

# Chapter 11 Orthogonal Functions and Fourier Series

複習：linear algebra 關於 orthogonal (正交) basis 的介紹

在 linear algebra 當中

(1) inner product  $(\mathbf{f}_1, \mathbf{f}_2) = \sum_n \mathbf{f}_1[n] \mathbf{f}_2[n]$

(2) orthogonal  $\sum_n \mathbf{f}_1[n] \mathbf{f}_2[n] = 0$

(3) 若  $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_N$  為 complete orthogonal set,

$$\mathbf{f}[n] = \sum_{m=1}^N a_m \mathbf{f}_m[n] \quad \text{where} \quad a_m = \frac{\sum_{n=1}^N \mathbf{f}[n] \mathbf{f}_m[n]}{\sum_{n=1}^N \mathbf{f}_m[n] \mathbf{f}_m[n]}$$

例如 在只有三個 entry 的情形下

$$\mathbf{f}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$f_{1[1]} \quad f_{1[2]} \quad f_{1[3]}$

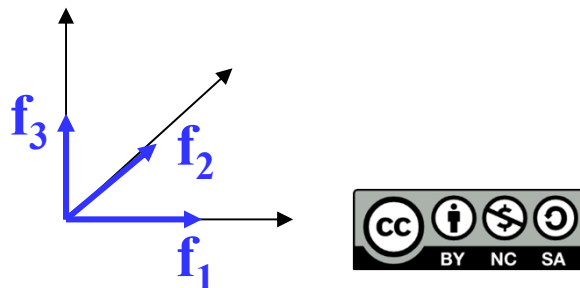
$$\mathbf{f}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$f_{2[1]} \quad f_{2[2]} \quad f_{2[3]}$

$$\mathbf{f}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$f_{3[1]} \quad f_{3[2]} \quad f_{3[3]}$

是一組 complete orthogonal set



問題：在 continuous 當中該如何定義 orthogonal?

## Section 11.1 Orthogonal Functions

### 11.1.1 綱要：熟悉幾個重要定義

- |                            |            |  |            |
|----------------------------|------------|--|------------|
| (1) <u>inner product</u>   | (page 347) | (7) <u>normalize</u>                   | (page 354) |
| (2) <u>orthogonal</u>      | (page 349) | (8) <u>complete</u>                    | (page 356) |
| (3) <u>orthogonal set</u>  | (page 351) | (9) <u>orthogonal series expansion</u> | (page 358) |
| (4) <u>square norm</u>     | (page 353) | (10) <u>generalized Fourier series</u> | (page 358) |
| (5) <u>norm</u>            | (page 353) | (11) <u>weight function</u>            | (page 360) |
| (6) <u>orthonormal set</u> | (page 353) |  |            |

↑  
With weighting functions, many definitions are changed.

- 學習方式：(1) 可以多和 linear algebra 當中的定義多比較
- (2) 複習三角函式的公式 (see page 363-364)

## 11.1.2 定義

(1) inner product on an interval  $[a, b]$

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx \quad (f_1, f_2 \text{ 為 real 時})$$

比較： discrete case  $(\mathbf{f}_1, \mathbf{f}_2) = \sum_n \mathbf{f}_1[n] \mathbf{f}_2[n]$

補充：more standard definition for inner product

$$(f_1, f_2) = \int_a^b f_1(x) f_2^*(x) dx$$

with conjugation

## Inner product 性質

$$(a) (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{f}_2, \mathbf{f}_1)^* \quad *: \text{conjugation}$$

$$(b) (k \mathbf{f}_1, \mathbf{f}_2) = k (\mathbf{f}_1, \mathbf{f}_2), k \text{ 為 scalar (或稱為constant)}$$

$$(c) (\mathbf{f}, \mathbf{f}) = 0 \text{ if and only if } \mathbf{f} = 0, \quad (\mathbf{f}, \mathbf{f}) > 0 \text{ if and only if } \mathbf{f} \neq 0,$$

$$(d) (\mathbf{f}_1 + \mathbf{f}_2, \mathbf{g}) = (\mathbf{f}_1, \mathbf{g}) + (\mathbf{f}_2, \mathbf{g})$$

discrete case 亦有這些性質

(2) orthogonal on an interval  $[a, b]$

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0 \quad (f_1, f_2 \text{ 為 real 時})$$

$$\text{或 } (f_1, f_2) = \int_a^b f_1(x) f_2^*(x) dx = 0 \quad (\text{more standard definition})$$

$$\text{比較： discrete case } \sum_n \mathbf{f}_1[n] \mathbf{f}_2[n] = 0$$

例子：當  $[a, b] = [-1, 1]$ ,

1 和  $x^k$  ( $k$  為奇數) 互為 orthogonal

$$\int_{-1}^1 1 \cdot x^k dx = \left. \frac{x^{k+1}}{k+1} \right|_{-1}^1 = \frac{1^{k+1} - (-1)^{k+1}}{k+1} = \frac{1-1}{k+1} = 0$$

注意：任何 even function 和任何 odd function 在  $[-b, b]$  之間必為 orthogonal,  $(a = -b)$

包括 Example 1 (text page 426) 的  $x^2$  和  $x^3$  在  $[-1, 1]$  之間也是 orthogonal

$$\int_{-b}^b f_1(x) f_2(x) dx = 0$$

↑            ↑  
even        odd

### (3) orthogonal set

有一組 functions  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$

$$\text{若 } \int_a^b \phi_m(x) \phi_n^*(x) dx = 0 \quad \text{for any } m \neq n$$

則  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  被稱作 orthogonal set on an interval  $[a, b]$



### Example 2 (text page 426)

Show that the set  $\{1, \cos x, \cos 2x, \cos 3x, \dots\}$  is an orthogonal set on the interval  $[-\pi, \pi]$

when one of the functions is 1

$$\int_{-\pi}^{\pi} 1 \cdot \cos nx dx = \frac{1}{n} \sin nx \Big|_{-\pi}^{\pi} = 0$$

when both the two functions are not 1

運用三角函式公式  
(pages 363-364)

$$\begin{aligned} & (m \neq n) \\ \int_{-\pi}^{\pi} \cos mx \cos nx dx &= \int_{-\pi}^{\pi} \frac{1}{2} (\cos(m+n)x + \cos(m-n)x) dx \\ &= \frac{\sin(m+n)x}{2(m+n)} \Big|_{-\pi}^{\pi} + \frac{\sin(m-n)x}{2(m-n)} \Big|_{-\pi}^{\pi} \\ &= \frac{\sin((m+n)\pi)}{2(m+n)} - \frac{\sin(-(m+n)\pi)}{2(m+n)} + \frac{\sin((m-n)\pi)}{2(m-n)} - \frac{\sin(-(m-n)\pi)}{2(m-n)} = 0 \end{aligned}$$

(4) square norm

$$\|f(x)\|^2 = (f(x), f(x)) = \int_a^b f(x) f^*(x) dx = \int_a^b |f(x)|^2 dx$$

比較： discrete case  $\sum_n \mathbf{f}[n] \mathbf{f}^*[n]$

(5) norm

$$\|f(x)\| = \sqrt{(f(x), f(x))} = \sqrt{\int_a^b f(x) f^*(x) dx}$$

(6) orthonormal set

對一個 orthogonal set, 若更進一步的滿足

$$\int_a^b \phi_n(x) \phi_n^*(x) dx = 1 \quad \text{for all } n$$

則被稱為 orthonormal set

## (7) normalize

將 norm 變為 1

$$\psi(x) \xrightarrow{\text{normalize}} v(x) = \frac{\psi(x)}{\|\psi(x)\|}$$

注意，此時

$$(v(x), v(x)) = \left( \frac{\psi(x)}{\|\psi(x)\|}, \frac{\psi(x)}{\|\psi(x)\|} \right) = \frac{1}{\|\psi(x)\|^2} (\psi(x), \psi(x)) = 1$$

