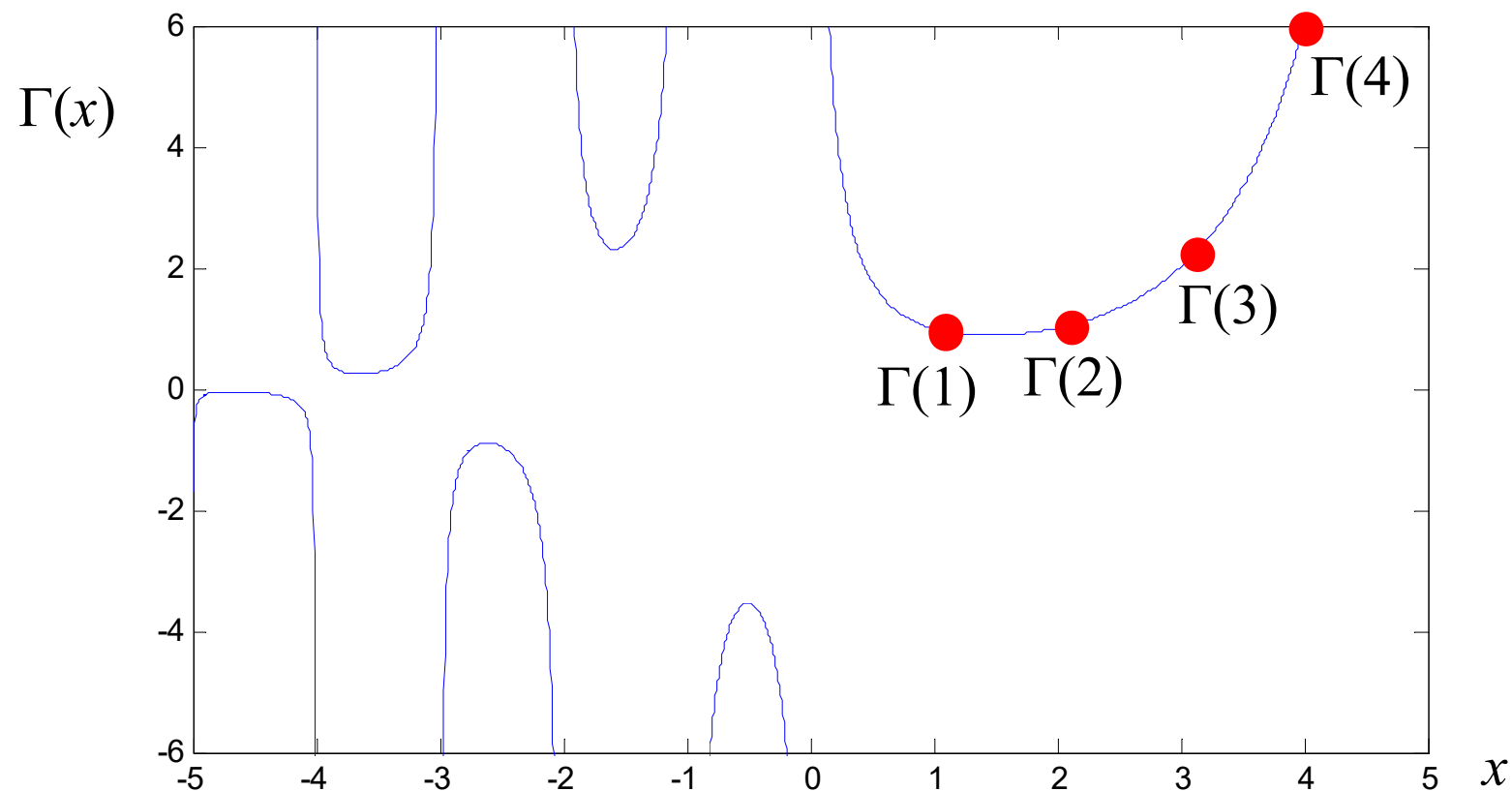
$$\Gamma(1) = 0! = 1$$

(2)  $\Gamma(x+1) = x\Gamma(x)$

參照課本 Appendix 1

(3)  $\Gamma(n) \rightarrow \infty$  when  $n$  is a negative integer or  $n = 0$

(4)  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$      $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$



$$c_{2n} = \frac{(-1)^n c_0}{2^{2n} n!(1+v)(2+v)(3+v)\cdots(n+v)} \quad \text{when } r = v$$

