多項式解

Chapter 6 Series Solutions of Linear Equations

當 DE 為 linear 且 coefficients 為 polynomials

→ **假設 DE 的 solutions 為 polynomial 的型態**

(和 Cauchy-Euler Method 以及 Taylor Series 的概念相近)

$$\boxed{y(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^n}$$ ☆☆

被稱作為 power series centered at x_0

- Power series 觀念的複習 (Sec. 6-1)
- 解法
  - x_0 is a non-singular point (Sec. 6-2)
  - x_0 is a singular point
    - regular singular point (Sec. 6-3)
    - irregular singular point
- Examples (Sec. 6-4) 只教不考

Section 6-1 Reviews of Power Series

6-1-1 定義

1. Power series 名1

$$\sum_{n=0}^{\infty} c_n (x-x_0)^n = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots\cdots\cdots$$

2. Convergence: $\lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-x_0)^n$ exists

☆1

測試方法：Ratio test (test for convergence)

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-x_0)^{n+1}}{c_n (x-x_0)^n} \right| = L$$

$L < 1$: convergent $L > 1$: divergent

$L = 1$: 不一定

名2

3. Radius of Convergence R

$L < 1$ if $|x - x_0| < R$ $L > 1$ if $|x - x_0| > R$

Ex 1: $|x-3| < 2$ convergent, $R=2$

Example 1 (text page 238)

For the Power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n n}$

ratio test

$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n} = 1$

$$\lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{2^{n+1}(n+1)}}{\frac{(x-3)^n}{2^n n}} \right| = |x-3| \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{|x-3|}{2}$$

$\sum_{n=1}^{\infty} \frac{1}{2n} - \frac{1}{2n-1} = \sum_{n=1}^{\infty} \frac{-1}{2n(2n-1)}$

$x<1,\ x>5$ divergent

$\frac{|x-3|}{2} < 1$ for $1 < x < 5$

Interval of convergence: (1, 5)

However, when $x = 1$, the power series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (convergent)

$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots\cdots$

When $x = 5$, the power series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$ (divergent)

Therefore, the interval of convergence is modified as: $1 \le x < 5$

Interval of convergence: [1, 5)

6-1-2 Maclaurin Series (Taylor Series)

☆2

$$y(x)=y(x_0)+\frac{y'(x_0)}{1!}(x-x_0)+\frac{y''(x_0)}{2!}(x-x_0)^2+\frac{y'''(x_0)}{3!}(x-x_0)^3+$$

$$+\frac{y^{(4)}(x_0)}{4!}(x-x_0)^4+\cdots\cdots$$

$$y(x)=\sum_{n=0}^{\infty}\frac{y^{(n)}(x_0)}{n!}(x-x_0)^n$$

☆3

$a_0 + a_0 r + a_0 r^2 + \cdots = \frac{a_0}{1-r}$

Maclaurin Series (Taylor Series)	Interval of Convergence
$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots\cdots$	$(-\infty, \infty)$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots\cdots$	$(-\infty, \infty)$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\cdots\cdots$	$(-\infty, \infty)$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots\cdots$	$(-1, 1]$
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots\cdots$ 公比=x，首項=1	$(-1, 1)$

$e^{2x} = ?$ $\cos 3x = ?$

Example 2 (text page 240)

Find a power series representation of $e^x \sin x$

☆4 多项式乘法

$$e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

$$= (1)x + (1)x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5 + \cdots\cdots$$

$$= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots\cdots$$

Section 6-2 Solutions about Ordinary Points

Suppose that the solution is $\sum_{n=0}^{\infty} c_n (x - x_0)^n$

6-2-1 方法適用情形

限制式

(1) Linear

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots\cdots + a_1(x)y' + a_0(x)y = g(x)$$

(2) x_0 is not a singular point

(3) It is better that $a_0(x)$, $a_1(x)$, …., $a_n(x)$, $g(x)$ are all polynomials.