可藉由 normalization, 將 orthogonal set 變成 orthonormal set

### Example 3 (text page 427)

Calculate the norms of  $\{1, \cos x, \cos 2x, \cos 3x, \dots\}$

$$\int_{-\pi}^{\pi} 1 \cdot 1 dx = x \Big|_{-\pi}^{\pi} = 2\pi$$

$$\int_{-\pi}^{\pi} \cos nx \cos nxdx = \int_{-\pi}^{\pi} \frac{1}{2} (\cos 2nx + 1) dx$$

運用三角函式公式  
(pages 363-364)

$$= \frac{\sin 2nx}{4n} + \frac{x}{2} \Big|_{-\pi}^{\pi} = \frac{\sin 2n\pi}{4n} + \frac{\pi}{2} - \frac{\sin(-2n\pi)}{4n} - \frac{(-\pi)}{2} = \pi$$

$$\|1\| = \sqrt{2\pi} \qquad \|\cos nx\| = \sqrt{\pi}$$

$\{1, \cos x, \cos 2x, \cos 3x, \dots\}$  can be normalized as an orthonormal set

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}, \dots \right\}$$

### (8) complete

若在 interval  $[a, b]$  之間，任何一個 function  $f(x)$  都可以表示成  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  的 linear combination

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \dots = \sum_{n=0}^{\infty} c_n\phi_n(x)$$

則  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  被稱作 complete

比較：在 linear algebra 當中，對 3-D vector 而言

$\mathbf{e}_1 = [1, 0, 0]$ ,  $\mathbf{e}_2 = [0, 1, 0]$ ,  $\mathbf{e}_3 = [0, 0, 1]$  為 complete

Any 3-D vector  $[a, b, c]$  can be expressed as  $a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$

## Complete:

For  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$

if we can find  $c_0, c_1, c_2, c_3, \dots$  such that

$$f(x) - \sum_{n=0}^{\infty} c_n \phi_n(x) = 0 \quad \text{for all } x$$

then  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  is a complete set.

**Example:**  $\{1, x-x_0, (x-x_0)^2, (x-x_0)^3, \dots\}$  is a **complete** set if  $f(x)$  is constrained to be a continuous function.

However,  $\{1, x-x_0, (x-x_0)^2, (x-x_0)^3, \dots\}$  are **not orthogonal**.

$\{1, \cos x, \cos 2x, \cos 3x, \dots\}$  are **orthogonal** but **not complete**.

(9)(10)

若  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  為 complete,  
可將  $f(x)$  表示成

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

Question: How to find  $c_n$ ?

當  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  不為 orthogonal,  $c_n$  不易算

當  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  為 orthogonal

$$\int_a^b f(x) \phi_n^*(x) dx = \sum_{m=0}^{\infty} c_m \int_a^b \phi_m(x) \phi_n^*(x) dx = c_n \int_a^b \phi_n(x) \phi_n^*(x) dx$$

$$c_n = \frac{\int_a^b f(x) \phi_n^*(x) dx}{\int_a^b \phi_n(x) \phi_n^*(x) dx}$$

被稱作 **(9) orthogonal series expansion**

$c_n$  被稱作 **(10) generalized Fourier series**

當  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  為 orthonormal

$$c_n = \int_a^b f(x) \phi_n^*(x) dx$$

Examples of Complete and Orthogonal Sets in the Continuous Case:



### 11.1.3 Orthogonal with Weight Function

(11-1) inner product with weight function

$$(f_1(x), f_2(x)) = \int_a^b w(x) f_1(x) f_2^*(x) dx$$

其中  $w(x)$  被稱作 weight function

加上了 weight function 後

(11-2) orthogonal 的定義改成

$$(f_m, f_n) = \int_a^b w(x) f_m(x) f_n^*(x) dx = 0 \quad \text{for } m \neq n$$

(11-3) square norm 的定義改成

$$\|f(x)\|^2 = \int_a^b w(x) f(x) f^*(x) dx$$

(11-4) norm 的定義改成

$$\|f(x)\| = \sqrt{\int_a^b w(x) f(x) f^*(x) dx}$$

(11-5) orthonormal 的定義改成

$$\int_a^b w(x) f_m(x) f_n^*(x) dx = 0 \quad \text{for } m \neq n$$

$$\int_a^b w(x) f_n(x) f_n^*(x) dx = 1$$

(11-6) normalize 的算法改成

$$v(x) = \frac{\psi(x)}{\|\psi(x)\|} = \frac{\psi(x)}{\sqrt{\int_a^b w(x) \psi(x) \psi^*(x) dx}}$$

(11-7) orthogonal series expansion of  $f(x)$  以及 generalize Fourier series 的算法改成

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

$$c_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x) dx}{\int_a^b w(x) \phi_n(x) \phi_n^*(x) dx}$$

## 11.1.4 三角函數表

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(要複習)

$\cos(a + b)$	$\cos a \cos b - \sin a \sin b$
$\sin(a + b)$	$\sin a \cos b + \cos a \sin b$
$\cos(a - b)$	$\cos a \cos b + \sin a \sin b$
$\sin(a - b)$	$\sin a \cos b - \cos a \sin b$
$\cos a \cos b$	$[\cos(a + b) + \cos(a - b)]/2$
$\sin a \sin b$	$[\cos(a - b) - \cos(a + b)]/2$
$\sin a \cos b$	$[\sin(a + b) + \sin(a - b)]/2$

$\cos(2a)$	$\cos^2 a - \sin^2 a$ or $1 - 2\sin^2 a$ or $2\cos^2 a - 1$
$\sin(2a)$	$2\sin a \cos a$
$\cos^2 a$	$[\cos(2a) + 1]/2$
$\sin^2 a$	$[1 - \cos(2a)]/2$

### 11.1.5 Section 11.1 需要注意的地方

(1) Norm 和 square of norm 要分清楚

做 normalization 時，要除以 norm

(2) 熟悉三角函數的公式

(i) 記住幾個，其他的就不難推算出來

(ii) 許多公式可以由  $\cos(a) = \frac{e^{ja} + e^{-ja}}{2}$  導出來

$$\sin(a) = \frac{e^{ja} - e^{-ja}}{2j} = \frac{je^{-ja} - je^{ja}}{2}$$

## Two well-known complete and orthogonal function set

(1) Fourier series, Fourier cosine series, Fourier sine series  
(Sections 11-2 and 11-3)

(2) Legendre polynomials (Section 6-4)

$$\int_{-1}^1 P_m(x) P_n(x) dx = 0 \quad \text{if } m \neq n$$

其他常用的 complete and orthogonal function set

Hermite polynomials (with weight function) (補充)

Chebyshev polynomials (with weight function) (補充)

## Section 11.2 Fourier Series

### 11.2.1 綱要

#### trigonometric functions

$$\left\{ 1, \cos \frac{\pi}{p} x, \cos \frac{2\pi}{p} x, \cos \frac{3\pi}{p} x, \dots, \sin \frac{\pi}{p} x, \sin \frac{2\pi}{p} x, \sin \frac{3\pi}{p} x, \dots \right\}$$

orthogonal set on the interval of  $[-p, p]$



proven on pages 372~374

$$\cos \frac{n\pi}{p} x \quad \text{週期} : \frac{2p}{n} \quad \text{頻率} : \frac{n}{2p}$$



