

4-6 Variation of Parameters

Solved by Matrices

4-6-1 方法的限制

The method can solve the particular solution for **any** linear DE

(1) May not have constant coefficients

(2) $g(x)$ may not be of the special forms

☆,

↑
the only constraint

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y = g(x)$$

4-6-2 Case of the 2nd order linear DE

$$a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y = g(x)$$

associated homogeneous equation: $a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y = 0$

Suppose that the solution of the associated homogeneous equation is

$$c_1y_1(x) + c_2y_2(x)$$

Then the **particular solution** is assumed as:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) \quad \star_2$$

(方法的基本精神)

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

代入原式後，總是可以簡化

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

$$y_p'' = u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''$$

代入 $\underline{y''(x) + P(x)y'(x) + Q(x)y = f(x)}$ (standard form) \star_3

$$P(x) = \frac{a_1(x)}{a_2(x)}, \quad Q(x) = \frac{a_0(x)}{a_2(x)}, \quad f(x) = \frac{g(x)}{a_2(x)}$$

$$\begin{aligned} y_p'' + P(x)y_p' + Q(x)y_p &= u_1''y_1 + 2u_1'y_1' + \underline{u_1y_1''} + u_2''y_2 + 2u_2'y_2' + u_2y_2'' \\ &\quad + P(\underline{u_1'y_1} + u_1y_1' + u_2'y_2 + u_2y_2') + Q(\underline{u_1y_1} + u_2y_2) \end{aligned}$$

zero zero

$$\begin{aligned} y_p'' + P(x)y_p' + Q(x)y_p &= u_1[\underline{y_1'' + Py_1' + Qy_1}] + u_2[\underline{y_2'' + Py_2' + Qy_2}] + y_1u_1'' \\ &\quad + 2u_1'y_1' + y_2u_2'' + 2u_2'y_2' + P[y_1u_1' + y_2u_2'] \end{aligned}$$

$$y_p'' + P(x)y_p' + Q(x)y_p = f(x), \quad y_p = u_1y_1 + u_2y_2$$

↓ 簡化

$$y_p'' + P(x)y_p' + Q(x)y_p = y_1u_1'' + 2u_1'y_1' + y_2u_2'' + 2u_2'y_2' + P[y_1u_1' + y_2u_2']$$

↓ 簡化

$$\frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1u_1'' + u_1'y_1' + y_2u_2'' + u_2'y_2' = f(x)$$

進一步簡化：

假設 $y_1u_1' + y_2u_2' = 0$

$$y_1'u_1' + y_2'u_2' = f(x)$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

聯立方程式

$$\begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = f(x) \end{cases}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$u_1' = \frac{\det \begin{bmatrix} 0 & y_2 \\ f(x) & y_2' \end{bmatrix}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}$$

$$u_2' = \frac{\det \begin{bmatrix} y_1 & 0 \\ y_1' & f(x) \end{bmatrix}}{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}}$$

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1 + y_2' u_2 = f(x) \end{cases} \begin{matrix} \rightarrow \\ \rightarrow \end{matrix} \begin{matrix} \star\star \\ \star\star \end{matrix} \begin{matrix} u_1' = \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \\ u_2' = \frac{W_2}{W} = \frac{y_1 f(x)}{W} \end{matrix} \rightarrow \begin{matrix} \boxed{u_1 = \int u_1'(x) dx} \\ \boxed{u_2 = \int u_2'(x) dx} \end{matrix} \begin{matrix} 235 \\ \star_5 \end{matrix}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$ $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

||: determinant \star_4 $f(x) = \frac{g(x)}{a_2(x)}$ \star_3

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

可以和 1st order case (page 63) 相比較

4-6-3 Process for the 2nd Order Case ☆☆

Step 2-1 變成 standard form ☆₃

$$y''(x) + P(x)y'(x) + Q(x)y = f(x)$$

Step 2-2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Step 2-3

$$u_1' = \frac{W_1}{W}$$

$$u_2' = \frac{W_2}{W}$$

Step 2-4

$$u_1 = \int u_1'(x) dx$$

$$u_2 = \int u_2'(x) dx ☆_5$$

Step 2-5

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

4-6-4 Examples

Example 1 (text page 162)

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Step 1: solution of $y'' - 4y' + 4y = 0$:

