

Chapter 12 Boundary-Value Problem in Rectangular Coordinates

- Role of Chapter 12:

Discuss the partial differential equation (PDE) for the case of two independent variables.

(x - y 座標) (Three or more independent variables 的問題在 Sec. 12-8 有討論，圓座標的問題在 Chapter 13 當中有討論，但皆不在這學期的上課範圍之中)

Solving the PDE by the methods of

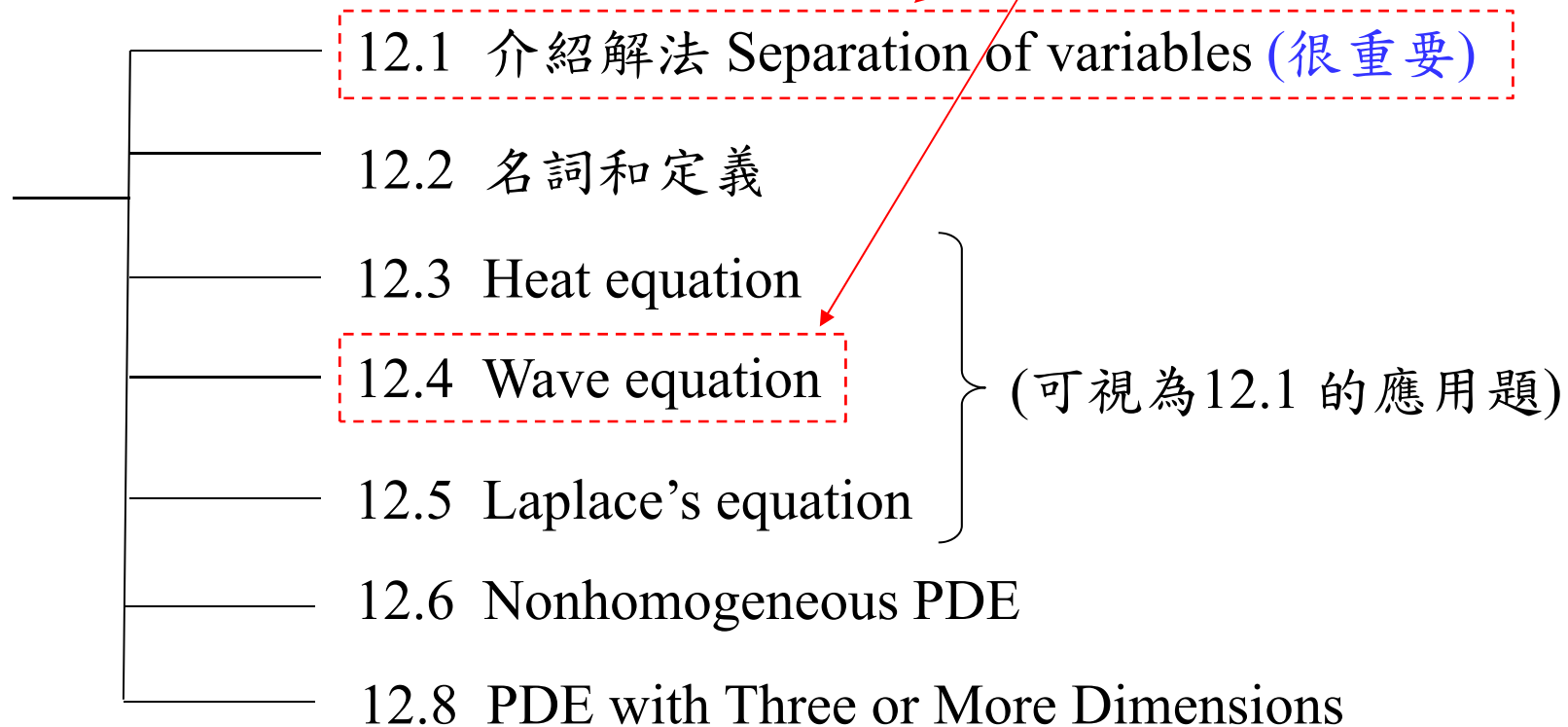
(1) separation of variables Chapter 12

(2) the Laplace / Fourier transform Chapter 14

(不在這學期的上課範圍)

本章的架構

只有這兩節是這學期授課範圍



兩大重點：

(1) 熟悉 separation of variables 解 PDE 的方法

(2) 名詞和定義

縮寫: boundary value problem (BVP)

initial value problem (IVP)

$$\text{例: } a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$

$$\text{BVP: } u(0,t) = 0 \quad u(L,t) = 0$$

$$\text{IVP: } u(x,0) = f(x) \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x)$$

partial differential equation (PDE)

ordinary differential equation (ODE)

Section 12.1 Separable Partial Differential Equations

12.1.1 Section 12.1 綱要

(1) linear second order partial differential equation for two independent variables

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G \quad 7 \text{ terms}$$

$B^2 - 4AC > 0$: hyperbolic, $B^2 - 4AC = 0$: parabolic

$B^2 - 4AC < 0$: elliptic

(2) Partial differential equation (PDE) 主要解法之一：

Separation of variables (see pages 422-424).