## (2) Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx \quad a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx$$

$$b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx$$

(紅色部分特別注意，勿記錯公式)

**(3) 名詞**

<u>trigonometric function</u>	(page 372)
<u>Fourier series (trigonometric series)</u>	(page 376)
<u>Fourier coefficients</u>	(page 376)
<u>partial sum</u>	(page 379)
<u>fundamental period</u>	(page 384)
<u>period extension</u>	(page 384)

物理意義：

Fourier Series == 對信號做頻率分析

「頻率」 (frequency) 是個常用字，以 Hz (每秒多少個週期) 為單位

說話聲音: 100~1200 Hz

人耳可聽見的聲音: 20~20000Hz

廣播 (AM):  $5 \times 10^5 \sim 1.6 \times 10^6$  Hz

廣播 (FM):  $8.8 \times 10^7 \sim 1.08 \times 10^8$  Hz

無線電視:  $7.6 \times 10^7 \sim 8.8 \times 10^7, 1.74 \times 10^8 \sim 2.16 \times 10^8$  Hz

行動通訊:  $5.1 \times 10^8$  Hz  $\sim 2.75 \times 10^{11}$  Hz

可見光:  $4 \times 10^{14}$  Hz  $\sim 8 \times 10^{14}$  Hz

測量頻率的方式: [Fourier series](#)  
[Fourier transform](#)

## 11.2.2 Trigonometric Functions

### trigonometric functions

$$\left\{ 1, \cos \frac{\pi}{p} x, \cos \frac{2\pi}{p} x, \cos \frac{3\pi}{p} x, \dots, \sin \frac{\pi}{p} x, \sin \frac{2\pi}{p} x, \sin \frac{3\pi}{p} x, \dots \right\}$$

Trigonometric functions is orthogonal on the interval of  $[-p, p]$

要用  $C_3^2 + 2 = 5$  次的 inner products 來證明

(1) 1 VS. Cosine

$$\int_{-p}^p 1 \cdot \cos \frac{\pi k}{p} x dx = \frac{p}{\pi k} \sin \frac{\pi k}{p} x \Big|_{-p}^p = \frac{p}{\pi k} \sin \pi k - \frac{p}{\pi k} \sin(-\pi k) = 0 - 0 = 0$$

(2) 1 VS. Sine

$$\int_{-p}^p 1 \cdot \sin \frac{\pi k}{p} x dx = -\frac{p}{\pi k} \cos \frac{\pi k}{p} x \Big|_{-p}^p = -\frac{p}{\pi k} \cos \pi k + \frac{p}{\pi k} \cos(-\pi k) = 0$$

## (3) Cosine VS. Sine

$$\begin{aligned}
\int_{-p}^p \cos \frac{\pi k}{p} x \cdot \sin \frac{\pi h}{p} x dx &= \int_{-p}^p \frac{1}{2} \left[ \sin \frac{\pi(h+k)}{p} x - \sin \frac{\pi(h-k)}{p} x \right] dx \\
&= \frac{p}{2\pi} \left[ -\frac{1}{h+k} \cos\left(\frac{\pi(h+k)x}{p}\right) + \frac{1}{h-k} \cos\left(\frac{\pi(h-k)x}{p}\right) \right] \Big|_{-p}^p \\
&= \frac{p}{2\pi} \left[ -\frac{1}{h+k} [\cos(\pi(h+k)) - \cos(-\pi(h+k))] \right. \\
&\quad \left. + \frac{1}{h-k} [\cos(\pi(h-k)) - \cos(-\pi(h-k))] \right] = 0 \quad (\text{when } h \neq k)
\end{aligned}$$

when ( $h = k$ )

$$\int_{-p}^p \cos \frac{\pi k}{p} x \cdot \sin \frac{\pi h}{p} x dx = \int_{-p}^p \frac{1}{2} \sin \frac{2\pi k}{p} x dx = -\frac{p}{4\pi k} \cos \frac{2\pi k}{p} \Big|_{-p}^p = 0$$

(4) Cosine VS. Cosine,  $k \neq h$

$$\begin{aligned} \int_{-p}^p \cos \frac{\pi k}{p} x \cdot \cos \frac{\pi h}{p} x dx &= \int_{-p}^p \frac{1}{2} \left[ \cos \frac{\pi(h+k)}{p} x + \cos \frac{\pi(h-k)}{p} x \right] dx \\ &= \frac{p}{2\pi} \left[ \frac{1}{h+k} \sin\left(\frac{\pi(h+k)x}{p}\right) + \frac{1}{h-k} \sin\left(\frac{\pi(h-k)x}{p}\right) \right]_{-p}^p \\ &= \frac{p}{2\pi} \left[ \frac{1}{h+k} [\sin(\pi(h+k)) - \sin(-\pi(h+k))] \right. \\ &\quad \left. + \frac{1}{h-k} [\sin(\pi(h-k)) - \sin(-\pi(h-k))] \right] = 0 \quad \text{when } h \neq k \end{aligned}$$

(5) Sine VS. Sine,  $k \neq h$

$$\begin{aligned} \int_{-p}^p \sin \frac{\pi k}{p} x \cdot \sin \frac{\pi h}{p} x dx &= \int_{-p}^p \frac{1}{2} \left[ \cos \frac{\pi(h-k)}{p} x - \cos \frac{\pi(h+k)}{p} x \right] dx \\ &= \frac{p}{2\pi} \left[ \frac{1}{h-k} \sin\left(\frac{\pi(h-k)x}{p}\right) - \frac{1}{h+k} \sin\left(\frac{\pi(h+k)x}{p}\right) \right]_{-p}^p = 0 \quad \text{when } h \neq k \end{aligned}$$

## Square norms of trigonometric functions

$$\|1\|^2 = \int_{-p}^p 1 \cdot 1 dx = x \Big|_{-p}^p = 2p$$

$$\left\| \cos \frac{\pi k}{p} x \right\|^2 = \int_{-p}^p \cos^2 \frac{\pi k}{p} x dx = \frac{1}{2} \int_{-p}^p (1 + \cos \frac{2\pi k}{p} x) dx = \frac{1}{2} \left( x + \frac{p}{2\pi k} \sin \frac{2\pi k}{p} x \right) \Big|_{-p}^p = p$$

$$\left\| \sin \frac{\pi k}{p} x \right\|^2 = \int_{-p}^p \sin^2 \frac{\pi k}{p} x dx = \frac{1}{2} \int_{-p}^p (1 - \cos \frac{2\pi k}{p} x) dx = \frac{1}{2} \left( x - \frac{p}{2\pi k} \sin \frac{2\pi k}{p} x \right) \Big|_{-p}^p = p$$



### 11.2.3 Fourier Series

The Fourier series is the orthogonal series expansion (see page 358) by trigonometric functions

(Fourier series 又被稱作 trigonometric series)

The Fourier Series of a function  $f(x)$  defined on the interval  $[-p, p]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx$$