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

$$m^2 - 4m + 4 = 0$$

$$m = 2, 2$$

$$W = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & 2x e^{2x} + e^{2x} \end{vmatrix}$$

$$= e^{2x} e^{2x} \begin{vmatrix} 1 & x \\ 2 & 2x+1 \end{vmatrix}$$

Step 2-2: $y_p = u_1 y_1 + u_2 y_2$, $y_1 = e^{2x}$, $y_2 = x e^{2x}$

$$W = \begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & 2x e^{2x} + e^{2x} \end{vmatrix} = e^{4x} \quad W_1 = \begin{vmatrix} 0 & x e^{2x} \\ (x+1)e^{2x} & 2x e^{2x} + e^{2x} \end{vmatrix} = -(x+1)x e^{4x}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x+1)e^{4x}$$

Step 2-3: $u'_1 = \frac{W_1}{W} = -x^2 - x$ $u'_2 = \frac{W_2}{W} = x + 1$

Step 2-4: \star_5

$$u_1 = \int u_1' dx = \int (-x^2 - x) dx = -\frac{1}{3}x^3 - \frac{1}{2}x^2 + c_1$$

$$u_2 = \int u_2' dx = \int (x+1) dx = \frac{1}{2}x^2 + x + c_2 \quad \text{简化}$$

Step 2-5: $y_p = \underbrace{\left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)}_{u_1} e^{2x} + \underbrace{\left(\frac{1}{2}x^2 + x\right)}_{u_2} x e^{2x} = \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)e^{2x}$

Step 3: $y = c_1 e^{2x} + c_2 x e^{2x} + \left(\frac{1}{6}x^3 + \frac{1}{2}x^2\right)e^{2x}$

It cannot be solved
by sec. 4-4

239

Example 2 (text page 163)

$$4y'' + 36y = \csc 3x$$

$$4m^2 + 36 = 0 \quad m = \pm i3$$

Step 1: solution of $4y'' + 36y = 0$: $y_c = c_1 \cos 3x + c_2 \sin 3x$

Step 2-1: standard form: $y'' + 9y = \csc 3x / 4$ $f(x) = \csc 3x / 4$ \star_3

Step 2-2: \star_3

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3 \sin 3x & 3 \cos 3x \end{vmatrix} = 3 \quad W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4} \csc 3x & 3 \cos 3x \end{vmatrix} = -1/4$$

$$W_2 = \begin{vmatrix} \cos 3x & 0 \\ -\sin 3x & \frac{1}{4} \csc 3x \end{vmatrix} = \frac{1}{4} \frac{\cos 3x}{\sin 3x}$$

Step 2-3: $u_1' = \frac{W_1}{W} = -\frac{1}{12}$ $u_2' = \frac{W_2}{W} = \frac{1}{12} \frac{\cos 3x}{\sin 3x} = \frac{1}{36} \frac{\frac{d}{dx} \sin 3x}{\sin 3x}$

Step 2-4: $u_1 = -\frac{x}{12}$ ~~\star_3~~ $u_2 = \frac{1}{36} \ln |\sin 3x|$ ~~\star_3~~

(未完待續)

注意 $\frac{1}{12} \int \frac{\cos 3x}{\sin 3x} dx$ 算法

Step 2-5: $y_p = -\frac{x}{12} \cos 3x + \frac{1}{36} \sin 3x \ln |\sin 3x|$

$u_1 \quad y_1 \qquad \qquad y_2 \quad u_2$

Step 3: $y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{12} \cos 3x + \frac{1}{36} \sin 3x \ln |\sin 3x|$

Note: 課本 Interval $(0, \pi/6)$ 應該改為 $(0, \pi/3)$

$\csc 3x \rightarrow \pm \infty$ if $3x = n\pi$ n is an integer
 $x = \frac{n\pi}{3}$
3
 singular points

Example 3 (text page 164)

$$y'' - y = 1/x$$

cannot be solved
by Sec 4-4

$$y_c = c_1 e^x + c_2 e^{-x}$$

$$f(x) = 1/x$$

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ 1/x & -e^{-x} \end{vmatrix} = -\frac{e^{-x}}{x}$$

$$W_2 = \begin{vmatrix} e^x & 0 \\ e^x & 1/x \end{vmatrix} = \frac{e^x}{x}$$

$$u_1' = \frac{W_1}{W} = \frac{e^{-x}}{2x} \quad u_2' = \frac{W_2}{W} = -\frac{e^x}{2x}$$

$$u_1 = \int \frac{e^{-x}}{2x} dx, \quad u_2 = -\int \frac{e^x}{2x} dx$$

Note: $\int \frac{e^x}{x} dx$ 沒有 analytic 的解

所以直接表示成 $\int_{x_0}^x \frac{e^t}{t} dt$ (複習 page 49)