(or can be expressed by Taylor series)

6-2-2 解法流程 ☆☆

☆☆

名1

Step 1 將 $y(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^n$ 代入 (x_0 必需為 ordinary point)

For simplification, we usually set $x_0 = 0$ if 0 is not a singular point.

不是 singular point 的
即為 ordinary point

Step 2 對齊 (一律變成 $(x - x_0)^k$)

Step 3 合併

Step 4 比較係數，將 c_n 之間的關係找出來

Step 5 Obtained independent solutions and general solution

6-2-3 例子

Example 5 (text page 246)

$$y'' - xy = 0$$

no singular point, $R \to \infty$

$|x - x_0| < R$, the solution is convergent for all x page 293

Set $y(x) = \sum_{n=0}^{\infty} c_n x^n$ since $P(x) = 0$ and $Q(x) = x$ are analytic at 0

Step 1 $$y'' + xy = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} c_n x^n = 0$$

Note! ☆2

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

set $k = n - 2$ $n = k+2$

set $k = n + 1$ $n = k-1$

Step 2 對齊 $$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k = 0$$

☆3

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k = 0$$

$k=0$ $\quad$ $k=1\sim\infty$

☆4

Step 3 $\quad 2c_2 + \sum_{k=1}^{\infty}[c_{k+2}(k+2)(k+1) - c_{k-1}]x^k = 0$

Step 4 $\quad \boxed{2c_2 = 0}$ $\qquad \boxed{c_{k+2}(k+2)(k+1) - c_{k-1} = 0}$

☆5

$c_2 = 0$ $\qquad$ 或 $\boxed{c_{k+2} = \dfrac{c_{k-1}}{(k+2)(k+1)}}$ $\quad k=1\sim\infty$

recurrence relation

c_0, c_1 給定之後

$k=1 \qquad c_3 = \dfrac{c_0}{2\cdot 3}$

$k=2 \qquad c_4 = \dfrac{c_1}{3\cdot 4}$

$k=3 \qquad c_5 = \dfrac{c_2}{4\cdot 5} = 0$

$$\boxed{c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)}}$$

以此類推，所有的 c_n 的值都可以算出來 (以 c_0 或 c_1 表示)

$$k=4 \qquad c_6 = \frac{c_3}{5\cdot 6} = \frac{c_0}{2\cdot 3\cdot 5\cdot 6}$$

$$k=5 \qquad c_7 = \frac{c_4}{6\cdot 7} = \frac{c_1}{3\cdot 4\cdot 6\cdot 7}$$

$$k=6 \qquad c_8 = \frac{c_5}{7\cdot 8} = 0$$

$$k=7 \qquad c_9 = \frac{c_6}{8\cdot 9} = \frac{c_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}$$

$$k=8 \qquad c_{10} = \frac{c_7}{9\cdot 10} = \frac{c_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}$$

$$k=9 \qquad c_{11} = \frac{c_8}{10\cdot 11} = 0$$

$$\vdots$$

$$\vdots$$

related to c_0

group 1 $c_0, c_3, c_6, c_9 \cdots\cdots$ $3k$

related to c_1

group 2 $c_1, c_4, c_7, c_{10} \cdots\cdots$ $3k+1$

group 3 $c_2, c_5, c_8, c_{11} \cdots\cdots$ $3k+2$

$\overset{\shortparallel}{0}$

Step 5

$$y(x)=\sum_{n=0}^{\infty}c_n x^n=\sum_{k=0}^{\infty}c_{3k}x^{3k}+\sum_{k=0}^{\infty}c_{3k+1}x^{3k+1}+\cancel{\sum_{k=0}^{\infty}c_{3k+2}x^{3k+2}}$$

$$=c_0\underbrace{\left[1+\frac{x^3}{2\cdot 3}+\frac{x^6}{2\cdot 3\cdot 5\cdot 6}+\frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}+\cdots\cdots\right]}_{y_1}$$

$$+c_1\underbrace{\left[x+\frac{x^4}{3\cdot 4}+\frac{x^7}{3\cdot 4\cdot 6\cdot 7}+\frac{x^{10}}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}+\cdots\cdots\right]}_{y_2}$$