$a_0, a_n, b_n$  被稱作 Fourier coefficients

$$a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx$$

$$b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx$$

Example 1 (text page 433)

$$f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ \pi - x & \text{for } 0 \leq x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \cos nx dx$$

$$= \frac{\pi - x}{\pi} \frac{1}{n} \sin nx \Big|_0^{\pi} - \frac{1}{\pi} \int_0^{\pi} (-1) \frac{1}{n} \sin nx dx$$

$$= -\frac{1}{n^2 \pi} \cos nx \Big|_0^{\pi} = \frac{1 - \cos n\pi}{n^2 \pi} = \frac{1 - (-1)^n}{n^2 \pi}$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{\pi} (\pi - x) \sin nx dx \\
 &= -\frac{\pi - x}{\pi} \frac{1}{n} \cos nx \Big|_0^{\pi} - \frac{1}{\pi} \int_0^{\pi} (-1) \left(-\frac{1}{n} \cos nx\right) dx \\
 &= \frac{1}{n} - \frac{1}{n^2 \pi} \sin nx \Big|_0^{\pi} = \frac{1}{n}
 \end{aligned}$$

$$\begin{aligned}
 f(x) &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left( \frac{1 - (-1)^n}{n^2 \pi} \cos \frac{n\pi}{p} x + \frac{1}{n} \sin \frac{n\pi}{p} x \right) \\
 &= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left( \frac{1 - (-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right)
 \end{aligned}$$

$$p = \pi$$

## 11.2.4 Sequence of Partial Sums

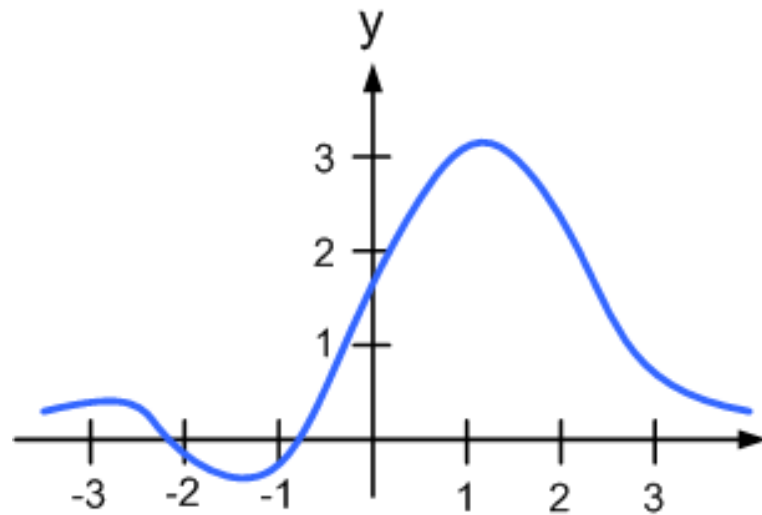
### Sequence of Partial Sums

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^N \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

$$f_1(x) = \lim_{N \rightarrow \infty} S_N(x)$$

$N$  越大，越能逼近原來的 function

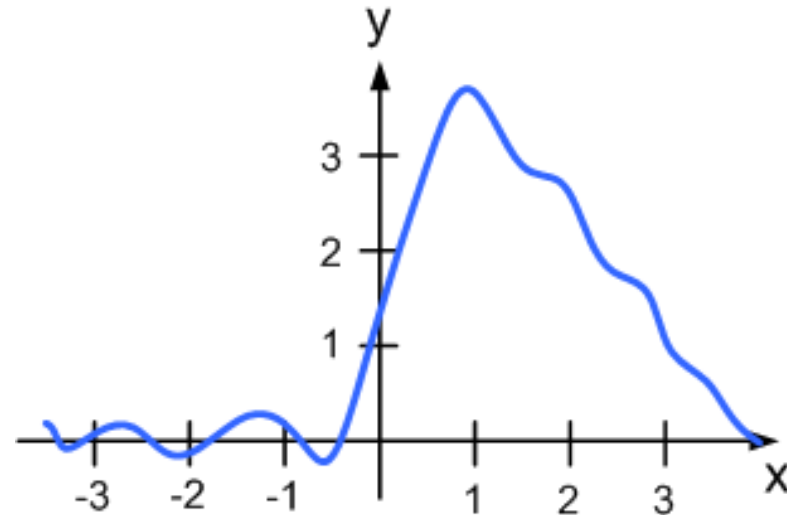
For Example 1



(a)  $S_3(x)$



$N = 3$

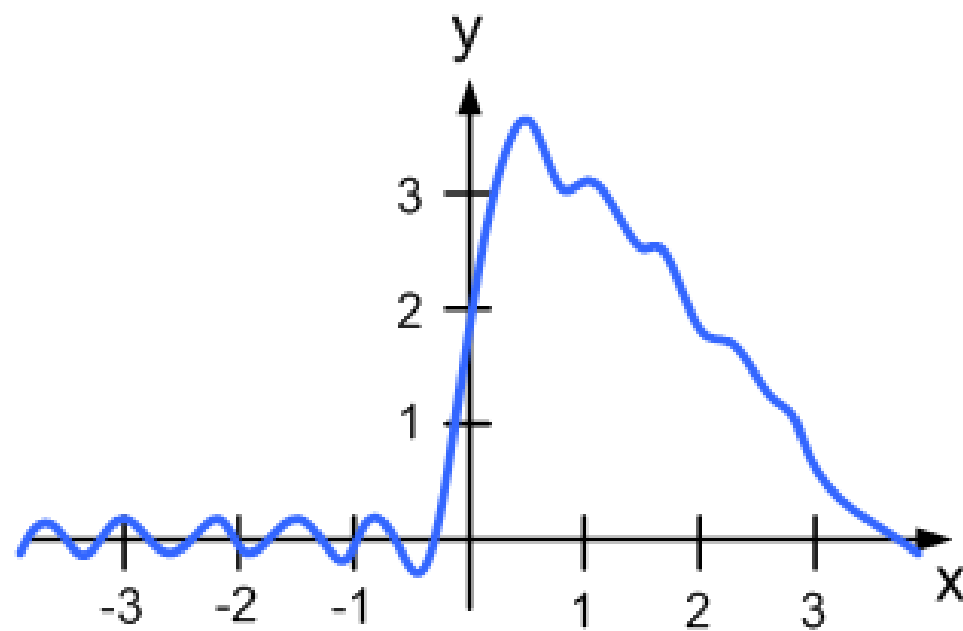


(b)  $S_8(x)$



$N = 8$

Fig. 11.2.3



(c)  $S_{15}(x)$



$N = 15$

Fig. 11.2.3

## 11.2.5 Conditions for Convergence

$$\text{If } a_0 = \frac{1}{p} \int_{-p}^p f(x) dx \quad a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx$$

$$b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx$$

$$f_1(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

$$f_1(x) = f(x) \quad \text{其實未必成立}$$

(1)  $f_1(x_0) = f(x_0)$  if  $f(x)$  is continuous at  $x_0$

$$(2) \quad f_1(x_0) = \frac{f(x_0+) + f(x_0-)}{2} \quad \text{if } f(x) \text{ is not continuous at } x_0$$

$$f(x_0+) = \lim_{h \rightarrow 0} f(x_0 + h) \quad f(x_0-) = \lim_{h \rightarrow 0} f(x_0 - h)$$

Example 1 的例子

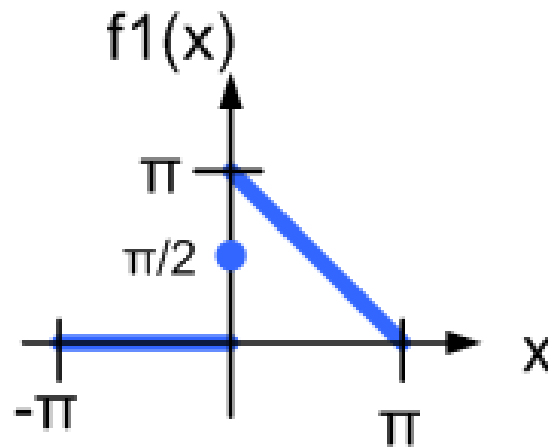
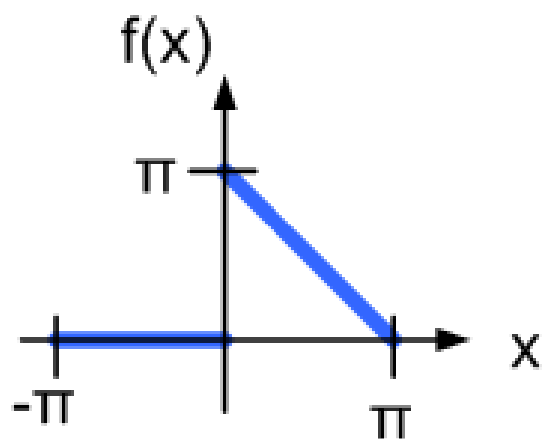