4-6-5 Case of the Higher Order Linear DE

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y = g(x)$$

Solution of the associated homogeneous equation:

$$y_c = c_1y_1(x) + c_2y_2(x) + c_3y_3(x) + \cdots + c_ny_n(x)$$

The **particular solution** is assumed as:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + u_3(x)y_3(x) + \cdots + u_n(x)y_n(x)$$

$$u'_k(x) = \frac{W_k}{W} \implies u_k(x) = \int u'_k(x) dx$$

Process of the Higher Order Case ★★

Step 2-1 變成 standard form ★₃

$$y^{(n)}(x) + \frac{a_{n-1}(x)}{a_n(x)} y^{(n-1)}(x) + \cdots + \frac{a_1(x)}{a_n(x)} y'(x) + \frac{a_0(x)}{a_n(x)} y = \frac{g(x)}{a_n(x)}$$

↗ f(x)

Step 2-2 Calculate W, W_1, W_2, \dots, W_n (see page 244)

Step 2-3 $u'_1 = \frac{W_1}{W}$ $u'_2 = \frac{W_2}{W}$ $u'_n = \frac{W_n}{W}$ ★★

Step 2-4 $u_1 = \int u'_1(x) dx$ $u_2 = \int u'_2(x) dx$ $u_n = \int u'_n(x) dx$ ★₅

Step 2-5 $y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) + \cdots + u_n(x) y_n(x)$

$$u'_k(x) = \frac{W_k}{W}$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y'_1 & y'_2 & y'_3 & \cdots & y'_n \\ y''_1 & y''_2 & y''_3 & \cdots & y''_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

page 149
Wronskian

☆☆

$$W_k = \begin{vmatrix} y_1 & y_2 & \cdots & y_{k-1} & 0 & y_{k+1} & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_{k-1} & 0 & y'_{k+1} & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{k-1}^{(n-2)} & 0 & y_{k+1}^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{k-1}^{(n-1)} & f(x) & y_{k+1}^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$\boxed{f(x) = g(x) / a_n(x)}$$

☆☆

☆₃

W_k : replace the k^{th} column of W by

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$

$$f(x) = \frac{g(x)}{a_n(x)}$$

For example, when $n = 3$,

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}$$

Exercise 30 $y''' + 4y' = \sec 2x$

auxiliary $m^3 + 4m = 0$
 $m = 0, \pm i2$

Complementary function: $y_c = c_1 + c_2 \cos 2x + c_3 \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ 1 & \cos 2x & \sin 2x \\ 0 & -2 \sin 2x & 2 \cos 2x \\ 0 & -4 \cos 2x & -4 \sin 2x \end{vmatrix} = 8$$

$$W = 1 \cdot \begin{vmatrix} -2 \sin 2x & 2 \cos 2x \\ -4 \cos 2x & -4 \sin 2x \end{vmatrix} = 8$$

$$W_1 = \begin{vmatrix} 0 & \cos 2x & \sin 2x \\ 0 & -2 \sin 2x & 2 \cos 2x \\ \sec 2x & -4 \cos 2x & -4 \sin 2x \end{vmatrix} = 2 \sec 2x$$

$$W_1 = \sec 2x \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} = 2 \sec 2x$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin 2x \\ 0 & 0 & 2 \cos 2x \\ 0 & \sec 2x & -4 \sin 2x \end{vmatrix} = -2$$

$$W_3 = \begin{vmatrix} 1 & \cos 2x & 0 \\ 0 & -2 \sin 2x & 0 \\ 0 & -4 \cos 2x & \sec 2x \end{vmatrix} = -2 \tan 2x$$

$$u_1' = \frac{W_1}{W} = \frac{\sec 2x}{4} \quad u_2' = \frac{W_2}{W} = \frac{-1}{4} \quad u_3' = \frac{W_3}{W} = \frac{-\tan 2x}{4}$$

$$u_1 = \frac{1}{8} \ln |\sec 2x + \tan 2x| \quad u_2 = \frac{-x}{4} \quad u_3 = \frac{1}{8} \ln |\cos 2x|$$