☆6 $y(x)=c_0y_1(x)+c_1y_2(x)$

1 is not the case where $k=0$

☆7

$$y_1(x)=1+\sum_{k=1}^{\infty}\frac{1}{2\cdot 3\cdots(3k-1)(3k)}x^{3k}$$

不寫在 Σ 裡面

x is not the case where $k=0$

$$y_2(x)=x+\sum_{k=1}^{\infty}\frac{1}{3\cdot 4\cdots(3k)(3k+1)}x^{3k+1}$$

☆12

ratio test:

for y_1, $\lim_{k\to\infty}\left|\frac{(k+1)^{th}\text{ term}}{k^{th}\text{ term}}\right|$

$=\lim_{k\to\infty}\left|\frac{x^3}{(3k+2)(3k+3)}\right|=0$ for all x

y_1 is convergent for all x

for y_2 $\lim_{k\to\infty}\left|\frac{x^3}{(3k+3)(3k+4)}\right|=0$

y_2 is convergent for all x

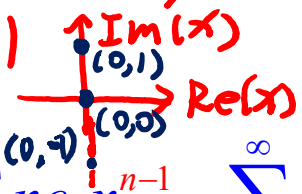
Example 6 (text page 248)

no real singular point
set $x_0 = 0$

$$(x^2+1)y'' + xy' - y = 0 \qquad y(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{(analytic at } x = 0\text{)}$$

Radius of convergence?

singular points: $x = \pm i$
if $x_0 = 0$, $R = 1$



$|x - x_0| < R$
$\Rightarrow |x| < 1$

Step 1 ☆2

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$k = n$ $\quad$ $k = n-2$ $\quad$ $k = n$ $\quad$ $k = n$

$n = k+2$

Step 2 ☆3

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} kc_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

k=0 k=1 k=1 k=0 k=1

Step 3

☆4 $\underbrace{2c_2 - c_0}_{k=0} \underbrace{+(6c_3 + c_1 - c_1)x}_{k=1}$

$$+\sum_{k=2}^{\infty}[k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k]x^k = 0$$

$$2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty}[(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}]x^k = 0$$

Step 4

☆5 $\boxed{2c_2 - c_0 = 0}$ $\boxed{6c_3 = 0}$ $\boxed{c_{k+2} = -\frac{k-1}{k+2}c_k}$

k=0 k=1 k=2, 3, ...

$\boxed{2c_2 - c_0 = 0}$ $\boxed{6c_3 = 0}$ $\boxed{c_{k+2} = \frac{1-k}{k+2} c_k}$

$c_2 = c_0/2$ $c_3 = 0$

$k \geq 2$

規律 $c_{2k} = \frac{(-1)^{k-1} 1\cdot 3\cdot 5\cdots(2k-3)}{2^k k!}$

☆7

c_0, c_1 給定之後

$c_2 = c_0/2$

$c_3 = 0$

k=2 $c_4 = -\frac{1}{4} c_2 = -\frac{1}{2\cdot 4} c_0 = -\frac{1}{2^2 \cdot 2!} c_0$

k=3 $c_5 = -\frac{2}{5} c_3 = 0$

k=4 $c_6 = -\frac{3}{6} c_4 = \frac{3}{2\cdot 4\cdot 6} c_0 = \frac{3}{2^3 \cdot 3!} c_0$

k=5 $c_7 = -\frac{4}{7} c_5 = 0$

k=6 $c_8 = -\frac{5}{8} c_6 = -\frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8} c_0 = -\frac{3\cdot 5}{2^4 \cdot 4!} c_0$

k=7 $c_9 = -\frac{6}{9} c_7 = 0$

$2\cdot 4\cdot 6\cdot 8$
$= 2^4(1\cdot 2\cdot 3\cdot 4)$

$\vdots$

$\vdots$

no constraint for c_1

Step 5 $y(x)=\sum_{n=0}^{\infty}c_n x^n=\sum_{k=0}^{\infty}c_{2k}x^{2k}+\sum_{k=0}^{\infty}c_{2k+1}x^{2k+1}$