名詞：real separation constant (page 422)

12.1.2 Linear Second Order Partial Differential Equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$$

independent variables: x, y dependent variables: $u(x, y)$, 簡寫成 u

homogeneous : $G(x, y) = 0$, nonhomogeneous : $G(x, y) \neq 0$

particular solution, general solution 的定義一如往昔

【Theorem 12.1.1】 Superposition Principle

If u_1, u_2, \dots, u_k are solutions of a **homogeneous** linear partial differential equation, then

$$u = c_1u_1 + c_2u_2 + \cdots + c_ku_k$$

is also a solution of the homogeneous linear partial differential equation.

12.1.3 Method of Separation of Variables

解 PDE with BVP (or IVP) 的方法

(1) method of separation of variables

若 PDE 當中有對 x 及對 y 的偏微分，

假設解為 $u(x, y) = X(x)Y(y)$

(2) using the Laplace transform (or Fourier, Fourier cosine transform, Fourier sine transform) (see Chapter 14，期末考範圍外)

共通的精神：PDE \longrightarrow ODE

Method of Separation of Variables 的流程

(Step 1) 假設解為 $u(x, y) = X(x)Y(y)$

解法關鍵



(Step 2) 將 $u(x, y) = X(x)Y(y)$ 代入 PDE，把 PDE 變成
“function of X ” = “function of Y ” = $-\lambda$
的型態，並得出 $X(x)$ 的 ODE 和 $Y(y)$ 的 ODE

λ 被稱為 real separation constant

Steps 3, 4, 5 要分成不同的 Cases 來解 ODEs

423

除了trivial 的情形外，所有可能的 cases 都要考慮

(Pre-Step) 考慮等於 0 的 initial / boundary conditions

(Step 3) 將 function of $X = -\lambda$ 的解算出，即為 $X(x)$

註：有時，先解 $Y(y)$ 會比較容易

(視 boundary (initial) conditions 而定)

(Step 4) 將 function of $Y = -\lambda$ 的解算出，即為 $Y(y)$

需注意的地方和 Step 3 相同

(Step 5) $u(x, y) = X(x)Y(y)$

(Step 6) 將所有可能的解全部加起來

(Step 7) 用 **非零的** boundary (initial) conditions 將 coefficients 求出

註：這一步經常會用到 Fourier series, Fourier cosine series
或 Fourier sine series

※ 若沒有 boundary (initial) conditions，Steps 6, 7 可以省略

Rules:

x 的 BVP (IVP) 簡單 \longrightarrow 先算 $X(x)$

y 的 BVP (IVP) 簡單 \longrightarrow 先算 $Y(y)$

沒有 BVP (IVP) \longrightarrow 先算 $X(x)$ 或 $Y(y)$ 皆可

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0$$

$$u(0, y) = 0 \quad u(L, y) = 0$$

$$u(x, 0) = f(x) \quad \left. \frac{\partial u}{\partial y} \right|_{y=0} = g(x)$$

此時先算 $X(x)$

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2}$$

$$u(0, y) = f(y) \quad u(L, y) = 0$$

$$\left. \frac{\partial}{\partial y} u(x, y) \right|_{y=0} = 0 \quad \left. \frac{\partial}{\partial y} u(x, y) \right|_{y=H} = 0$$

此時先算 $Y(y)$

Example 2 (text page 462)

$$\frac{\partial^2 u}{\partial x^2} = 4 \frac{\partial u}{\partial y}$$

(Step 1) 假設解為 $u(x, y) = X(x)Y(y)$ (解法關鍵)

(Step 2) 將 $u(x, y) = X(x)Y(y)$ 代入 $\frac{\partial^2 u}{\partial x^2} = 4 \frac{\partial u}{\partial y}$

$$X''(x)Y(y) = 4X(x)Y'(y)$$

$$\frac{X''(x)}{4X(x)} = \frac{Y'(y)}{Y(y)}$$

$$\text{令 } \frac{X''(x)}{4X(x)} = \frac{Y'(y)}{Y(y)} = -\lambda$$

real separation constant

(解法關鍵)

$$X''(x) + 4\lambda X(x) = 0 \quad Y'(y) + \lambda Y(y) = 0$$

$$X''(x) + 4\lambda X(x) = 0 \quad Y'(y) + \lambda Y(y) = 0$$

Case 1 for Steps 3, 4, 5 $\lambda = 0$

(Step 3-1) $X''(x) = 0$

auxiliary function $m^2 = 0$ roots : 0, 0

$$X(x) = c_1 + c_2x$$

(Step 4-1) $Y'(y) = 0$ $Y(y) = c_3$

(Step 5-1) $u(x, y) = X(x)Y(y) = (c_1 + c_2x)c_3 = A_1 + B_1x$

$$A_1 = c_1c_3 \quad B_1 = c_2c_3$$

Case 2 for Steps 3, 4, 5 $\lambda < 0$

為了方便起見，令 $\lambda = -\alpha^2$

(Step 3-2) $X''(x) - 4\alpha^2 X(x) = 0$ roots of the auxiliary function: $2\alpha, -2\alpha$