Set  $c_0 = \frac{1}{2^v \Gamma(1+v)}$

$$\begin{aligned} c_{2n} &= \frac{(-1)^n}{2^{2n+v} n!(1+v)(2+v)(3+v)\cdots(n+v)\Gamma(1+v)} \\ &= \frac{(-1)^n}{2^{2n+v} n!(2+v)(3+v)\cdots(n+v)\Gamma(2+v)} && \Gamma(2+v) = (1+v)\Gamma(1+v) \\ &= \frac{(-1)^n}{2^{2n+v} n!(3+v)\cdots(n+v)\Gamma(3+v)} && \Gamma(3+v) = (2+v)\Gamma(2+v) \\ &\vdots \\ &= \frac{(-1)^n}{2^{2n+v} n!\Gamma(n+v+1)} \end{aligned}$$

同理，當  $r = -\nu$  set  $c_0 = \frac{1}{2^{-\nu} \Gamma(1-\nu)}$

$$c_{2n} = \frac{(-1)^n}{2^{2n-\nu} n! \Gamma(n-\nu+1)}$$

Two independent solutions of the Bessel's equation

代入  $\sum_{n=0}^{\infty} c_n x^{n+r}$

When  $r = \nu$   $J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$

When  $r = -\nu$   $J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$

稱作 Bessel functions of the first kind of order  $\nu$  and  $-\nu$

注意，兩個 roots 的差為  $2\nu$

(1) 當  $2\nu$  不為整數時，Bessel's equation 的解即為

$$c_1 J_\nu(x) + c_2 J_{-\nu}(x) \quad (\text{也可表示成 } c_1 J_\nu(x) + c_2 Y_\nu(x))$$

(2) 當  $2\nu$  為整數，但  $\nu = m + 1/2$  ( $m$  是一個整數) 時，Bessel's equation 的解亦為  $c_1 J_\nu(x) + c_2 J_{-\nu}(x)$  (也可表示成  $c_1 J_\nu(x) + c_2 Y_\nu(x)$ )

(3) 當  $2\nu$  為整數，且  $\nu$  是一個整數時，Bessel's equation 的解為

$$c_1 J_\nu(x) + c_2 Y_\nu(x)$$

$Y_\nu(x)$ : Bessel function of the second kind of order  $\nu$

(見後頁)

(Note 2)



$Y_\nu(x)$ : Bessel function of the **second kind** of order  $\nu$

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

當  $m$  為整數時， $Y_m(x)$  定義成

$$Y_m(x) = \lim_{\nu \rightarrow m} \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

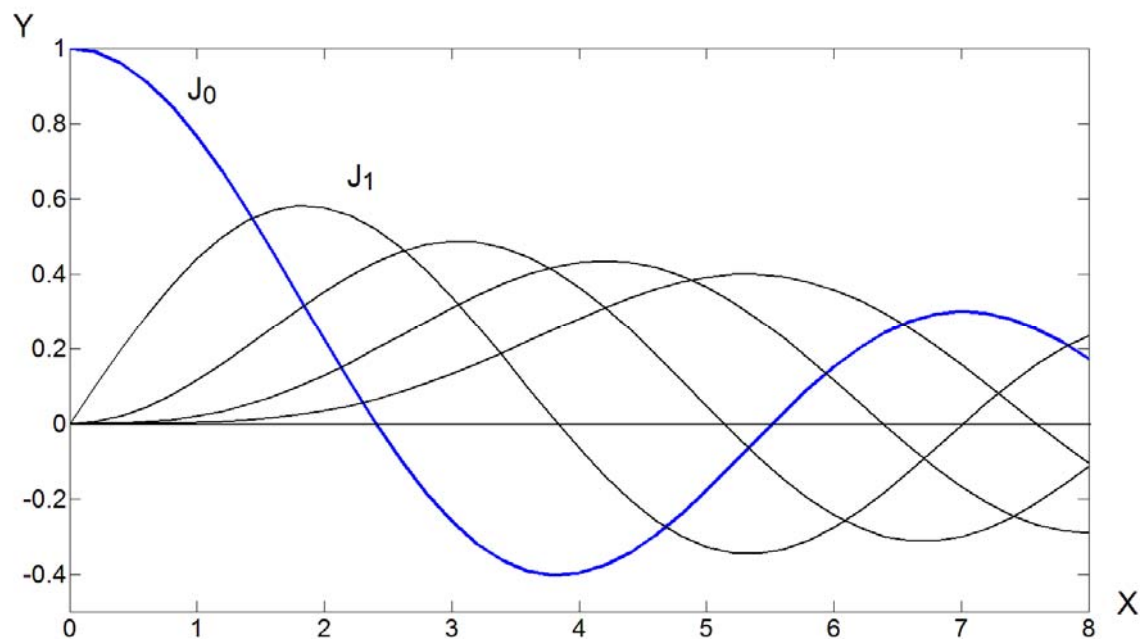
用 L'Hopital's rule 來算

$$Y_m(x) = \lim_{\nu \rightarrow m} \frac{-\pi \sin \nu\pi J_\nu(x) + \cos \nu\pi \frac{\partial}{\partial \nu} J_\nu(x) - \frac{\partial}{\partial \nu} J_{-\nu}(x)}{\pi \cos \nu\pi}$$