☆6

$$=c_0\underbrace{\left[1+\frac{x^2}{2}-\frac{x^4}{2^2\cdot 2!}+\frac{3x^6}{2^3\cdot 3!}-\frac{3\cdot 5x^8}{2^4\cdot 4!}+\frac{3\cdot 5\cdot 7x^{10}}{2^5\cdot 5!}\cdots\cdots\right]}_{y_1}+c_1\underbrace{x}_{y_2}$$

☆8

$$y(x)=c_0y_1(x)+c_1y_2(x)$$

☆7-1 不寫在 Σ 裡面 $1, \frac{x^2}{2}$ are not the cases where $n=0, n=1$

☆7 $y_1(x)=1+\frac{1}{2}x^2+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^{2n}$ $|x|<1$ (Why?)

$y_2(x)=x$

☆12 ratio test:

y_1 is convergent for $|x|<1$

y_2 is convergent (only one term)

for y_1

$$\lim_{x\to\infty}\left|\frac{(k+1)^{th}\text{ term}}{k^{th}\text{ term}}\right|$$

$$=\lim_{k\to\infty}\left|\frac{(2k-1)x^2}{2(k+1)}\right|=\lim_{k\to\infty}\left|\frac{2k-1}{2k+2}x^2\right|=\lim_{k\to\infty}|x^2|$$

Example 8 (text page 250)

$$y'' + (\cos x)y = 0$$ ☆9

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\cdots\cdots$$

Step1 $$y(x) = \sum_{n=0}^{\infty} c_n x^n$$ set $x_0 = 0$

多項式乘法

$$\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\cdots\cdots\right) \sum_{n=0}^{\infty} c_n x^n = 0$$

直接乘開來比較係數

n=2 n=3 n=4 n=5

$$2c_2 + c_0 + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{c_0}{2}\right)x^2 + \left(20c_5 + c_3 - \frac{c_1}{2}\right)x^3$$

$$+ \cdots\cdots = 0$$

$$2c_2 + c_0 + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{c_0}{2}\right)x^2 + \left(20c_5 + c_3 - \frac{c_1}{2}\right)x^3 + \cdots\cdots = 0$$

$$2c_2 + c_0 = 0 \Rightarrow c_2 = -c_0 / 2$$

$$6c_3 + c_1 = 0 \Rightarrow c_3 = -c_1 / 6$$

$$12c_4 + c_2 - c_0 / 2 = 0 \Rightarrow 12c_4 - c_0 = 0 \Rightarrow c_4 = c_0 / 12$$

$$20c_5 + c_3 - c_1 / 2 = 0 \Rightarrow 20c_5 - 2c_1 / 3 = 0 \Rightarrow c_5 = c_1 / 30$$

$$\vdots$$

$$\vdots$$

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots\cdots\cdots \qquad y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots\cdots\cdots$$

$$y(x) = c_0 y_1(x) + c_1 y_2(x)$$

$n = 2k$ $\quad$ $n = 2k+1$

6-2-4 定義

名之

1. Analytic at x_0: If a function can be expressed as a power series and the radius of convergence of the power series is nonzero

簡單的判斷 $f(x)$ 在 x_0 是否為analytic 方法

必要 (1) $f(x_0)$ should be neither ∞ nor $-\infty$

(2) $f^{(m)}(x_0)$ should be neither ∞ nor $-\infty$

$m = 1, 2, 3, \ldots\ldots\ldots\ldots$

$f(x)$

ex: $=(x-1)^{\frac{1}{2}}$

is not analytic at $x=1$

$f'(x) = \frac{1}{2}(x-1)^{-\frac{1}{2}}$

$(x-x_0)^{\alpha}$ is not analytic at x_0 if α is not an integer

2. Ordinary Point and Singular Point:

- For the 2nd order linear DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \longrightarrow y'' + P(x)y' + Q(x)y = 0$$

Definition 6.2.1 考1

☆10 x_0 is an ordinary point of the 2nd order linear DE if both $P(x)$ and $Q(x)$ are **analytic** at x_0

Otherwise, x_0 is a singular point.