Fig. 11-2-1



## 11.2.6 Period Extension

$$f_1(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

fundamental period:  $2p$

在 interval  $x \in [-p, p]$  以外的地方

$$f_1(x + 2p) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi}{p} x + 2n\pi\right) + b_n \sin\left(\frac{n\pi}{p} x + 2n\pi\right) \right)$$

$$f_1(x + 2p) = f_1(x) \quad (\text{period Extension})$$

- ◆  $f_1(x)$  是個週期為  $2p$  的函式 (這是  $f_1(x)$  和  $f(x)$  第二個不同的地方)

Example 1 的例子

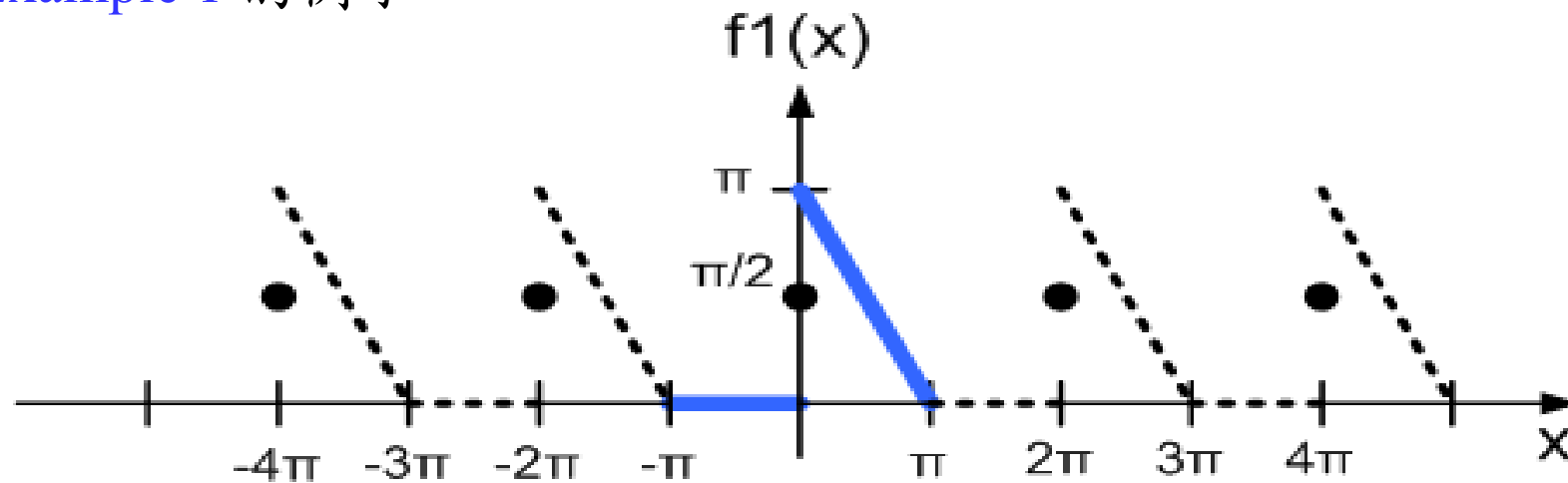


Fig. 11.2.2



對一個非週期的函式，Fourier series expansion 的結果不適用於  $x \notin [-p, p]$  的區域

但是週期函式則可

### 11.2.7 Section 11.2 需要注意的地方

(1) Fourier series 的公式 (常背錯)

(a) 第一項是  $a_0/2$ ，而非  $a_0$

(b) 算  $a_0, a_n, b_n$  時，積分後別忘了除以  $p$

( $p$  是 interval width 的一半)

(2) 背熟三角函式公式

(3) 熟悉  $\int_a^b u(t)v'(t)dt = u(t)v(t)\Big|_a^b - \int_a^b u'(t)v(t)dt$

(在計算 Fourier coefficients 會常用到，如 Example 1)

(4) 當  $n$  為整數時,  $\cos n\pi = (-1)^n$  習慣這種表示法

(5) 正確而言,  $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$

近似於

因為當  $f_1(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$

$f_1(x)$  和  $f(x)$  之間有二個不同的地方

(a) 在 discontinuous 的地方  $f_1(x_0) = [f(x_0+) + f(x_0-)]/2$

(b)  $f_1(x)$  為 periodic,  $f_1(x) = f_1(x+p)$

然而, 習慣上, 還是寫成  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$

# 數學史上最美麗的詩篇 --- 傅立葉級數

Clerk Maxwell

悲傷的傅立葉

## Section 11.3 Fourier Cosine and Sine Series

### 11.3.1 綱要

(1)

 $f(x)$  is evenFourier cosine series (或 cosine series)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$$

Fourier Series

$$a_0 = \frac{2}{p} \int_0^p f(x) dx \quad a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx$$

比較 page 368

 $f(x)$  is oddFourier sine series (或 sine series)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

- (2) 重要名詞：Fourier cosine series, cosine series (page 395)  
Fourier sine series, sine series (page 396)  
Gibb's Phenomenon (page 399)

(3) Half-range extension:  $[0, L]$

- (a) cosine series:  $f(x) = f(-x)$ , interval is changed into  $[-L, L]$ , set  $p = L$   
(b) sine series:  $f(x) = -f(-x)$ , interval is changed into  $[-L, L]$ , set  $p = L$   
(c) Fourier series: (i) interval  $[-p, p]$  is replaced by  $[0, L]$ ,  
(ii)  $p$  is replaced by  $L/2$

(4) One of the applications: **Solving particular solution**

(See pages 407-412)