$$X(x) = d_1 e^{2\alpha x} + d_2 e^{-2\alpha x}$$

通常將解改寫成 $X(x) = c_4 \cosh(2\alpha x) + c_5 \sinh(2\alpha x)$

(Step 4-2) $\frac{Y'(y)}{Y(y)} = \alpha^2$ $Y'(y) - \alpha^2 Y(y) = 0$

$$Y'(y) - \alpha^2 Y(y) = 0 \quad Y(y) = c_6 e^{\alpha^2 y}$$

(Step 5-2) $u(x, y) = X(x)Y(y) = A_2 e^{\alpha^2 y} \cosh(2\alpha x) + B_2 e^{\alpha^2 y} \sinh(2\alpha x)$

$$A_2 = c_4 c_6$$

$$B_2 = c_5 c_6$$

Case 3 for Step 3 $\lambda > 0$

為了方便起見，令 $\lambda = \alpha^2$

(Step 3-3) $X''(x) + 4\alpha^2 X(x) = 0$ roots of the auxiliary function: $j2\alpha, -j2\alpha$

$$X(x) = c_7 \cos(2\alpha x) + c_8 \sin(2\alpha x)$$

(Step 4-3) $\frac{Y'(y)}{Y(y)} = -\alpha^2 \quad Y'(y) + \alpha^2 Y(y) = 0 \quad Y(y) = c_9 e^{-\alpha^2 y}$

(Step 5-3) $u(x, y) = A_3 e^{-\alpha^2 y} \cos(2\alpha x) + B_3 e^{-\alpha^2 y} \sin(2\alpha x)$

若要處理 boundary conditions，或著想得到 general solution，
要將所有可能的解都加起來

(Step 6)

$$u(x, y) = A_1 + B_1 x + \sum_{\alpha > 0} [A_{2,\alpha} e^{\alpha^2 y} \cosh(2\alpha x) + B_{2,\alpha} e^{\alpha^2 y} \sinh(2\alpha x)] + \sum_{\alpha > 0} [A_{3,\alpha} e^{-\alpha^2 y} \cos(2\alpha x) + B_{3,\alpha} e^{-\alpha^2 y} \sin(2\alpha x)] \quad \alpha \text{ 是任意實數}$$

(註：nonseparable 的解在這一步得到)

Exercise Problem 5

$$x \frac{\partial u}{\partial x} = y \frac{\partial u}{\partial y}$$

Exercise Problem 9

$$k \frac{\partial^2 u}{\partial x^2} - u = \frac{\partial u}{\partial t}$$

$$k > 0$$

12.1.4 Classification

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0$$

$B^2 - 4AC > 0$ \longrightarrow The PDE is said to be **hyperbolic** (雙曲線)

$B^2 - 4AC = 0$ \longrightarrow The PDE is said to be **parabolic** (拋物線)

$B^2 - 4AC < 0$ \longrightarrow The PDE is said to be **elliptic** (橢圓形)