Theorem 6.2.1

☆11

If x_0 is an ordinary point of the 2nd order homogeneous linear DE, then we can find two linearly independent solutions in the form of a power series centered at x_0, i.e.,

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

- For the kth order linear DE

$$a_k(x)y^{(k)} + a_{k-1}(x)y^{(k-1)} + \cdots\cdots + a_1(x)y' + a_0(x)y = 0$$

$$\longrightarrow \quad y^{(k)} + P_{k-1}(x)y^{(k-1)} + \cdots\cdots + P_1(x)y' + P_0(x)y = 0$$

$$P_m(x) = \frac{a_m(x)}{a_k(x)}$$

Extension of Definition 6.2.1 ☆ 10-1

x_0 is an ordinary point of the kth order linear DE if $P_0(x)$, $P_1(x)$, $P_2(x)$, ……. , $P_{k-1}(x)$, are **analytic** at x_0

Otherwise, x_0 is a singular point .

Extension of Theorem 6.2.1 ☆ 11-1

If x_0 is an ordinary point of the kth order homogeneous linear DE, then we can find $\boxed{k \text{ linearly independent solutions}}$ in the form of a power series centered at x_0 , i.e.,

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

6-2-5 Interval of Convergence 的判斷方法

☆12 ratio test

判斷方法一：找出 $\lim_{n\to\infty}\left|\frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n}\right|<1$ 的條件

判斷方法二 ☆12-1

（較快速，但較不精準

找出的收斂的範圍有時會比實際的收斂範圍小）

$$|x-x_0|<R$$

其中 R 是 x_0 和最近的 singular point 的距離

Singular point can be a complex number , see Example 6

超過這個範圍未必不為 convergence

6-2-6 思考

(1) 對於 nonhomogeneous 的情形.....

$$a_n(x)y^{(n)}(x)+a_{n-1}(x)y^{(n-1)}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g(x)$$

(2) 這方法還可以用在什麼情形？

6-2-7 本節需注意的地方

(1) 要了解幾個重要定義： (a) convergence, (b) radius of convergence, (c) analytic at x_0, (d) singular point, (e) ordinary point

(2) 複習一下 Taylor (Maclaurin) series (如 page 276)

(3) Index 的地方計算要小心

(a) 先都化成 x^k 再合併，(b) 頭幾項可能要獨立出來

(c) Index 對齊計算要小心

(4) n^{th} order homogeneous linear DE 要有 n 個 linearly independent 解

(5) 有時要考慮 interval of convergence

Section 6-3 Solutions about Singular Points

$$\boxed{y^{(n)}+P_{n-1}(x)y^{(n-1)}+\cdots\cdots+P_1(x)y'+P_0(x)y=0}$$ (standard form)

Note! ☆1-1

假設解為 $y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+r}$

6-3-1 方法適用情形

☆2 限制

(1) Linear

(2) $(x-x_0)P_{n-1}(x)$, $(x-x_0)^2P_{n-2}(x)$, ……, $(x-x_0)^{n-1}P_1(x)$, $(x-x_0)^nP_0(x)$ are **analytic** at x_0

x_0 is regular singular

(比較： Section 6-2 要求 $P_{n-1}(x)$, $P_{n-2}(x)$, ……, $P_1(x)$, $P_0(x)$ are **analytic** at x_0)

(3) It is better that $P_0(x)$, $P_1(x)$, …., $P_{n-1}(x)$ are all polynomials.