#### 6.4.1.4 Bessel function of the 1<sup>st</sup> kind (order $m$ 為整數時)的性質

(1)  $J_0(0) = 1$ ,  $J_m(0) = 0$  for  $m \neq 0$

(2) Zero crossing 的位置，隨著  $m$  增加而越來越遠 (見 Table 6.4.1)



$$(3) \quad J_m(-x) = (-1)^m J_m(x) \quad \text{when } m \text{ is an integer}$$

$$(4) \quad J_{-m}(x) = (-1)^m J_m(x) \quad \text{when } m \text{ is an integer}$$

$$(5) \quad \frac{d}{dx} [x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x)$$

$$(6) \quad \frac{d}{dx} [x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x)$$

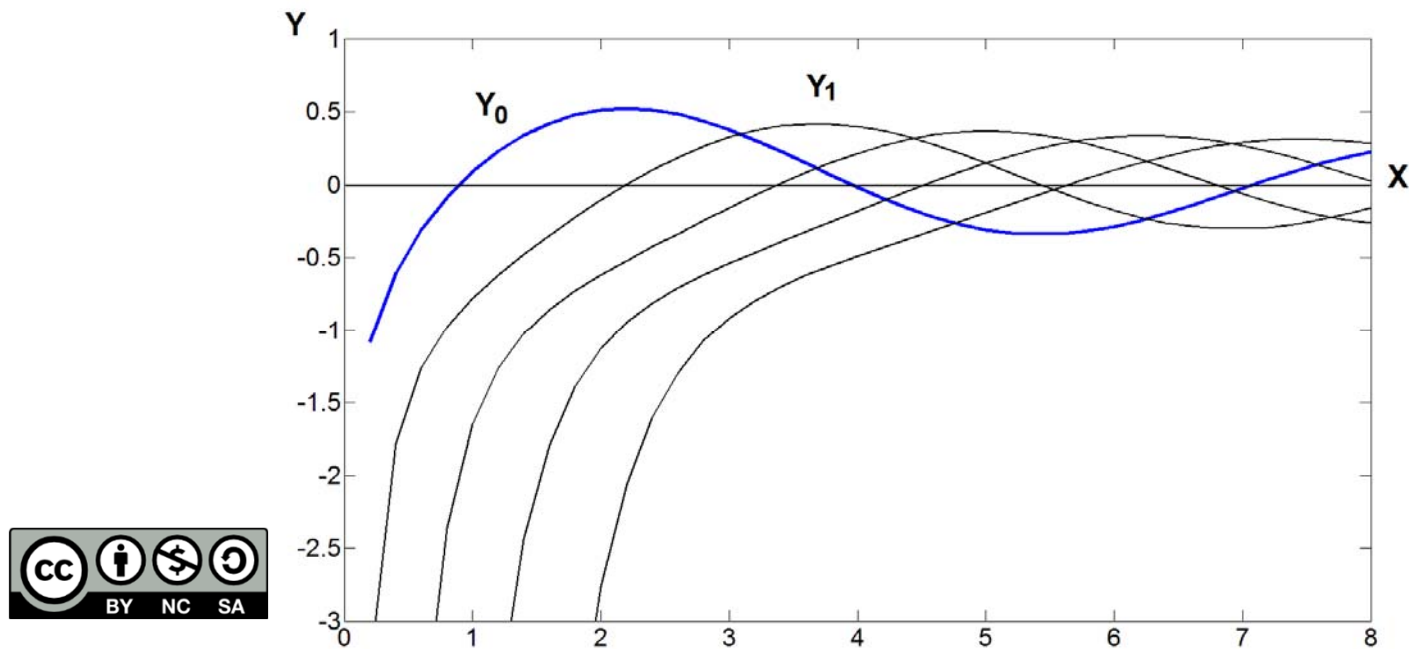
見 Example 6, text pages 268, 269

### 6.4.1.5 Bessel function of the 2<sup>nd</sup> kind (order $m$ 為整數時)的性質

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(1)  $\lim_{x \rightarrow 0} Y_m(x) = -\infty$

