Summary Tables for Formulas

(1) The Methods to Solve First Order Differential Equations

Name of Methods	Suitable Conditions	Key Formula of the Method
(1) Direct Integral	$\frac{dy}{dx}=f(x)$	$y(x)=\int f(x)dx+c$
(2) Separable Variables	$\frac{dy}{dx}=g(x)h(y)$	$\int \frac{dy}{h(y)}=\int g(x)dx$
(3) Linear DE	$\frac{dy}{dx}+P(x)y=f(x)$	$\frac{d}{dx}\left[e^{\int P(x)dx}y\right]=e^{\int P(x)dx}f(x)$
(4) Exact Equation	$M(x,y)dx+N(x,y)dy$ $M_y=N_x$	First, solve $\frac{\partial f}{\partial x}=M(x,y)$, then solve $\frac{\partial f}{\partial y}=N(x,y)$ (or exchange the order)
(5) Exact Equation (Integration Factor I)	$M(x,y)dx+N(x,y)dy$ $\frac{M_y-N_x}{M}$ is independent of x	$\mu(y)=e^{\int \frac{(N_x-M_y)}{M}dy}$, solving $\mu(y)M(x,y)dx+\mu(y)N(x,y)dy$
(6) Exact Equation (Integration Factor II)	$M(x,y)dx+N(x,y)dy$ $\frac{M_y-N_x}{N}$ is independent of y	$\mu(x)=e^{\int \frac{(M_y-N_x)}{N}dx}$, solving $\mu(x)M(x,y)dx+\mu(x)N(x,y)dy$
(7) Homogeneous equation	$M(x,y)dx+N(x,y)dy$ $M(tx,ty)=t^{\alpha}M(x,y)$ $N(tx,ty)=t^{\alpha}N(x,y)$	Set $u=y/x,\quad (y=xu)$ $dy=udx+xdu$
(8) Bernoulli's Equation	$\frac{dy}{dx}=-P(x)y+f(x)y^{n}$	Set $u=y^{1-n}$ $\frac{dy}{dx}=\frac{1}{1-n}u^{\frac{n}{1-n}}\frac{du}{dx}$
(9) $Ax+By+C$	$\frac{dy}{dx}=f(Ax+By+C)$	Set $u=Ax+By+c$ $Bdy=du-Adx$

(2) The Methods to Solve Higher Order Linear Differential Equations

(A) Solving the Complementary Function for Linear DEs		
Name of Methods	Suitable Conditions	Key Formula of the Method
(1) Reduction of Order	(i) linear, (ii) 2nd order, (iii) one nontrivial solution $y_1(x)$ has been known	$y_2(x)=y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ where $P(x)$ is the coefficient of y' in the *standard form*.
(2) Auxiliary Function	(i) linear, (ii) constant coefficients	$\sum_{n=0}^{N} a_n y^{(n)}(x)=0 \rightarrow \sum_{n=0}^{N} a_n m^n=0$
	(a) If m_q is a root	$e^{m_q x}$ is one of the independent solutions.
	(b) If m_q is a root with multiplicity k	$e^{m_q x}, xe^{m_q x}, \cdots, x^{k-1}e^{m_q x}$ are k of the independent solutions
	(c) If $\alpha \pm j\beta$ are the roots	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ are two of the independent solutions
	(d) If $\alpha \pm j\beta$ are the roots with multiplicity k	$e^{\alpha x}\cos(\beta x), xe^{\alpha x}\cos(\beta x), \cdots\cdots, x^{k-1}e^{\alpha x}\cos(\beta x)$ $e^{\alpha x}\sin(\beta x), xe^{\alpha x}\sin(\beta x), \cdots\cdots, x^{k-1}e^{\alpha x}\sin(\beta x)$ are $2k$ of the independent solutions

(B) Solving the Particular Solution for Linear DEs $\sum_{n=0}^{N} P_n\left(x\right) y^{(n)}\left(x\right) = g\left(x\right)$		
Name of Methods	Suitable Conditions	Key Formula of the Method
(1) Form Rule	(i) linear, (ii) constant coefficients, (iii) $g(x)$, $g'(x)$, $g''(x)$, …. have finite number of terms	See PowerPoint Page 195
(2) Annihilator Method	(i) linear, (ii) constant coefficients, (iii) $g(x)$, $g'(x)$, $g''(x)$, …. have finite number of terms	If $L_1 g\left(x\right) = 0$, original DE: $Ly = g$, $L = \sum_{n=0}^{N} a_n D^n$, Particular solutions: Satisfy $L_1 L y = 0$, but $Ly \neq 0$
	$g(x)$	Annihilator
	$\left(a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots + a_0\right) e^{\alpha x}$	$\left[D - \alpha\right]^{n+1}$
	$\left(a_n x^n + a_{n-1} x^{n-1} + \cdots\cdots + a_0\right) e^{\alpha x} \left(b_1 \cos \beta x + b_2 \sin \beta x\right)$	$\left[D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right]^{n+1}$
	$g_1(x) + g_2(x) + \dots\dots + g_k(x)$	$L_k L_{k-1} \cdots L_2 L_1$ if $L_h[g_h(x)] = 0$

(3) Formulas for Trigonometric and Hyperbolic Functions

(1) $\cos(x) =$	$\frac{e^{ix}+e^{-ix}}{2}$
(2) $\sin(x) =$	$\frac{e^{ix}-e^{-ix}}{i2}$
(3) $\cosh(x) =$	$\frac{e^{x}+e^{-x}}{2}$
(4) $\sinh(x) =$	$\frac{e^{x}-e^{-x}}{2}$
(5) $\cos(a+b) =$	$\cos(a)\cos(b) - \sin(a)\sin(b)$
(6) $\sin(a+b) =$	$\sin(a)\cos(b) + \cos(a)\sin(b)$
(7) $\cos(2a) =$	$\cos^2(a) - \sin^2(a)$
(7a) $\cos^2(a) =$	$(\cos(2a) + 1)/2$
(7b) $\sin^2(a) =$	$(1 - \cos(2a))/2$
(8) $\sin(2a) =$	$2\sin(a)\cos(a)$
(9) $\sinh(0) =$	0
(10) $\cosh(0) =$	1
(11) $\frac{d}{dx}\cosh x =$	$\sinh x$, $\left.\frac{d}{dx}\cosh x\right\|_{x=0} = 0$
(12) $\frac{d}{dx}\sinh x =$	$\cosh x$, $\left.\frac{d}{dx}\sinh x\right\|_{x=0} = 1$

(4) Integrals

(1) $1/x$	$\ln\|x\| + c$
(2) $\cos(x)$	$\sin(x) + c$
(3) $\sin(x)$	$-\cos(x) + c$
(4) $\tan(x)$	$-\ln\|\cos(x)\| + c$
(5) $\cot(x)$	$\ln\|\sin(x)\| + c$
(6) a^x	$a^x/\ln(a) + c$
(7) $\frac{1}{x^2 + a^2}$	$\frac{1}{a}\arctan\frac{x}{a} + c$
(8) $1/\sqrt{a^2 - x^2}$	$\arcsin(x/a) + c$
(9) $-1/\sqrt{a^2 - x^2}$	$\arccos(x/a) + c$
(10) $\sec^2 x$	$\tan x + c$
(11) $\sec x \tan x$	$\sec x + c$
(12) $x\, e^{ax}$	$\frac{e^{ax}}{a}\left(x - \frac{1}{a}\right) + c$
(13) $x^2\, e^{ax}$	$\frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right) + c$