## 11.3.2 Even and Odd Functions

even function:  $f(x) = f(-x)$

odd function:  $f(x) = -f(-x)$

Example

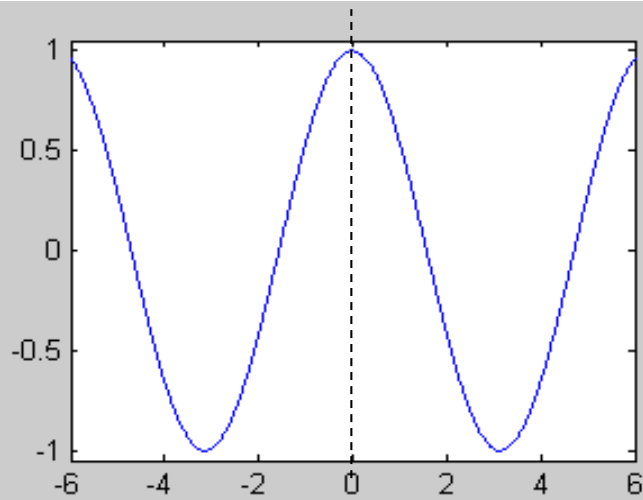
$1, x^2, x^4, x^6, x^8 \dots$  are even

$x, x^3, x^5, x^7, x^9 \dots$  are odd



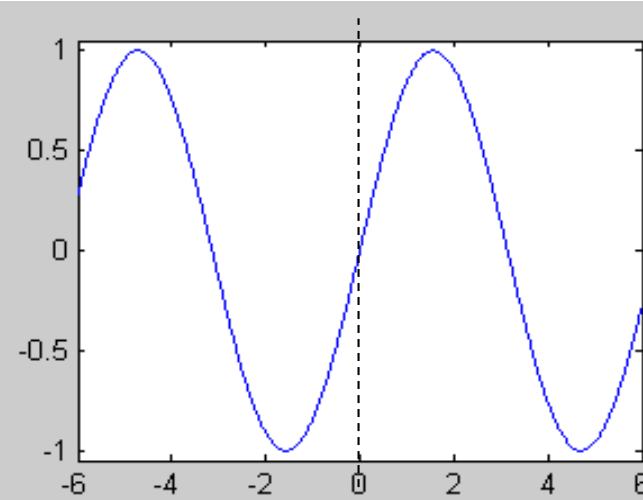
Cosine functions are even

$$\cos(t)$$



Sine functions are odd

$$\sin(t)$$



Several properties about even and odd functions

(a) The **product** of **two even functions** is **even**

$$\text{例} : x^2 \cdot x^4 = x^6$$

(b) The **product** of **two odd functions** is **even**

$$\text{例} : x \cdot x = x^2$$

(c) The **product** of **an even function** and **an odd function** is **odd**

$$\text{例} : x \cdot x^2 = x^3$$

(d) The **sum** (or **difference**) of **two even function** is still **even**

(e) The **sum** (or **difference**) of **two odd function** is still **odd**

(f) If  $f(x)$  is even, then  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

(g) If  $f(x)$  is odd, then  $\int_{-a}^a f(x) dx = 0$

(Proof):

$$\begin{aligned} \int_{-a}^a f(x) dx &= \int_{-a}^0 f(x) dx + \int_0^a f(x) dx \\ &= -\int_a^0 f(-x_1) dx_1 + \int_0^a f(x) dx \quad (\text{令 } x_1 = -x, dx_1 = -dx) \\ &= \int_0^a f(-x_1) dx_1 + \int_0^a f(x) dx \end{aligned}$$

When  $f(x) = f(-x)$

$$\int_{-a}^a f(x) dx = \int_0^a f(x_1) dx_1 + \int_0^a f(x) dx = 2 \int_0^a f(x) dx$$

When  $f(x) = -f(-x)$

$$\int_{-a}^a f(x) dx = \int_0^a -f(x_1) dx_1 + \int_0^a f(x) dx = 0$$

### 11.3.3 Fourier Cosine and Sine Series

(1) The Fourier series of an **even function** on the interval  $(-p, p)$  is the **cosine series** (或稱作 **Fourier cosine series**)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$$

$$a_0 = \frac{2}{p} \int_0^p f(x) dx \qquad a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx$$

和之前 Fourier series 不一樣的地方有三個

- 適用情形：
- (1)  $f(x)$  is even
  - (2) Half range extension (page 401)

(2) The Fourier series of an **odd function** on the interval  $(-p, p)$  is the **sine series** (或稱作 **Fourier sine series**)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

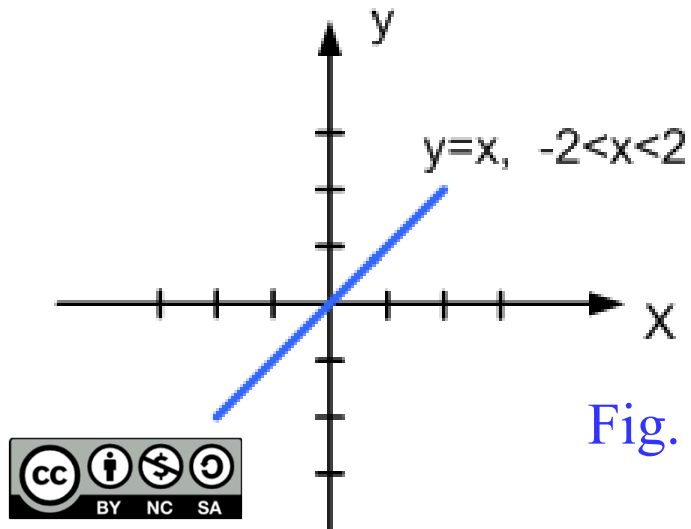
和之前 Fourier series 不一樣的地方有三個  
(是哪三個)

適用情形：(1)  $f(x)$  is odd

(2) Half range extension (page 401)

### Example 1 (text page 438)

Expand  $f(x) = x$ ,  $-2 < x < 2$  in a Fourier series



$f(x)$  is odd

$\therefore$  expand  $f(x)$  by a Fourier sine series

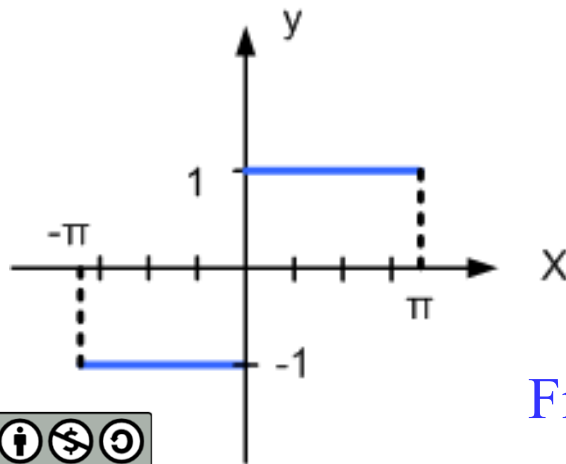
Fig. 11.3.3

$$\begin{aligned}
 b_n &= \frac{2}{2} \int_0^2 x \sin \frac{n\pi}{2} x dx = -\frac{2}{n\pi} x \cos \frac{n\pi}{2} x \Big|_0^2 + \frac{2}{n\pi} \int_0^2 \cos \frac{n\pi}{2} x dx \\
 &= -\frac{2}{n\pi} 2 \cos n\pi + 0 + \frac{4}{n^2 \pi^2} \sin \frac{n\pi}{2} x \Big|_0^2 = -\frac{4}{n\pi} (-1)^n + 0 - 0 = \frac{4}{n\pi} (-1)^{n+1}
 \end{aligned}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin \frac{n\pi}{2} x$$

Example 2 (text page 438)

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 \leq x < \pi \end{cases}$$



odd function, 使用 sine series

Fig. 11.3.5



$$b_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin \frac{n\pi}{\pi} x dx = -\frac{2}{\pi} \frac{\cos nx}{n} \Big|_0^{\pi} = \frac{2}{\pi} \frac{1 - (-1)^n}{n}$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx$$

### 11.3.4 Gibbs Phenomenon

Example 2 的結果  $f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin nx$

partial sum  $S_N(x) = \frac{2}{\pi} \sum_{n=1}^N \frac{1 - (-1)^n}{n} \sin nx$

當  $N$  不為無限大，在 discontinuities 附近會有 “overshooting”