6-3-2 定義

☆1

合為n

$(x-x_0)^k P_{n-k}(x)$

Singular Points 分成二種

- If x_0 is a singular point but $(x-x_0)P_{n-1}(x)$, $(x-x_0)^2P_{n-2}(x)$, ……, $(x-x_0)^{n-1}P_1(x)$, $(x-x_0)^nP_0(x)$ are analytic at x_0

x_0 : regular singular point 名1

- If $(x-x_0)P_{n-1}(x)$, $(x-x_0)^2P_{n-2}(x)$, ……, $(x-x_0)^{n-1}P_1(x)$, $(x-x_0)^nP_0(x)$ are not analytic at x_0

x_0 : irregular singular point 名2

standard: $y'' + \frac{3(x-2)}{(x^2-4)^2} y' + \frac{5}{(x^2-4)^2} y = 0$

Example 1 (text page 253)

$$(x^2-4)^2 y'' + 3(x-2)y' + 5y = 0$$

$$P_1(x) = \frac{3}{(x-2)(x+2)^2} \qquad P_0(x) = \frac{5}{(x-2)^2(x+2)^2}$$

$$(x-2)P_1 = \frac{3}{(x+2)^2} \qquad (x-2)^2 P_0 = \frac{5}{(x+2)^2}$$

$x = 2$ is a regular singular point

$x = -2$ is a irregular singular point

$$(x+2)P_1 = \frac{3}{(x-2)(x+2)} \qquad (x+2)^2 P_0 = \frac{5}{(x-2)^2}$$

6-3-3 解法

解法的關鍵： ☆☆

假設解為 $y(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$ 不同一

Theorem 6.3.1 Frobenius' Theorem ☆3

若 x_0 是 linear DE 當中的一個 regular singular point

則這個 linear DE 至少有一個解是 $y(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$ 的型態

不能保證所有的解

比較 page= 291, 292

☆☆

(Process of the Frobenius Method)

名3

不同一

Step 1 將 $y(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$ 代入

Step 2 Power 對齊(以指數較小的為準) ☆5

Step 3 合併 ☆6

Step 4 算出 r ☆7 不同二

Step 5 比較係數，將 c_n 之間的關係找出來

Step 6 將 Step 4 得出的 r 代入 Step 5

得出所有的 independent solutions 及 general solution

Step 7 (見後頁)

(Step 7)

當 (1) r 有重根

或 (2) r 的根之間的差為整數，且從 Step 6 得出來的解不為 independent 時

不同左

用 $y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$ 和長除法找出 $y_2(x)$

(參考 Section 6-3 的 Examples 4, 5)

當 r 的根之間的差為整數，但從 Step 6 得出來的解為 independent 時，不需進行這個步驟

6-3-4 範例

Example 2 (text page 255)

$x_0 = 0$ is singular

$y'' + \frac{1}{3x}y' - \frac{1}{3x}y_1 = 0$

$xP_1 = \frac{1}{3}$ $x^2 P_0 = -\frac{x}{3}$

$P_1 = \frac{1}{3x}$, $P_0 = -\frac{1}{3x}$ $x_0 = 0$ is a regular singular point

$$3xy'' + y' - y = 0$$

Step 1 將 $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ 代入

☆4

$$3\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} c_n (n+r)x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

不同二 always $n = 0 \sim \infty$

$n = 0 \sim \infty$

Step 2 Power 對齊

☆5

$n = k$ $n = k$ $n = k - 1$

$k = n + 1$

$$3\sum_{k=0}^{\infty} c_k (k+r)(k+r-1)x^{k+r-1} + \sum_{k=0}^{\infty} c_k (k+r)x^{k+r-1} - \sum_{k=1}^{\infty} c_{k-1} x^{k+r-1} = 0$$

$$3\sum_{k=0}^{\infty} c_k (k+r)(k+r-1)x^{k+r-1} + \sum_{k=0}^{\infty} c_k (k+r)x^{k+r-1} - \sum_{k=1}^{\infty} c_{k-1} x^{k+r-1} = 0$$

k=0 k≥1 k=0 k≥1

☆6 Step 3 合併

$$[3c_0 r(r-1) + c_0 r]x^{r-1} + \sum_{k=1}^{\infty}[3c_k (k+r)(k+r-1) + c_k (k+r) - c_{k-1}]x^{k+r-1} = 0$$

☆7 Step 4 算出 r (Note: r is unknown)