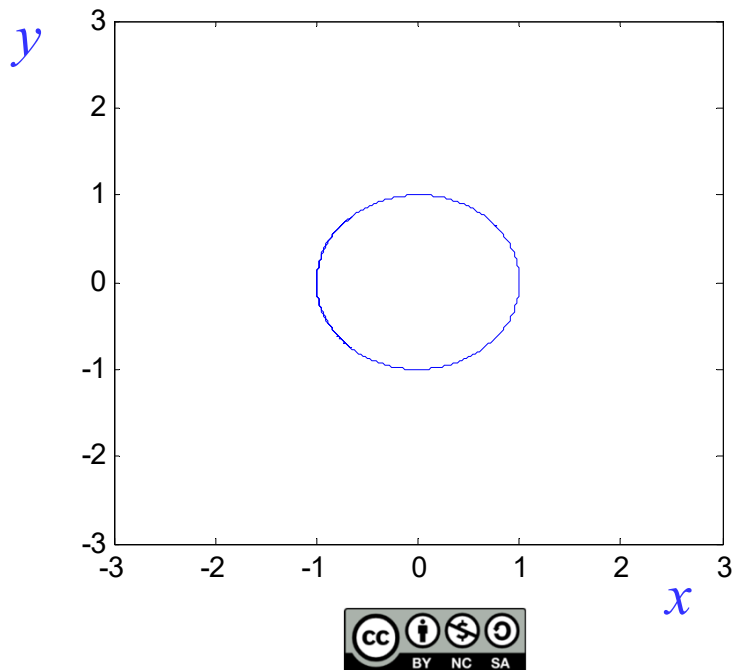
這些命名方式，是根據 2 次多項式在 x - y 平面上的軌跡

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

當 $x^2 + y^2 - 1 = 0$

$$x^2 + y^2 = 1$$

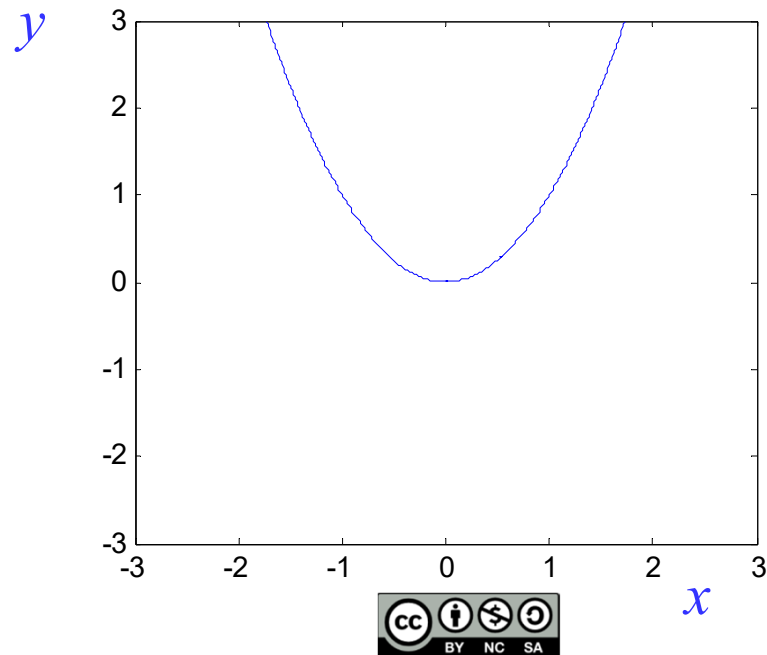
$$B^2 - 4AC = -4 < 0$$



當 $x^2 - y = 0$

$$y = x^2$$

$$B^2 - 4AC = 0$$

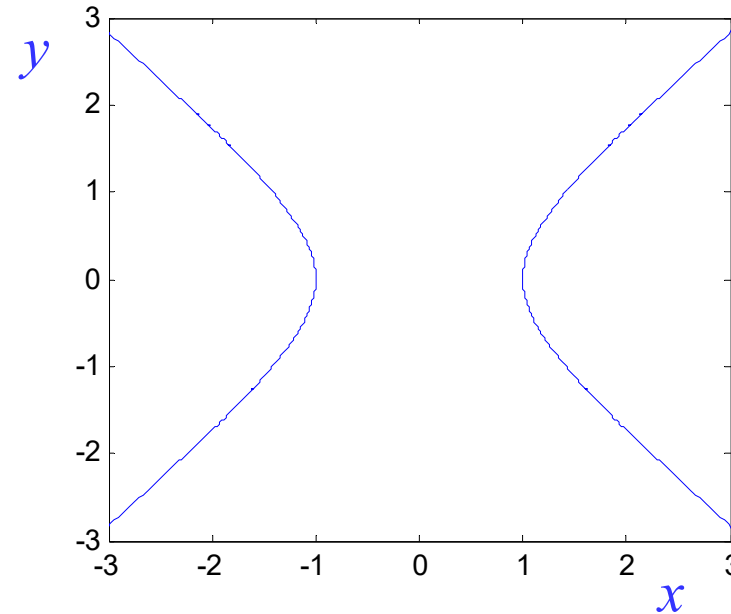


當

$$x^2 - y^2 - 1 = 0$$

$$x^2 - y^2 = 1$$

$$B^2 - 4AC = 4 > 0$$



記憶秘訣：只要清楚幾個「特例」，就可以記住當

$$B^2 - 4AC < 0, \quad B^2 - 4AC = 0, \quad B^2 - 4AC > 0$$

的時候，應該是什麼圖形

Example 3 (text page 463)

$$3 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(1) 本節除了定義以外，只有兩個重點：classification of equations 以及 method of separation of variables.

(2) 然而，method of separation of variables 解法的流程，稍有些複雜，需要熟悉 (Sections 12-4, 12-5 都將用這個方法)

關鍵：記住**第一步** $u(x, y) = X(x)Y(y)$

第二步 function of $X =$ function of $Y = -\lambda$

(3) Method of separation of variables 在計算時，會分成很多個 cases.