“overshooting” 的大小不會隨著  $N$  而變小

但寬度會越來越窄，越來越靠近 discontinuities 的地方

這種現象，稱作 Gibb's phenomenon



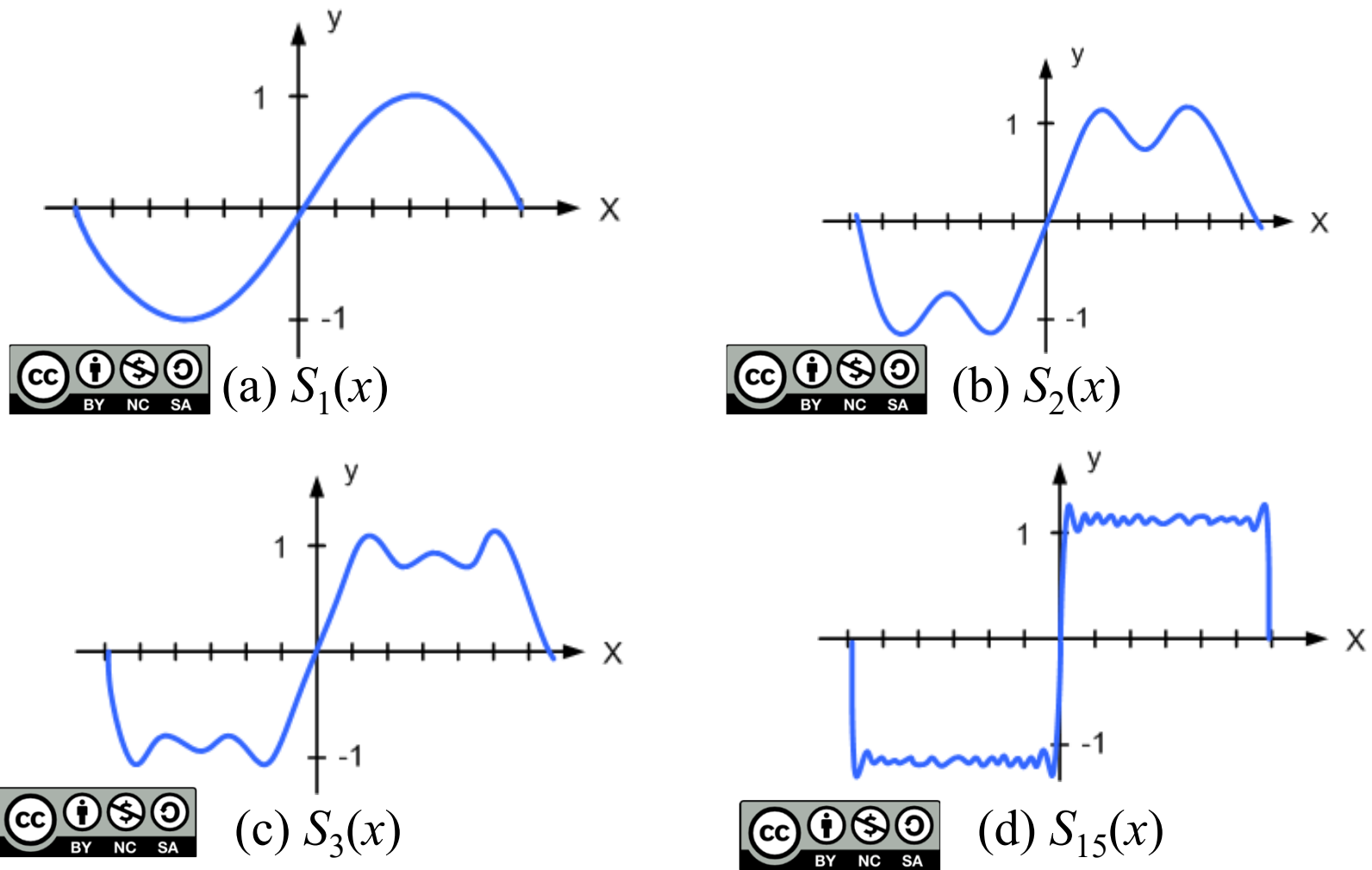


Fig. 11.3.6

### 11.3.5 Half Range Extension

之前的例子： $f(x)$  is defined in the interval of  $-p < x < p$

若問題改成

Expand  $f(x)$ ,  $0 < x < L$

$f(x)$  只有在  $0 < x < L$  當中有定義  
對範圍之外的  $f(x)$  用猜測的方式)

(a) In a cosine series, suppose that  $f(x) = f(-x)$  and  $f(x) = f(x+2L)$

(i) Interval:  $[-L, L]$ , (ii) 所有公式的  $p$  由  $L$  取代, (iii) 結果是 even

(b) in a sine series, suppose that  $f(x) = -f(-x)$  and  $f(x) = f(x+2L)$

(i) Interval:  $[-L, L]$ , (ii) 所有公式的  $p$  由  $L$  取代, (iii) 結果是 odd

(c) in a Fourier series, suppose that  $f(x) = f(x+L)$

(i) Interval:  $[0, L]$ , (ii) 所有公式的  $p$  由  $L/2$  取代  
(和前面兩個不同)

如 Example 3 (text page 440),

$$f(x) = x^2, \quad 0 < x < L$$

(a) in a cosine series

假設  $f(x) = f(-x)$  for  $-L < x < 0$ , (假設  $f(x)$  是一個 **even function**)  
interval 變為  $(-L, L)$

原本 cosine series 公式

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$$

$$a_0 = \frac{2}{p} \int_0^p f(x) dx$$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx$$

現在只不過將  $p$  改成  $L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

$$a_0 = \frac{2}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x dx$$

(b) in a sine series

假設  $f(x) = -f(-x)$  for  $-L < x < 0$ , (假設  $f(x)$  是一個 **odd function**)  
interval 變為  $(-L, L)$

原本 sine series 公式

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

現在只不過將  $p$  改成  $L$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx$$

(c) in a Fourier series interval 仍為  $(0, L)$

原本 Fourier series 公式

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right) \quad a_0 = \frac{1}{p} \int_{-p}^p f(x) dx$$

$$a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx \quad b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx$$

現在 (1) 將 interval  $[-p, p]$  換為  $[0, L]$ , (2) 將  $p$  換為  $L/2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2n\pi}{L} x + b_n \sin \frac{2n\pi}{L} x \right) \quad a_0 = \frac{2}{L} \int_0^L f(x) dx$$

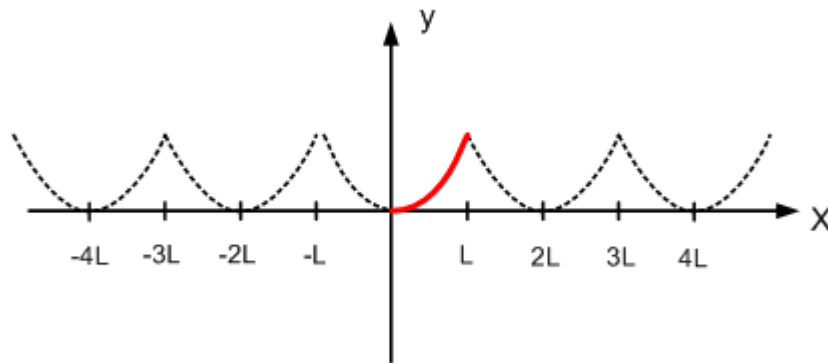
$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi}{L} x dx \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{2n\pi}{L} x dx$$