(2) Zero crossing 的位置，隨著  $m$  增加而越來越遠



### 6.4.1.6 Bessel's equation 的變型

$$x^2 y'' + xy' + (x^2 - v^2)y = 0 \quad \text{解： } c_1 J_v(x) + c_2 Y_v(x)$$

$$(A) \quad x^2 y'' + xy' + (\alpha^2 x^2 - v^2)y = 0 \quad \text{解： } c_1 J_v(\alpha x) + c_2 Y_v(\alpha x)$$

Proof: Set  $t = \alpha x$

$$\frac{dy}{dx} = \frac{dt}{dx} \frac{dy}{dt} = \alpha \frac{dy}{dt}$$

$$\text{Similarly, } \frac{d^2 y}{dx^2} = \frac{dt}{dx} \frac{d}{dt} \left( \frac{dy}{dx} \right) = \alpha \frac{d}{dt} \left( \alpha \frac{dy}{dt} \right) = \alpha^2 \frac{d^2 y}{dt^2}$$

$$\text{原式} = x^2 y'' + xy' + (\alpha^2 x^2 - v^2)y = \frac{t^2}{\alpha^2} \alpha^2 \frac{d^2 y}{dt^2} + \frac{t}{\alpha} \alpha \frac{dy}{dt} + (\alpha^2 \frac{t^2}{\alpha^2} - v^2)y$$

$$= \boxed{t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - v^2)y = 0} \rightarrow \text{對 } t \text{ 而言是 Bessel equation}$$

$$y = c_1 J_v(t) + c_2 Y_v(t) = c_1 J_v(\alpha x) + c_2 Y_v(\alpha x)$$

(B) modified Bessel equation of order  $\nu$

$$x^2 y'' + xy' - (x^2 + \nu^2)y = 0 \quad \text{解： } c_1 I_\nu(x) + c_2 K_\nu(x)$$

其中  $I_\nu(x) = i^{-\nu} J_\nu(ix)$  稱作是 **modified** Bessel function of the first kind of order  $\nu$

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi} \quad \text{稱作是 } \mathbf{modified} \text{ Bessel function of the second kind of order } \nu$$

當  $\nu$  為整數時，也是取 limit

$$(C) \quad x^2 y'' + (1 - 2a)xy' + (b^2 c^2 x^{2c} + a^2 - p^2 c^2)y = 0$$

$$\text{解：} \quad y = x^a \left[ c_1 J_p(bx^c) + c_2 Y_p(bx^c) \right]$$

式子有點複雜，但可以用來解許多物理上的問題

Example 4 (text page 266)

$$\begin{array}{c} xy'' + 3y' + 9y = 0 \\ \downarrow \\ x^2 y'' + 3xy' + 9xy = 0 \end{array}$$

## 6.4.2 Legendre's Equation

### 6.4.2.1 Legendre's Equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$$y(x) = \sum_{k=0}^{\infty} c_k x^k \quad \text{代入，得出 (過程見text pages 270, 271)}$$

Two linearly independent solutions are

$$y_1(x) = c_0 \left[ 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} x^6 + \dots \right]$$

$$y_2(x) = c_0 \left[ x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} x^7 + \dots \right]$$



(a) When  $n$  is not an integer, both the two solutions have infinite number of terms.

(b) When  $n$  is an even integer,  $y_1(x)$  has finite number of terms.

In  $y_1(x)$ , the coefficient of  $x^k$  is zero when  $k > n$ .

(c) When  $n$  is an odd integer,  $y_2(x)$  has finite number of terms.

In  $y_2(x)$ , the coefficient of  $x^k$  is zero when  $k > n$ .

$y_1(x)$  when  $n$  is an even integer and  $y_2(x)$  when  $n$  is an odd integer are called the Legendre polynomials (denoted by  $P_n(x)$ ).

通常選

$$c_0 = (-1)^{n/2} \frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n} \quad c_1 = (-1)^{(n-1)/2} \frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)}$$

(讓  $P_n(1)$  一律等於 1)

When  $n$  is a non-negative integer, one of the solutions is the  $n^{\text{th}}$  order polynomial.