(4) Separation of variables 要解 BVP 和 IVP 時，需要將每個 cases 得出來的解都加起來 (Step 6)

(5) 為了方便解決 BVP 或 IVP，經常將 $d_1 e^{2\alpha x} + d_2 e^{-2\alpha x}$

改寫成 $c_4 \cosh(2\alpha x) + c_5 \sinh(2\alpha x)$

(6) Hyperbolic, parabolic, elliptic 的條件，可以用幾個 special cases 來記

Section 12.4 Wave Equation

12.4.1 本節綱要

要解決的問題 (one-dimensional wave equation)

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad 0 < x < L \quad t > 0$$

BVP and IVP

$$u(0, t) = 0 \quad u(L, t) = 0 \quad \text{for } t > 0$$

$$u(x, 0) = f(x) \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x) \quad \text{for } 0 < x < L$$

例子見 page 440

解法見 page 441-449

實際上，Sections 12.4 可看成是 Section 12.1 的 [method of separation of variables](#) 的練習題

(可見得 [method of separation of variables](#) 有多重要)

名詞：

standing waves (page 450)

normal modes (page 450)

first standing wave (page 451)

[fundamental frequency](#) (page 451)

nodes (page 453)

overtones (page 453)

12.4.2 物理意義

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

u : 高度

$$u(0, t) = 0$$

$$u(L, t) = 0$$

$\frac{\partial u}{\partial t}$: 速度

$$u(x, 0) = f(x) \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x)$$

$\frac{\partial^2 u}{\partial t^2}$: 加速度

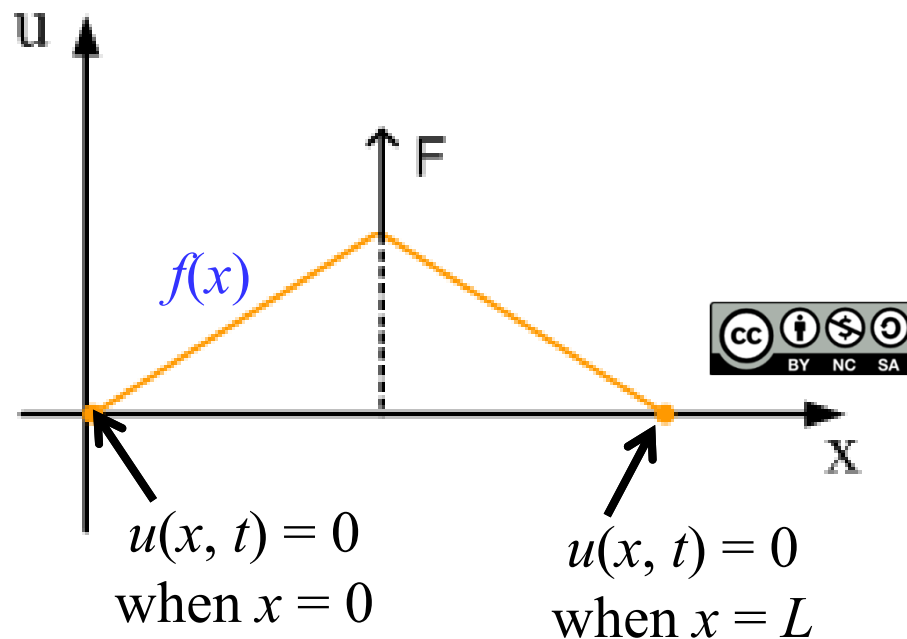


Fig. 12.2.4

12.4.3 Solutions for Wave Equations (自己挑戰解解看)

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad 0 < x < L \quad t > 0$$

四大條件 $u(0, t) = 0$ $u(L, t) = 0$ for $t > 0$

$$u(x, 0) = f(x) \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x) \quad \text{for } 0 < x < L$$

求解 (使用 method of separation of variables)

(Step 1) 假設解為 $u(x, t) = X(x)T(t)$

(Step 2) 將 $u(x, y) = X(x)T(t)$ 代入 $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$

$$a^2 X''(x)T(t) = X(x)T''(t) \quad \frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)}$$

$$\text{令 } \frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda$$

$$\text{得出2個 ODEs } X''(x) + \lambda X(x) = 0 \quad T''(t) + a^2 \lambda T(t) = 0$$

(Steps 3, 4, 5 的前處理)

(1) 因為 x 的 boundary condition 較簡單，所以先解 $X(x)$

(2) 分成 $\lambda = 0$, $\lambda < 0$, $\lambda > 0$ 三個 cases

(3) 由於 $u(0, t) = 0$ for all $t > 0$ $u(0, t) = X(0)T(t) = 0$

$T(t)$ 不可為 0 (否則 $u(x, t) = X(x)T(t) = 0$ for any x, t)

所以 $X(0) = 0$

同理，由 $u(L, t) = 0$ 可以立即判斷 $X(L) = 0$

$X''(x) + \lambda X(x) = 0$ subject to $X(0) = 0$ and $X(L) = 0$

$$X''(x) + \lambda X(x) = 0$$

subject to

$$X(0) = 0$$

and

$$X(L) = 0$$

$$T''(t) + a^2 \lambda T(t) = 0$$

Case 1 for Steps 3, 4, 5 $\lambda = 0$

(Step 3-1) $X''(x) = 0$ $X(x) = d_1 x + d_0$

根據 boundary conditions

$$\begin{array}{ccc} d_0 = 0 & \implies & d_0 = 0 \\ d_1 L + d_0 = 0 & & d_1 = 0 \end{array} \quad X(x) = 0$$