$\int \sec(ax) dx = \frac{1}{a} \ln |\sec(ax) + \tan(ax)| + C$

$$y(x) = c_1 + c_2 \cos 2x + c_3 \sin 2x$$

$$+ \frac{1}{8} \ln |\sec 2x + \tan 2x| + \frac{-x}{4} \cos 2x + \frac{1}{8} (\ln |\cos 2x|) \sin 2x$$

for $-\pi/4 < x < \pi/4$

$\sec 2x$

Note: $-\pi/4, \pi/4$ are singular points

singular points

$x: \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}, \pm \frac{5\pi}{4}, \dots$

4-6-7 本節需注意的地方

(1) 養成先解 associated homogeneous equation 的習慣

(2) 記熟幾個重要公式

(3) 這裡 $||$ 指的是 determinant

(4) 算出 $u_1'(x)$ 和 $u_2'(x)$ 後別忘了作積分

特別要小心

(5) $f(x) = g(x)/a_n(x)$ (和 1st order 的情形一樣，使用 standard form)

(6) 計算 $u_1'(x)$ 和 $u_2'(x)$ 的積分時， $+c$ 可忽略

因為我們的目的是算 particular solution y_p

y_p 是任何一個能滿足原式的解

(7) 這方法解的範圍，不包含 $a_n(x) = 0$ 的地方

4-7 Cauchy-Euler Equation

4-7-1 解法限制條件

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = g(x)$$

限制條件中,

not constant coefficients

but the coefficients of $y^{(k)}(x)$ have the form of $a_k x^k$

a_k is some constant

associated homogeneous
equation

particular solution

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = 0$$

4-7-2 解法

Associated homogeneous equation of the Cauchy-Euler equation

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \cdots + a_1 x y'(x) + a_0 y = 0$$

Guess the solution as $y(x) = x^m$, then ☆☆ 比較 Sec 4-3 $y = e^{mx}$

$$\begin{aligned}
 & k=h \quad a_n x^n m(m-1)(m-2)\cdots(m-n+1) x^{m-n} + \\
 & k=h-1 \quad a_{n-1} x^{n-1} m(m-1)(m-2)\cdots(m-n+2) x^{m-n+1} + \\
 & k=h-2 \quad a_{n-2} x^{n-2} m(m-1)(m-2)\cdots(m-n+3) x^{m-n+2} + \\
 & \quad \vdots \\
 & k=1 \quad + a_1 x m x^{m-1} \rightarrow a_1 m x^m \\
 & k=0 \quad + a_0 x^m = 0
 \end{aligned}$$

$$\begin{aligned}
 & y = x^m \text{ 代 } \lambda a_k x^k y^{(k)}(x) \\
 & a_k x^k m(m-1)\cdots(m-k+1) x^{m-k} \\
 & = a_k m(m-1)\cdots(m-k+1) x^m
 \end{aligned}$$

$k=0 \sim n$

Delete x^m on the previous page

$$\begin{aligned}
 & a_n m(m-1)(m-2)\cdots(m-n+1) \\
 & + a_{n-1} m(m-1)(m-2)\cdots(m-n+2) \\
 & + a_{n-2} m(m-1)(m-2)\cdots(m-n+3) \\
 & \vdots \\
 & + a_1 m \\
 & + a_0 = 0
 \end{aligned}$$

\Rightarrow \star_2
 auxiliary function

比較: 和 constant coefficient

時有何不同? Sec 4-3 page 180

$$\frac{d^k}{dx^k} \Rightarrow m^k$$

規則: 把 $x^k \frac{d^k}{dx^k}$ 變成 $\frac{m!}{(m-k)!} = m(m-1)\cdots(m-k+1)$

$$m! = 1 \cdot 2 \cdot 3 \cdots (m-1) m$$

4-7-3 For the 2nd Order Case

$$a_2 x^2 y''(x) + a_1 x y'(x) + a_0 y = 0$$

$x=0$ is a singular point

auxiliary function: $\frac{m!}{(m-2)!} = m(m-1)$

$$a_2 m(m-1) + a_1 m + a_0 = 0$$

$$a_2 m^2 + (a_1 - a_2)m + a_0 = 0$$

roots

$$m_1 = \frac{a_2 - a_1 + \sqrt{(a_1 - a_2)^2 - 4a_2 a_0}}{2a_2}$$

$$m_2 = \frac{a_2 - a_1 - \sqrt{(a_1 - a_2)^2 - 4a_2 a_0}}{2a_2}$$

★ 3-1

[Case 1]: $m_1 \neq m_2$ and m_1, m_2 are real

two independent solution of the homogeneous part:

