2-4 Exact Equations

2-4-1 方法的條件

任何 first order DE 皆可改寫成

$N(x,y)\frac{dy}{dx}+M(x,y)=0$

$$M(x,y)dx+N(x,y)dy=0$$ 的型態

(1) 當 $\boxed{\frac{\partial}{\partial y}M(x,y)=\frac{\partial}{\partial x}N(x,y)}$ ☆1-1 成立時， 限制條件

可以用本節的 Exact Equation 的方法來解

(2) 當 $\boxed{\dfrac{\frac{\partial}{\partial y}M(x,y)-\frac{\partial}{\partial x}N(x,y)}{M(x,y)}}$ is independent of x

或 $\boxed{\dfrac{\frac{\partial}{\partial y}M(x,y)-\frac{\partial}{\partial x}N(x,y)}{N(x,y)}}$ is independent of y

可以用Modified Exact Equation Method 來解 (見講義 2-4-5)

2-4-2 方法的來源

- Review the concept of partial differentiation

$$\underline{df(x,y)} = \underline{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}$$ ☆2

- Specially, when $f(x, y) = c$ where c is some constant,

等高線

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

補充：

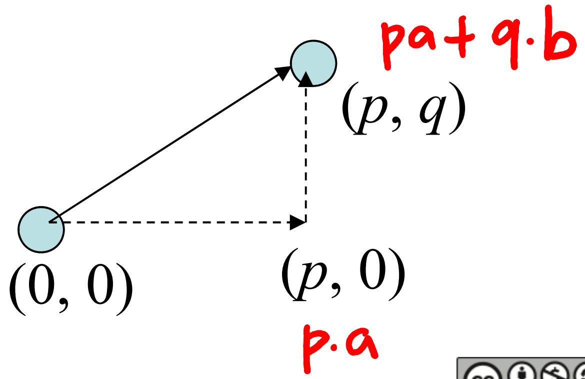
$$df(x,y,z)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz$$

$$df(x_1,x_2,x_3,\cdots,x_k)=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+\frac{\partial f}{\partial x_3}dx_3+\cdots\cdots+\frac{\partial f}{\partial x_k}dx_k$$

思考：假設一個人在山坡的某處。若往東走，每走 1 公尺，高度會增加 a 公尺。若往北走，每走 1 公尺，高度會增加 b 公尺。假設這人現在所在的位置是(0, 0)。那麼這人的東北方，座標為 (p, q) 的地方，高度會比 (0, 0) 高多少？

$$a\times p+b\times q$$

$$df(x,y)=\underbrace{\boxed{\frac{\partial f(x,y)}{\partial x}}}_{a}\underbrace{\boxed{dx}}_{p}+\underbrace{\boxed{\frac{\partial f(x,y)}{\partial y}}}_{b}\underbrace{\boxed{dy}}_{q}$$



[Definition 2.4.1] We can express any 1st order DE as

$$\boxed{M(x,y)dx + N(x,y)dy = 0}$$

• If there exists some function $f(x, y)$ that satisfies

$$\boxed{\frac{\partial f(x,y)}{\partial x} = M(x,y)} \quad \text{and} \quad \boxed{\frac{\partial f(x,y)}{\partial y} = N(x,y)},$$

★3

then we call the 1st order DE the **exact equation**.

• The method for checking whether the DE is an exact equation:

$$\boxed{\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}}$$

★1-1

(Proof): If $\frac{\partial f(x,y)}{\partial x} = M(x,y)$ and $\frac{\partial f(x,y)}{\partial y} = N(x,y)$,

then $$\frac{\partial M(x,y)}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = \frac{\partial N(x,y)}{\partial x}$$

For the exact equation,

$$M(x,y)\,dx+N(x,y)\,dy=0$$

$$\downarrow$$

$$\frac{\partial f(x,y)}{\partial x}dx+\frac{\partial f(x,y)}{\partial y}dy=0$$

$$\downarrow$$

from page 102

可改寫成 $df(x,y)=0,$

$$\downarrow$$

$$f(x,y)=c$$

☆4

2-4-3 解法 ☆☆: $Mdx + Ndy = 0$

Step 0: check whether $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The Method for Solving the Exact Equation (A):

$$\boxed{\frac{\partial f(x,y)}{\partial x} = M(x,y)} \qquad \boxed{\frac{\partial f(x,y)}{\partial y} = N(x,y)}$$

(1)

☆5

$$f(x,y) = \int M(x,y)\,dx + \boxed{g(y)} \xrightarrow{(2)} \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y) = N(x,y)$$

g(y) is a constant for x

(2)

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

(2)

$$g(y) = \int g'(y)\,dy$$

(3) 代入

(3)

$$\boxed{\int M(x,y)\,dx + g(y) = c}$$

f(x,y) = c

(4)

further computation

Solution

double N

(0) Check $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$

整理

Previous Step: Check whether $\boxed{\dfrac{\partial M(x,y)}{\partial y}=\dfrac{\partial N(x,y)}{\partial x}}$ is satisfied.