這個 case 得出 trivial solution $u(x, t) = X(x)T(t) = 0$

$u(x, 0) = f(x)$ 將無法滿足 $\lambda = 0$ 時無解

無需再解 Step 4-1, Step 5-1

Case 2 of Steps 3, 4, 5: $\lambda < 0$

(Step 3-2) 令 $\lambda = -\alpha^2$

$$X''(x) - \alpha^2 X(x) = 0$$

Solution: $X(x) = d_2 e^{\alpha x} + d_3 e^{-\alpha x}$

較易處理 boundary conditions

可改寫成 $X(x) = d_4 \cosh(\alpha x) + d_5 \sinh(\alpha x)$

根據 boundary conditions $X(0) = 0$ and $X(L) = 0$

$$\begin{array}{l} d_4 = 0 \\ d_4 \cosh(\alpha L) + d_5 \sinh(\alpha L) = 0 \end{array} \quad \Longrightarrow \quad \begin{array}{l} d_4 = 0 \\ d_5 = 0 \end{array} \quad X(x) = 0$$

這個 case 得出 trivial solution $u(x, t) = X(x)T(t) = 0$

$u(x, 0) = f(x)$ 將無法滿足

$\lambda < 0$ 時無解

無需再解 Step 4-2, Step 5-2

Case 3 of Steps 3, 4, 5: $\lambda > 0$

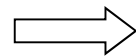
(Step 3-3) 令 $\lambda = \alpha^2$

$$X''(x) + \alpha^2 X(x) = 0$$

Solution: $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$

根據 boundary conditions $X(0) = 0$ and $X(L) = 0$

$$c_1 = 0$$



$$c_1 = 0$$

$$c_1 \cos \alpha L + c_2 \sin \alpha L = 0$$

$$\alpha = \frac{n\pi}{L} \quad n \text{ 是任意正整數}$$

$$c_2 = \text{any nonzero constant}$$

特別注意：

不可直接由 $\begin{cases} c_1 = 0 \\ c_1 \cos \alpha L + c_2 \sin \alpha L = 0 \end{cases}$ 就斷言 $\begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$

應該看看是否有適當的 α , 讓第二個式子等於零

$$X(x) = c_2 \sin \frac{n\pi}{L} x$$

n 是任意正整數, c_2 是任意數

$$\alpha = \frac{n\pi}{L} \quad \lambda = \alpha^2 = \frac{n^2 \pi^2}{L^2}$$

(Step 4-3) $T''(t) + a^2 \lambda T(t) = 0$

$$T''(t) + \frac{a^2 n^2 \pi^2}{L^2} T(t) = 0$$

Solution: $T(t) = c_3 \cos\left(\frac{na\pi}{L} t\right) + c_4 \sin\left(\frac{na\pi}{L} t\right)$ n 是任意正整數

(Step 5-3)

$$\begin{aligned} u_n(x, t) &= X(x)T(t) = c_2 \sin\left(\frac{n\pi}{L} x\right) \left[c_3 \cos\left(\frac{na\pi}{L} t\right) + c_4 \sin\left(\frac{na\pi}{L} t\right) \right] \\ &= \sin\left(\frac{n\pi}{L} x\right) \left[A_n \cos\left(\frac{na\pi}{L} t\right) + B_n \sin\left(\frac{na\pi}{L} t\right) \right] \quad n \text{ 是任意正整數} \end{aligned}$$

$$A_n = c_2 c_3, \quad B_n = c_2 c_4,$$

注意： $u_n(x, t) = \sin\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{na\pi}{L}t\right) + B_n \sin\left(\frac{na\pi}{L}t\right)\right]$

只是其中一個解，因為 n 是任意正整數

要將這些解加起來

(Step 6)

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{na\pi}{L}t\right) + B_n \sin\left(\frac{na\pi}{L}t\right)\right]$$

(Step 6)

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \left[A_n \cos\left(\frac{na\pi}{L}t\right) + B_n \sin\left(\frac{na\pi}{L}t\right) \right]$$

討論：為什麼 n 是從 1 加到 ∞ ，而非由 $-\infty$ 加到 ∞ ？

因為 $\sin\left(\frac{n\pi}{L}x\right) = -\sin\left(\frac{-n\pi}{L}x\right)$, $\cos\left(\frac{na\pi}{L}t\right) = \cos\left(\frac{-na\pi}{L}t\right)$,

$$\sin(0) = 0$$

可證明 $\sum_{n=-\infty}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \left[C_n \cos\left(\frac{na\pi}{L}t\right) + D_n \sin\left(\frac{na\pi}{L}t\right) \right]$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \left[A_n \cos\left(\frac{na\pi}{L}t\right) + B_n \sin\left(\frac{na\pi}{L}t\right) \right]$$

$$A_n = C_n - C_{-n}$$

$$B_n = D_n + D_{-n}$$

(Step 7)
$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \left[A_n \cos\left(\frac{na\pi}{L}t\right) + B_n \sin\left(\frac{na\pi}{L}t\right) \right]$$