$$x^{m_1} \quad \text{and} \quad x^{m_2}$$

$$y_c = c_1 x^{m_1} + c_2 x^{m_2}$$

[Case 2]: $m_1 = m_2$ $(a_1 - a_2)^2 - 4a_2 a_0 = 0$

★ 3-2

Use the method of reduction of order Sec 4-2

$$y_1 = x^{m_1} \quad m_1 = \frac{a_2 - a_1}{2a_2}$$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx = x^{m_1} \int \frac{e^{-\int \frac{a_1}{a_2 x} dx}}{x^{2m_1}} dx$$

Note 1: 原式 $\longrightarrow y''(x) + \frac{a_1}{a_2 x} y'(x) + \frac{a_0}{a_2 x^2} y = 0, \quad P(x) = \frac{a_1}{a_2 x}$

Note 2: 此時 $m_1 = m_2 = \frac{a_2 - a_1}{2a_2}$

$$y_2(x) = x^{m_1} \int \frac{e^{-\int \frac{a_1}{a_2 x} dx}}{x^{2m_1}} dx = x^{m_1} \int \frac{e^{-\frac{a_1}{a_2} \ln|x|}}{x^{2m_1}} dx = x^{m_1} \int \frac{|x|^{-\frac{a_1}{a_2}}}{x^{2m_1}} dx \quad m_1 = \frac{a_2 - a_1}{2a_2}$$

$$= (-1)^{\frac{a_1}{a_2}} x^{m_1} \int x^{-\frac{a_1}{a_2}} x^{\frac{a_1 - a_2}{a_2}} dx = x^{m_1} \int x^{-1} dx = \underline{x^{m_1} \ln|x|}$$

If $y_2(x)$ is a solution of a homogeneous DE

then $c y_2(x)$ is also a solution of the homogeneous DE

If we constrain that $x > 0$, then
if $x < 0$

$$y_2 = x^{m_1} \ln x$$

$$y_2 = x^{m_1} \ln(-x) = x^{m_1} \ln|x|$$

$x=0$ is a
singular point

solution interval $x \in (0, \infty)$

$$\star_{3-2} \boxed{y_c = c_1 x^{m_1} + c_2 x^{m_1} \ln x}$$

The formulas in Sec 4-7
are similar to those in Sec 4-3
but x is replaced by $\ln x$

Sec 4-3: 重根 乘 x

Sec 4-7 重根 乘 $\ln x$

[Case 3]: $m_1 \neq m_2$ and m_1, m_2 are the form of $(a_1 - a_2)^2 - 4 a_2 a_0 < 0$

$$\star_{3-3} \quad m_1 = \alpha + j\beta \quad m_2 = \alpha - j\beta$$

two independent solution of the homogeneous part:

$$x^{\alpha+j\beta} \quad \text{and} \quad x^{\alpha-j\beta}$$

$$y_c = C_1 x^{\alpha+j\beta} + C_2 x^{\alpha-j\beta}$$

$$x^{\alpha+j\beta} = (e^{\ln x})^{\alpha+j\beta} = e^{(\alpha+j\beta)\ln x} = e^{\alpha \ln x} e^{j\beta \ln x}$$

$$= x^\alpha (\cos(\beta \ln x) + j \sin(\beta \ln x))$$

$$\text{同理 } x^{\alpha-j\beta} = x^\alpha (\cos(\beta \ln x) - j \sin(\beta \ln x))$$

$$y_c = x^\alpha [(C_1 + C_2) \cos(\beta \ln x) + j(C_1 - C_2) \sin(\beta \ln x)]$$

$$y_c = x^\alpha [c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$$

$$c_1 = C_1 + C_2$$

$$c_2 = j(C_1 - C_2)$$

$$\star_{3-3} \quad \text{page 183, } x \rightarrow \ln x$$

Example 1 (text page 167)