Step 1: Solve $\dfrac{\partial f(x,y)}{\partial x}=M(x,y) \longrightarrow f(x,y)=\int M(x,y)\,dx+g(y)$

Step 2: 將 $f(x,y)$ 代入 $\dfrac{\partial f(x,y)}{\partial y}=N(x,y)$ ，以解出 $g(y)$

Step 3: Substitute $g(y)$ into

$$f(x,y)=\int M(x,y)\,dx+g(y)=c$$

Step 4: Further computation and obtain the solution

Extra Steps: (a) Consider the initial value problem

The Method for Solving the Exact Equation (B):

$$\boxed{\frac{\partial f(x,y)}{\partial x} = M(x,y)}$$

$$\boxed{\frac{\partial f(x,y)}{\partial y} = N(x,y)}$$

(1)

$$f(x,y) = \int N(x,y)\,dy + h(x) = c$$

★5

$h(x)$ is a constant for y

(2)

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int N(x,y)\,dy + h'(x) = M(x,y)$$

(2)

$$h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy$$

(2)

$$h(x) = \int h'(x)\,dx$$

(2)

(3) 代入

(3)

$$\boxed{\int N(x,y)\,dy + h(x) = c}$$

f = c

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(4)

further computation

Solution

(0) Check $\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y)$

2-4-4 例子

[Example 1] (text page 67)

It can be solved by Sections 2-2, 2-3

$$\boxed{2xydx+(x^2-1)dy=0}$$

$$\frac{dy}{dx}+\frac{2xy}{x^2-1}=0$$

$M(x,y)=2xy$

$N(x,y)=x^2-1$ $\dashrightarrow$ Step 0: check whether it is exact

$$\frac{\partial M}{\partial y}=2x=\frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial x}=2xy \qquad \frac{\partial f}{\partial y}=x^2-1$$

Step 1

Step 2

Step 2

$$f(x,y)=x^2y+g(y)$$

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$$\frac{\partial f}{\partial y}=x^2+g'(y)=x^2-1$$

singular points: ± 1

Step 3

Step 2

Step 3

$$g'(y)=-1$$

Step 2

$$x^2y-y=c$$

Step 4

$$g(y)=-y$$

$-y+c_1$

$$\boxed{y=c/(x^2-1)}$$

$(-1,1)$, $(1,\infty)$, or $(-\infty,-1)$

思考: 是否有其他的方法可以解 Example 1?

$$\frac{\partial M}{\partial y} = 2e^{2y} - 1\cos xy + xy\sin xy$$
$$\frac{\partial N}{\partial x} : 2e^{2y} - 1\cos xy + xy\sin xy$$
Exact!

[Example 2] (text page 67)

$$\boxed{(\overset{M}{e^{2y} - y\cos xy})dx + (\overset{N}{2xe^{2y} - x\cos xy + 2y})dy = 0}$$

$$M(x,y) = e^{2y} - y\cos xy \qquad N(x,y) = 2xe^{2y} - x\cos xy + 2y$$

Step 0:

check for exact

$$\frac{\partial M}{\partial y} = 2e^{2y} - \cos xy + xy\sin xy = \frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x\cos xy + 2y$$

Step 1

$$f(x,y) = xe^{2y} - \sin xy + y^2 + h(x)$$

☆5

Step 2

$$\frac{\partial f}{\partial x} = e^{2y} - y\cos xy$$

$$e^{2y} - y\cos xy + h'(x) = e^{2y} - y\cos xy$$

$$h'(x) = 0$$

Step 2

$$h(x) = \cancel{c_1}\ 0$$

Step 3

Step 3

$$f(x,y) = xe^{2y} - \sin xy + y^2$$

Step 4

☆4 $$c = xe^{2y} - \sin xy + y^2$$

$$xe^{2y} - \sin xy + y^2 + c = 0$$ implicit solution

要注意

(a) 自行由另一個方向 $f(x,y)=\int M(x,y)dx+g(y)$ 來練習，

看是否得出同樣的結果。

(b) 得出的解 $xe^{2y}-\sin xy+y^2+c=0$ 為 implicit solution

(c) 思考：何時用 $f(x,y)=\int M(x,y)dx+g(y)$

何時用 $f(x,y)=\int N(x,y)dy+h(x)$