Suppose that $c_0 \neq 0$

$$c_0\left(3r(r-1)+r\right)=0$$

不同之

$$3r(r-1)+r=0 \qquad 3r^2-2r=0 \qquad r(3r-2)=0$$

$\Downarrow$

為4 indicial equation

$$\boxed{r=0 \quad or \quad 2/3}$$

Step 5

☆8

$$3c_k (k+r)(k+r-1) + c_k (k+r) - c_{k-1} = 0$$

$$c_k = \frac{1}{(k+r)(3k+3r-2)} c_{k-1}$$

$$c_k = \frac{1}{(k+r)(3k+3r-2)} c_{k-1} \qquad k=1 \sim \infty$$

Step 6 ☆9

當 $r=0$ $\boxed{c_k = \frac{c_{k-1}}{k(3k-2)}}$

$n!$ $1\cdot 4\cdot 7\cdots(3n-2)$

$k=1$ $c_1 = \frac{c_0}{1\cdot 1}$

$k=2$ $c_2 = \frac{c_1}{2\cdot 4} = \frac{c_0}{2!1\cdot 4}$

$k=3$ $c_3 = \frac{c_2}{3\cdot 7} = \frac{c_0}{3!1\cdot 4\cdot 7}$

$k=4$ $c_4 = \frac{c_3}{4\cdot 10} = \frac{c_0}{4!1\cdot 4\cdot 7\cdot 10}$

$\vdots$

$$\boxed{c_n = \frac{c_0}{n!1\cdot 4\cdot 7\cdots\cdots(3n-2)}}$$

☆10

當 $r=2/3$ $\boxed{c_k = \frac{1}{(3k+2)k} c_{k-1}}$

$5\cdot 8\cdot 11\cdots(3n+2)$ $n!$

$k=1$ $c_1 = \frac{c_0}{5\cdot 1}$

$k=2$ $c_2 = \frac{c_1}{8\cdot 2} = \frac{c_0}{2!5\cdot 8}$

$k=3$ $c_3 = \frac{c_2}{11\cdot 3} = \frac{c_0}{3!5\cdot 8\cdot 11}$

$k=4$ $c_4 = \frac{c_3}{14\cdot 4} = \frac{c_0}{4!5\cdot 8\cdot 11\cdot 14}$

$\vdots$

$$\boxed{c_n = \frac{c_0}{n!5\cdot 8\cdot 11\cdots\cdots(3n+2)}}$$

Solution of Example 2 (別忘了將最後的解寫出)

$$\sum_{n=0}^{\infty} c_n x^{n+r} = x^r \sum_{n=0}^{\infty} c_n x^n$$

$$y = C_1 y_1 + C_2 y_2$$

replace c_0 by C_1, C_2

$r=0$

$$y_1(x) = x^0 \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!\,1\cdot 4\cdot 7\cdots\cdots(3n-2)} x^n \right]$$

(Set $c_0 = 1$)

1 is not the special case of $n=0$

不同四 x^r ☆11

$r=\frac{2}{3}$

$$y_2(x) = x^{2/3} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!\,5\cdot 8\cdot 11\cdots\cdots(3n+2)} x^n \right]$$

☆12

1 is not the special case of $n=0$

$x \in (0, \infty)$

(別忘了寫出 x 的範圍)

Note: $x_0=0$ is a singular point

Examples 4, 5 (text pages 258, 259)

$$xy'' + y = 0$$

$y'' + 0 \cdot y' + \frac{y}{x} = 0$

$P_1 = 0 \quad P_0 = \frac{1}{x}$

$xP_1 = 0 \quad x^2 P_0 = 1$

$\therefore x_0 = 0$ is regular singular

Step 1 將 $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ ☆☆ 代入

$$\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

☆4

Step 2 對齊 $n = k$ $n = k-1$ $k = n+1$

☆5

$$\sum_{k=0}^{\infty} c_k (k+r)(k+r-1) x^{k+r-1} + \sum_{k=1}^{\infty} c_{k-1} x^{k+r-1} = 0$$