$$x^2 y''(x) - 2xy'(x) - 4y = 0$$

$$m(m-1) - 2m - 4 = 0 \quad m^2 - 3m - 4 = 0 \quad m = 4, -1$$

$$C_1 x^4 + C_2 x^{-1}$$

Example 2 (text page 168)

$$4x^2 y''(x) + 8xy'(x) + y = 0$$

$$4m(m-1) + 8m + 1 = 0 \quad 4m^2 + 4m + 1 = 0$$

$$m = -\frac{1}{2}, -\frac{1}{2}$$

$$C_1 x^{-\frac{1}{2}} + C_2 x^{-\frac{1}{2}} \ln x$$

Example 3 (text page 169)

$$4x^2 y''(x) + 17y = 0$$

$$y(1) = -1 \quad y'(1) = -\frac{1}{2}$$

$$4m(m-1) + 17 = 0$$

$$4m^2 - 4m + 17 = 0$$

$$(2m-1)^2 = -16$$

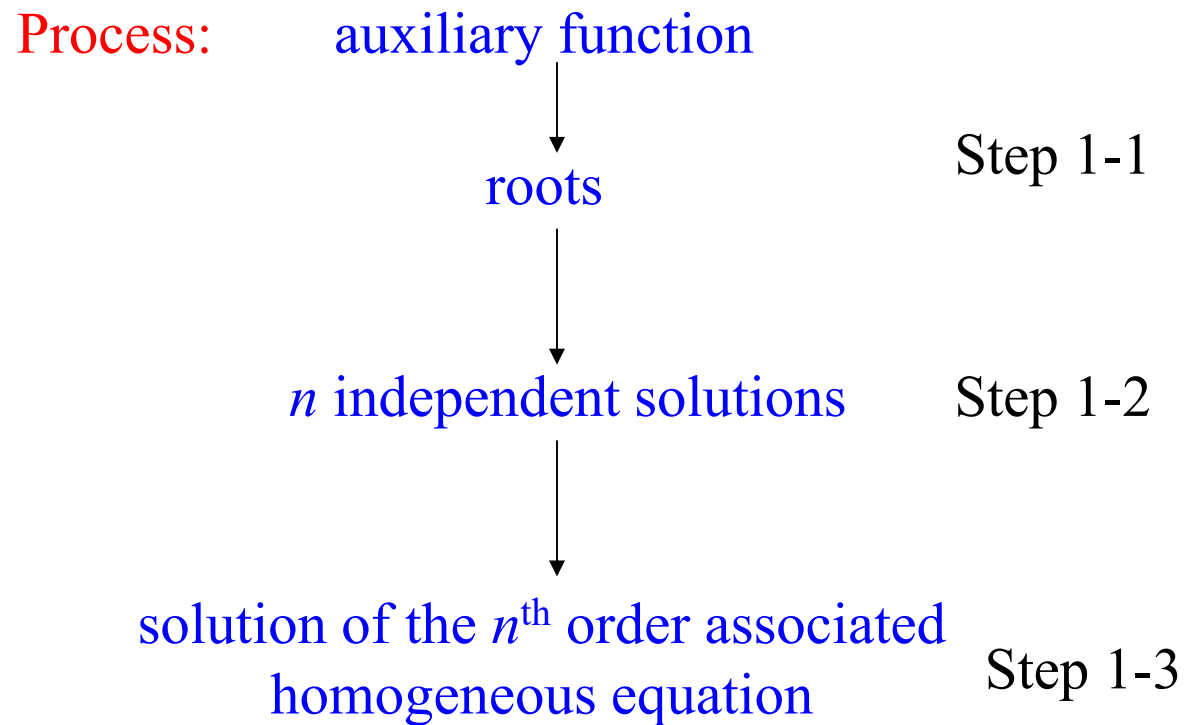
$$2m-1 = \pm i4$$

$$m = \frac{1 \pm i4}{2}$$

$$\alpha = \frac{1}{2}, \beta = 2$$

$$x^{\frac{1}{2}} (C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x))$$

4-7-4 For the Higher Order Case



★4-1

(1) 若 auxiliary function 在 m_0 的地方 只有一個根

$$x^{m_0}$$

是 associated homogeneous equation 的其中一個解

★4-2

(2) 若 auxiliary function 在 m_0 的地方 有 k 個重根

$$x^{m_0}, x^{m_0} \ln x, x^{m_0} (\ln x)^2, \dots, x^{m_0} (\ln x)^{k-1}$$

皆為 associated homogeneous equation 的解

★4-3

(3) 若 auxiliary function 在 $\alpha + j\beta$ 和 $\alpha - j\beta$ 的地方各有一個根
(未出現重根)