由 initial conditions

$$u(x,0) = f(x) \qquad \left. \frac{\partial u}{\partial t} \right|_{t=0} = g(x)$$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \qquad g(x) = \sum_{n=1}^{\infty} B_n \frac{na\pi}{L} \sin\left(\frac{n\pi}{L}x\right)$$

也就是說， A_n 是 $f(x)$ 的 Fourier sine series (Sec. 11-3, page 389),

$B_n \frac{na\pi}{L}$ 是 $g(x)$ 的 Fourier sine series

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx$$

$$B_n \frac{na\pi}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi}{L} x dx \qquad B_n = \frac{2}{na\pi} \int_0^L g(x) \sin \frac{n\pi}{L} x dx$$

12.4.4 名詞

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \left[A_n \cos\left(\frac{na\pi}{L}t\right) + B_n \sin\left(\frac{na\pi}{L}t\right) \right]$$

$$u(x, t) = u_1(x, t) + u_2(x, t) + u_3(x, t) + \dots$$

其中

$$u_n(x, t) = \sin\left(\frac{n\pi}{L}x\right) \left[A_n \cos\left(\frac{na\pi}{L}t\right) + B_n \sin\left(\frac{na\pi}{L}t\right) \right]$$

$$= C_n \sin\left(\frac{n\pi}{L}x\right) \left[\sin\left(\frac{na\pi}{L}t + \phi_n\right) \right]$$

$$C_n = \sqrt{A_n^2 + B_n^2} \quad \cos \phi_n = \frac{B_n}{C_n} \quad \sin \phi_n = \frac{A_n}{C_n}$$

$u_n(x, t)$ 被稱作 standing waves (駐波) 或 normal modes

$n = 1$ 時， $u_1(x, t)$ 被稱作 **first standing wave** 或 **first normal mode** 或 **fundamental mode of vibration**

$$u_1(x, t) = C_1 \sin\left(\frac{\pi}{L}x\right) \left[\sin\left(\frac{a\pi}{L}t + \phi_1\right) \right]$$

$$u_1\left(x, t + \frac{2L}{a}\right) = C_1 \sin\left(\frac{\pi}{L}x\right) \left[\sin\left(\frac{a\pi}{L}t + 2\pi + \phi_1\right) \right] = u_1(x, t)$$

$$\text{對於 } t \text{ 而言，週期} = \frac{2L}{a} \quad \text{頻率} = 1/\text{週期} = \frac{a}{2L}$$