Note: Since  $f(x) \cos \frac{2n\pi}{L} x = f(x+L) \cos \frac{2n\pi(x+L)}{L}$

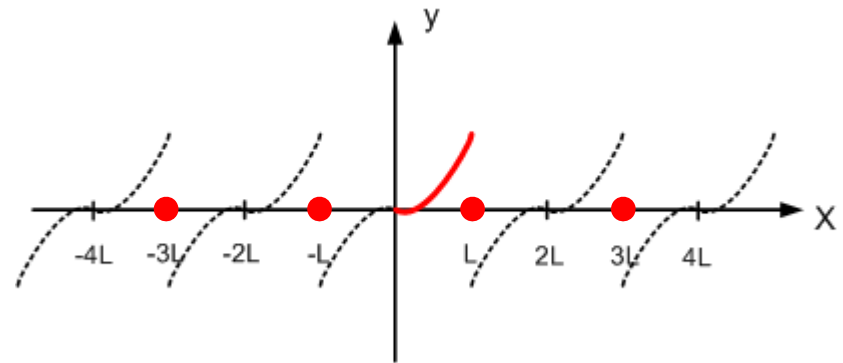
$$a_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos \frac{2n\pi}{L} x dx = \frac{2}{L} \int_0^L f(x) \cos \frac{2n\pi}{L} x dx$$

Example 3,  $f(x) = x^2$ ,  $0 < x < L$

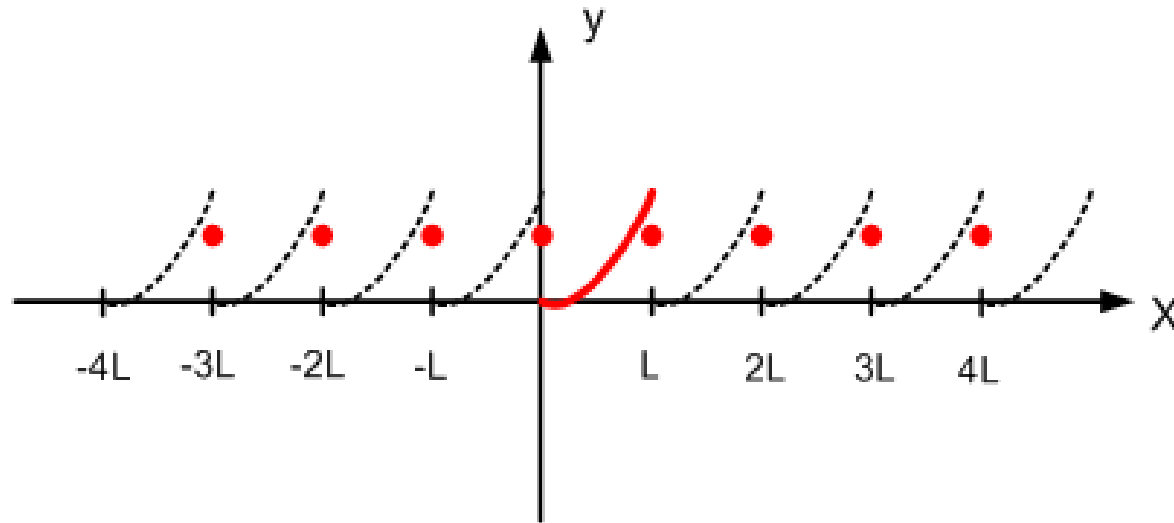
將三個方法的結果畫成圖形



cosine series



sine series



**Fourier series**

### 11.3.6 Solving Particular Solutions (第四個方法)

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = f(t)$$

$$f(t) = f(t + 2p)$$

方法的限制

(註：以下的步驟不包含解 homogeneous solution  
homogeneous solution 還是需要用 Section 4-3 的方法來解)

(Step 1) 將  $f(t)$  表示成 Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} t + b_n \sin \frac{n\pi}{p} t \right)$$

或 cosine series (當  $f(t)$  為 even)

或 sine series (當  $f(t)$  為 odd)



(Step 2) 假設 particular solution 的型態為

$$y_p(t) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi}{p} t + B_n \sin \frac{n\pi}{p} t \right)$$

(Step 3) 代回原式，比較係數，將  $A_0, A_n, B_n$  解出來

若所假設的 particular solution 和 homogeneous solution 有相同的地方，則要乘上  $t$

Example 4 (text page 441)

$$\frac{1}{16} \frac{d^2 x}{dt^2} + 4x = f(t) \quad \begin{aligned} f(t) &= \pi t & \text{for } -1 < t < 1 \\ f(t) &= f(t-2) \end{aligned}$$

(相關的物理定理見 Section 5-1)

(Step 1) 假設  $f(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$  (因為  $f(t)$  是 odd)

$$\begin{aligned} b_n &= 2 \int_0^1 \pi t \sin(n\pi t) dt \\ &= -2 \frac{t}{n} \cos(n\pi t) \Big|_0^1 + \int_0^1 \frac{2}{n} \cos(n\pi t) dt \\ &= -2 \frac{1}{n} (-1)^n - 0 + \frac{2}{n^2 \pi} \sin(n\pi t) \Big|_0^1 = \frac{2}{n} (-1)^{n+1} \end{aligned}$$

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin n\pi t$$

(Step 2) 假設 particular solution 為

$$x_p(t) = \sum_{n=1}^{\infty} (A_n \cos n\pi t + B_n \sin n\pi t)$$

$(p = 1)$

思考：為什麼這裡可以沒有常數項  $A_0$ ?

(Step 3) 將  $x_p(t)$  和 Step 1 的結果代入  $\frac{1}{16} \frac{d^2 x}{dt^2} + 4x = f(t)$

$$\sum_{n=1}^{\infty} \left( -\frac{1}{16} A_n n^2 \pi^2 \cos n\pi t - \frac{1}{16} B_n n^2 \pi^2 \sin n\pi t \right) + \sum_{n=1}^{\infty} (4A_n \cos n\pi t + 4B_n \sin n\pi t) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin n\pi t$$

$$-\frac{1}{16} A_n n^2 \pi^2 + 4A_n = 0 \quad \Longrightarrow \quad A_n = 0$$

$$-\frac{1}{16} B_n n^2 \pi^2 + 4B_n = \frac{2}{n} (-1)^{n+1} \quad \Longrightarrow \quad B_n = \frac{32(-1)^{n+1}}{n(64 - n^2 \pi^2)}$$

Therefore, the particular solution is:

$$x_p(t) = \sum_{n=1}^{\infty} \frac{32(-1)^{n-1}}{n(64 - n^2 \pi^2)} \sin n\pi t$$

General solution:

$$x(t) = c_1 \cos(8t) + c_2 \sin(8t) + \sum_{n=1}^{\infty} \frac{32(-1)^{n-1}}{n(64 - n^2\pi^2)} \sin n\pi t$$

注意：由於  $\frac{1}{16} \frac{d^2x}{dt^2} + 4x = f(t)$  當中並沒有一次，三次，五次....微分項，所以 particular solution 不可能會有 cosine terms

所以，在 Step 2 當中，可以直接假設

$$x_p(t) = \sum_{n=1}^{\infty} B_n \sin n\pi t$$

### 11.3.7 Section 11.3 需要注意的地方

(1) 公式一些地方易記錯

for cosine series and sine series,

$$a_0 = \frac{2}{p} \int_0^p f(x) dx \quad a_n = \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx$$
$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

(2) Fourier series 的 half-range extension 和 cosine series 及 sine series 不同

$p$  is replaced by  $L/2$ ,  $[-p, p]$  is replaced by  $[0, L]$

(3) Half range extension 和 solving particular solution 這兩個部分較複雜，需要特別注意，並且多練習例題

## Exercise for Practice

Section 11-1 3, 5, 6, 8, 14, 17, 19, 20, 21, 22, 23

Section 11-2 2, 5, 9, 10, 12, 16, 19, 22, 23, 24

Section 11-3 14, 16, 18, 21, 22, 23, 28, 29, 33, 36, 37, 43, 46, 47a, 48a,  
49, 52

Review 11 6, 12, 13, 14, 15, 17, 18