$$x^\alpha \cos(\beta \ln x), \quad x^\alpha \sin(\beta \ln x)$$

是 associated homogeneous equation 的其中二個解

★4-4

(4) 若 auxiliary function 在 $\alpha + j\beta$ 和 $\alpha - j\beta$ 的地方皆有 k 個重根

$$x^\alpha \cos(\beta \ln x), \quad x^\alpha \cos(\beta \ln x) \ln x, \quad x^\alpha \cos(\beta \ln x) (\ln x)^2, \quad \dots, \quad x^\alpha \cos(\beta \ln x) (\ln x)^{k-1}$$

$$x^\alpha \cos(\beta \ln x) (\ln x)^{k-1}$$

$$x^\alpha \sin(\beta \ln x), \quad x^\alpha \sin(\beta \ln x) \ln x, \quad x^\alpha \sin(\beta \ln x) (\ln x)^2, \quad \dots, \quad x^\alpha \sin(\beta \ln x) (\ln x)^{k-1}$$

$$x^\alpha \sin(\beta \ln x) (\ln x)^{k-1}$$

是 associated homogeneous equation 的其中 $2k$ 個解

Example 4 (text page 169)

$$x^3 y'''(x) + 5x^2 y''(x) + 7xy'(x) + 8y = 0$$

auxiliary function

$$m(m-1)(m-2) + 5m(m-1) + 7m + 8 = 0$$

$$m^3 - 3m^2 + 2m + 5m^2 - 5m + 7m + 8 = 0$$

$$m^3 + 2m^2 + 4m + 8 = 0$$

$$(m+2)(m^2+4) = 0$$

$$m = -2, \pm i2$$

$$y = c_1 x^{-2} + c_2 \cos(2 \ln x) + c_3 \sin(2 \ln x)$$

4-7-5 Nonhomogeneous Case

To solve the nonhomogeneous Cauchy-Euler equation:

Method 1: (See Example 5) ★₅

(1) Find the complementary function (general solutions of the associated homogeneous equation) from the rules on pages 252-255, 259-260.

(2) Use the method in Sec. 4-6 (Variation of Parameters) to find the particular solution.

(3) Solution = complementary function + particular solution

★₆

Method 2: See Example 6 , 很重要

Set $x = e^t$, $t = \ln x$

Example 5 (text page 169, illustration for method 1)

$$x^2 y''(x) - 3xy'(x) + 3y = 2x^4 e^x$$

Step 1 solution of the associated homogeneous equation

auxiliary function

$$m(m-1) - 3m + 3 = 0 \quad m^2 - 4m + 3 = 0 \quad m_1 = 1$$

$$y_c = c_1 x + c_2 x^3$$

Step 2-2 Particular solution $W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3$

Sec 4-6
page 236

standard $m_2 = 3$

$y''(x) - \frac{3}{x}y' + \frac{3}{x^2}y = \underline{\underline{2x^2 e^x}}$
↑
 $f(x)$

$$W_1 = \begin{vmatrix} 0 & x^3 \\ \underline{2x^2 e^x} & 3x^2 \end{vmatrix} = -2x^5 e^x \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & \underline{2x^2 e^x} \end{vmatrix} = 2x^3 e^x$$

↑ Note!

Step 2-3 $u_1' = \frac{W_1}{W} = -x^2 e^x \quad u_2' = \frac{W_2}{W} = e^x$

Step 2-4 $u_1 = \int u_1' dx = -x^2 e^x + 2x e^x - 2e^x$ ~~+~~

$$u_2 = \int u_2' dx = e^x$$

Step 2-5 $y_p = u_1 y_1 + u_2 y_2 = 2x^2 e^x - 2x e^x$

Step 3 $y = c_1 x + c_2 x^3 + 2x^2 e^x - 2x e^x$

Example 6 (text page 170, illustration for method 2)

$$x^2 y''(x) - xy'(x) + y = \ln x$$

Set $x = e^t$, $t = \ln x$ (Step 1)