(Note 3)

由  $y_1(x)$

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

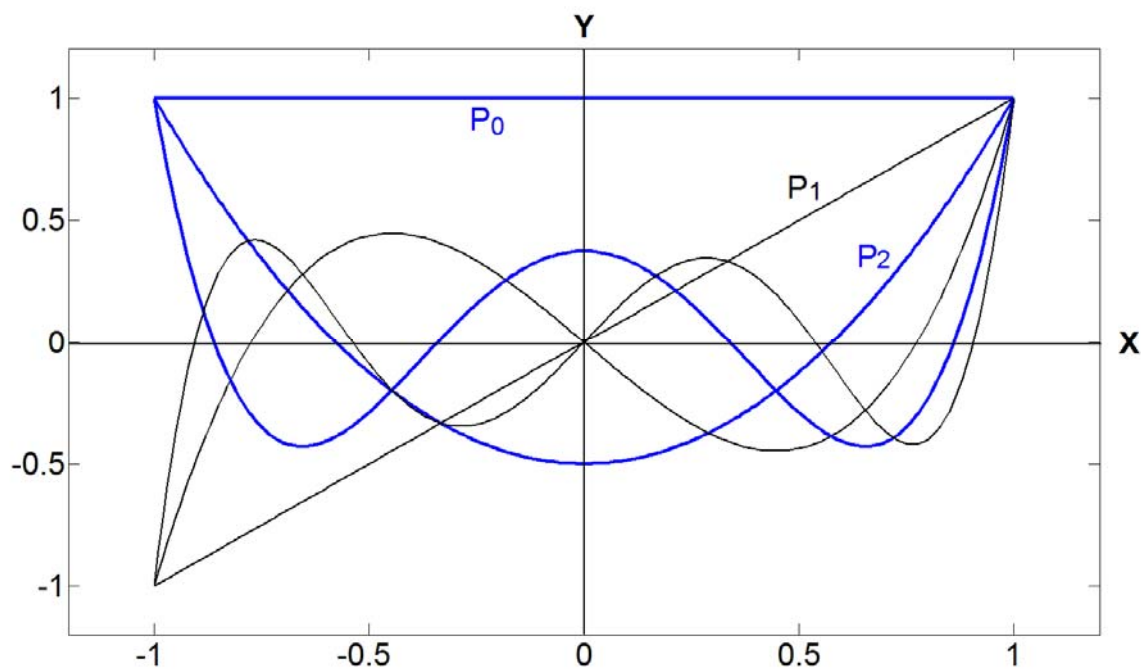
由  $y_2(x)$

$$P_1(x) = x$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

## Legendre polynomials



Interval:

$$x \in [-1, 1]$$



### 6.4.2.2 Properties of Legendre Polynomials

(1)  $P_n(-x) = (-1)^n P_n(x)$     **even / odd symmetry**

(2)  $P_n(1) = 1$        $P_n(-1) = (-1)^n$

(3)  $P_n(0) = 0$     when  $n$  is odd

(4)  $P'_n(0) = 0$     when  $n$  is even

(5)  $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$     **recursive relation**

(6)  $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$     **Rodrigues' formula**

$$(7) \quad \int_{-1}^1 P_m(x) P_n(x) dx = 0 \quad \text{If } m \neq n \quad \text{orthogonality property}$$

(Note 4)

(8) 若任何在  $x \in [-1, 1]$  區間為 continuous 的函式  $f(x)$

皆可表示為

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

$$\text{由於} \quad \int f(x) P_m(x) dx = \sum_{n=0}^{\infty} a_n \int P_n(x) P_m(x) dx = a_n \int P_m(x) P_m(x) dx$$

根據 orthogonality property

$$\text{所以} \quad a_n = \frac{\int f(x) P_m(x) dx}{\int P_m(x) P_m(x) dx}$$

Orthogonality property 才是 Legendre polynomials 最重要的性質

## Review of Chapter 6

解法適用範圍：Linear DE，且 coefficients 最好為 polynomials

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \dots + P_1(x)y' + P_0(x)y = 0$$

- 當  $P_m(x)$  在  $x = x_0$  時為 analytic

$x_0$  為 ordinary point  $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$  代入求解

- 當  $P_m(x)$  在  $x = x_0$  時不為 analytic

但是  $(x - x_0)^{n-m} P_m(x)$  在  $x = x_0$  時為 analytic

$x_0$  為 regular singular point  $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$  代入求解

有時，另一個解為  $Cy_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}$

## Exercise for practice

Sec. 6-1: 4, 9, 12, 20, 22, 24, 30, 32, 34

Sec. 6-2: 2, 10, 13, 18, 20, 22, 24, 26, 27

Sec. 6-3: 4, 9, 13, 16, 22, 24, 26, 28, 29, 31, 33, 36

Review 6 6, 7, 10, 14, 19, 20