$f_1 = \frac{a}{2L}$ 被稱作 **fundamental frequency (基頻)** 或 **first harmonic**

以此類推， $u_2(x, t)$ 被稱作 **second standing wave**

$u_3(x, t)$ 被稱作 **third standing wave**

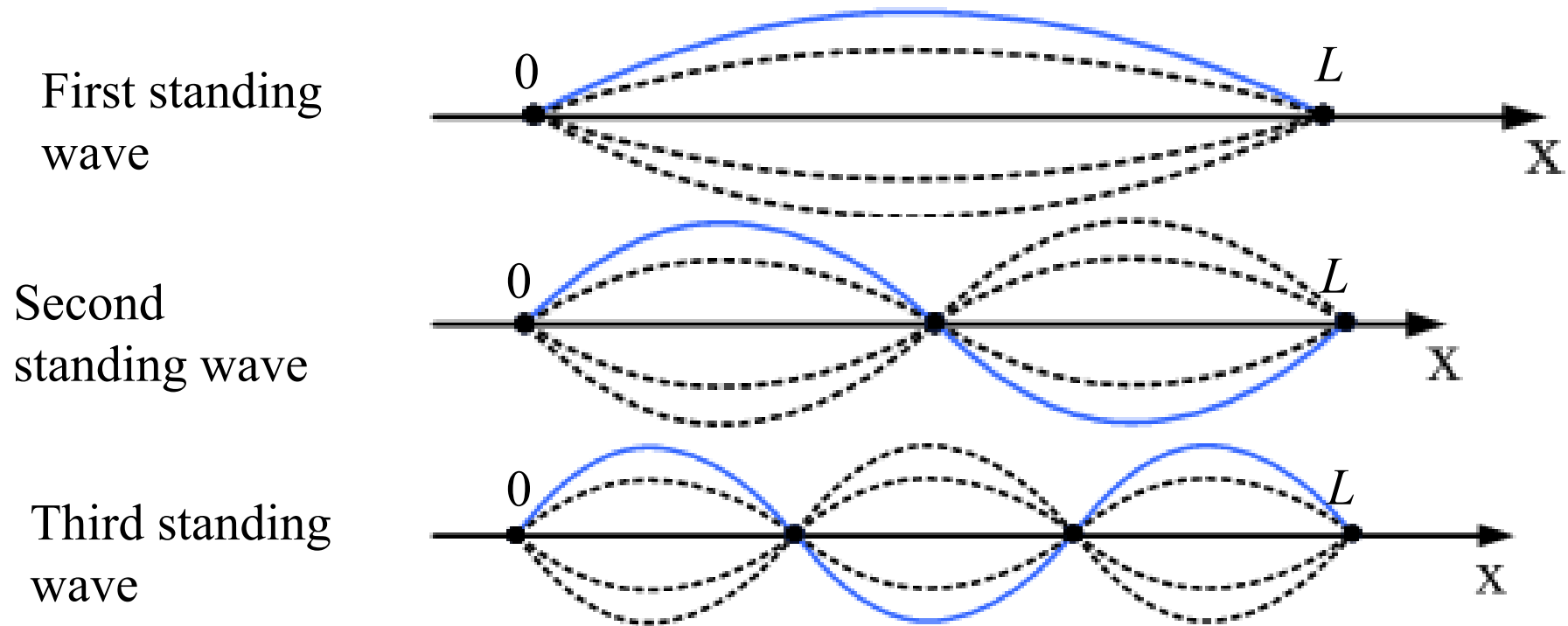


Fig. 12.4.2



$$u_n(x, t) = C_n \sin\left(\frac{n\pi}{L}x\right) \left[\sin\left(\frac{na\pi}{L}t + \phi_n\right) \right]$$

$x = \frac{L}{n}$ 時，無論 t 等於多少， $u_n\left(\frac{L}{n}, t\right) = 0$

$x = \frac{L}{n}$ 是 n^{th} standing wave 的 **node** (節點)

$$u_n(x, t) = u_n\left(x, t + \frac{2L}{an}\right)$$

$u_n(x, t)$ 的頻率 = 1/週期 = $n \frac{a}{2L}$

$f_n = n \frac{a}{2L} = nf_1$ 被稱作 **overtones** (泛音)

12.4.5 Sections 12.4 需要注意的地方

(1) Method of separation of variables 解 PDE 的過程雖然長，但是把握住講義 pages 422-424 的 7 個 steps，並練習幾次，就可以熟悉。

(這些對大二下和大三上的電磁學很重要)

(2) 雖然概念不難，但是計算過程很長且繁雜

所以一定要多研究簡化運算、快速判斷的方法

(3) 有沒有注意到，

若 boundary conditions 出現 $u(0, y) = 0, u(L, y) = 0,$

最後的解總是和 sine 有關 $X(x) = c_2 \sin \frac{n\pi}{L} x$ 週期為 $2L/n$

若 boundary conditions 出現 $\left. \frac{\partial u}{\partial x} \right|_{x=0} = 0 \quad \left. \frac{\partial u}{\partial x} \right|_{x=L} = 0$

最後的解總是和 cosine 或 constant 有關

$$X(x) = c_1 \quad \text{or} \quad X_n(x) = c_1 \cos \frac{n\pi}{L} x \quad \text{週期也為 } 2L/n$$

(4) 經驗足夠後，看到 $u(x, y)$ 的 boundary conditions

出現 $u(a, y) = 0 \longrightarrow$ 就知道 $X(a) = 0$ ，

看到 $u(x, b) = 0 \longrightarrow$ 就知道 $Y(b) = 0$ 。

看到 $\left. \frac{\partial u}{\partial x} \right|_{x=a} = 0 \longrightarrow$ 就知道 $X'(a) = 0$ ，

看到 $\left. \frac{\partial u}{\partial y} \right|_{y=b} = 0 \longrightarrow$ 就知道 $Y'(b) = 0$

(5) 對於 wave equations 而言， $X(x)$ 和 $T(t)$ 的解有相同的型態

如果 $X(x)$ 為 sine & cosine, $T(t)$ 也為 sine & cosine

對於 Laplace's equations 而言， $X(x)$ 和 $Y(y)$ 的解型態不同

如果 $X(x)$ 為 sine & cosine, $Y(y)$ 為 sinh & cosh

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

(6) 要熟悉 $\cosh(x)$, $\sinh(x)$ 的性質

(7) Method of separation of variables 在計算上容易出錯的地方

(以講義 pages 441-449 wave equations 為例)

$$(a) \quad \frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda$$

(b) Steps 3, 4, 5 要考慮所有 cases

(c) 不可直接由 $c_1 = 0$ 及 $c_1 \cos \alpha L + c_2 \sin \alpha L = 0$ 判斷 $c_1 = c_2 = 0$

因為 α 可以是 $\pi n/L$, 如講義 page 445 所述

(d) 在 Step 6, 要將所有可能的解加起來, 才是 $u(x, t)$ 的一般解

如講義 pages 429, 447 所述

Exercise for Practice

Section 12-1 3, 6, 9, 10, 12, 14, 16, 18, 22, 23, 30, 32

Section 12-4 1, 4, 7, 10, 11, 15, 17, 21, 23

Review 12 1, 2, 5, 13

Happy New Year!

祝各位期末考順利，寒假愉快！