$$\frac{dy}{dx} = \frac{dt}{dx} \frac{dy}{dt} = \frac{1}{x} \frac{dy}{dt} \quad (\text{chain rule})$$

$$x \frac{dy}{dx} = \frac{dy}{dt} = D_t y$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{dt}{dx} \frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{1}{x} \frac{d}{dt} \left(\frac{1}{x} \frac{dy}{dt} \right)$$

$$= \frac{1}{x^2} \frac{d^2 y}{dt^2} + \frac{1}{x} \left(\frac{d}{dt} \frac{1}{x} \right) \left(\frac{dy}{dt} \right) = \frac{1}{x^2} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right)$$

$$\begin{aligned} x^2 \frac{d^2 y}{dx^2} &= \frac{d^2 y}{dt^2} - \frac{dy}{dt} \\ &= (D_t^2 - D_t) y \\ &= D_t(D_t - 1) y \end{aligned}$$

Therefore, the original equation is changed into

$$\frac{d^2}{dt^2} y(t) - 2 \frac{d}{dt} y(t) + y(t) = t$$

linear, constant coefficients (Step 2)

This process can be simplified using the auxiliary function.

Sec. 4-3, Sec. 4-4

266

$$\frac{d^2}{dt^2} y(t) - 2 \frac{d}{dt} y(t) + y(t) = t$$

$$y_p(t) = At + B$$

$$m^2 - 2m + 1 = 0 \quad m = 1, 1$$
$$-2A + At + B = t, \quad A = 1, B = 2$$

$$\Rightarrow y(t) = c_1 e^t + c_2 t e^t + t + 2$$

(Step 3)

$$\Rightarrow y(x) = c_1 x + c_2 x \ln x + \ln x + 2$$

(別忘了 $t = \ln x$ 要代回來)

★ 6-1

(Step 4)

Note 1: 以此類推

auxiliary for $x^k \frac{d^k}{dx^k}$

$$x^k \frac{d^k y}{dx^k} = (D_t - k + 1) \cdots (D_t - 1) D_t y$$

D_t means $\frac{d}{dt}$

(Step 2)

★ 6-2

(Step 2) Determine the auxiliary function, then replace m by D_t

Note 2: 簡化計算的小技巧: 使用 Cauchy-Euler equation 的 auxiliary function

4-7-6 本節要注意的地方

(1) 本節公式記憶的方法：

把 Section 4-3 的 e^x 改成 x ， x 改成 $\ln(x)$

把 auxiliary function 的 m^n 改成 $m(m-1)(m-2)\cdots(m-n+1)$

(2) 如何解 particular solution?

Variation of Parameters 的方法

(3) 解的範圍將不包括 $x=0$ 的地方 (Why?)

Extra Problems:

How do we solve

$$(1) \quad xy''(x) + y'(x) = 0 \rightarrow x^2 y''(x) + xy'(x) = 0$$

★ 7-1

$$(2) \quad (x-1)^2 y''(x) + y(x) = 0$$

★ 7-2

x^m is replaced by $(x-1)^m$

auxiliary $m(m-1) + 1 = 0$

$$m^2 - m + 1 = 0$$

$$m = \frac{1 \pm i\sqrt{3}}{2}$$

$$(x-1)^{\frac{1}{2}} \left(C_1 \cos\left(\frac{\sqrt{3}}{2} \ln|x-1|\right) + C_2 \sin\left(\frac{\sqrt{3}}{2} \ln|x-1|\right) \right)$$

$\alpha = \frac{1}{2}, \beta = \frac{\sqrt{3}}{2}$

還有很多 linear DE 沒有辦法解，怎麼辦

- (1) numerical approach (Section 4-9-3)
- (2) using special function (Chap. 6)
- (3) Laplace transform and Fourier transform (Chaps. 7, 11, 14)
- (4) 查表 (table lookup)

- (1) 即使用了 Section 4-7 的方法，大部分的 DE 還是沒有辦法解
- (2) 所幸，自然界真的有不少的例子是 linear DE
甚至是 constant coefficient linear DE

Exercises for practicing

Section 4-6 4, 5, 8, 13, 14, 17, 18, 21, 25, 28, 29, 34

Section 4-7 11, 17, 18, 20, 21, 24, 32, 35, 36, 37, 40, 42

Review 4 27, 28, 29, 30, 32, 42