$k=0$ $k \geq 1$

Step 3 合併

☆6

$$c_0 r(r-1) x^{r-1} + \sum_{k=1}^{\infty} [c_k (k+r)(k+r-1) + c_{k-1}] x^{k+r-1} = 0$$

$$c_0 r(r-1)x^{r-1} + \sum_{k=1}^{\infty}[c_k(k+r)(k+r-1) + c_{k-1}]x^{k+r-1} = 0$$

Step 4 $r(r-1) = 0$ ← $c_0 \neq 0$

☆7 $r = 0 \quad or \quad 1$ ← indicial equation 名4

Step 5 $c_k = -\dfrac{c_{k-1}}{(k+r)(k+r-1)}$ $k = 1, 2, 3, \cdots \infty$

☆8

$(-1)^n$

Step 6 當 $r = 1$ $c_k = -\dfrac{c_{k-1}}{(k+1)k}$

☆9 $(n+1)!$ ← $n!$

☆10 $c_n = (-1)^n \dfrac{c_0}{(n+1)!n!}$

$(-1)^n$: from -1

$(n+1)!$: from $k+1$, $k = 1 \sim n$

$n!$: from k, $k = 1 \sim n$

$0! = 1$, $1! = 1$

☆11 x^r

$$y_1(x) = x\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!(n+1)!}x^n\right] = x\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}x^n\right]$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}x^{n+1}$$

1 is a special case of $n=0$

☆12

$y_1 = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$

Step 6　當 $r=0$　$c_k = -\dfrac{c_{k-1}}{(k-1)k}$　$k=2,3,\cdots\infty$

$(-1)^n$　$n!$　$1\cdot2\cdot3\cdots n-1=(n-1)!$

$k=1$ 時不能算

☆13 此時，應該根據 Step 3，由

$$c_k(k+r)(k+r-1)+c_{k-1}=c_1\cdot 0+c_0=0$$

($k=1, r=0$ 代入)

c_0 必需等於 0, c_1 可為任意值

$$c_2=-\frac{c_1}{1\cdot 2}\qquad c_3=-\frac{c_2}{2\cdot 3}=\frac{c_1}{1\cdot 2\cdot 2\cdot 3}\qquad \cdots\cdots\cdots\cdots$$

這地方容易犯錯，要小心

$$c_n=\frac{(-1)^{n-1}c_1}{(n-1)!\cdot n!}$$

set $c_1=1$

x^r

$$y_2(x)=x^0\left[x+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n-1)!n!}x^n\right]=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!n!}x^n=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!(m+1)!}x^{m+1}$$

$m=n-1$

x is the special case of $n=1$

$y_2(x)=y_1(x)$ ☆3

因為前頁算出來的 $y_2(x)$ 等於 $y_1(x)$，

只好另外用 Sec. 4-2 “reduction of order” 的方法求解

☆14

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx$$

P(x): coefficient of y'(x) in the standard form.

不同五

In this example, P(x): 0

註：若算出的 $y_2(x)$ 等於 $y_1(x)$ 乘上一個 constant, 則 $y_2(x)$ 和 $y_1(x)$ 仍視同 linearly dependent，仍需要用上方 “reduction of order” 的方法來求另外一個解

$$y_2(x)=y_1(x)\int\frac{e^{-\int 0dx}}{y_1^2(x)}dx=y_1(x)\int\frac{dx}{[x-\frac{1}{2}x^2+\frac{1}{12}x^3-\frac{1}{144}x^4+\cdots\cdots]^2}$$

$$=y_1(x)\int\frac{dx}{[x^2-x^3+\frac{5}{12}x^4-\frac{7}{72}x^5+\cdots\cdots]}$$

long division

長除法 ☆15

$$=y_1(x)\int\left[\frac{1}{x^2}+\frac{1}{x}+\frac{7}{12}+\frac{19}{72}x+\cdots\cdots\right]dx$$

Note!

$$=y_1(x)\left[-\frac{1}{x}+\ln x+\frac{7}{12}x+\frac{19}{144}x^2+\cdots\cdots\right]$$

思考：為何不是 $\ln|x|$？

$$=y_1(x)\ln x+y_1(x)\left[-\frac{1}{x}+\frac{7}{12}x+\frac{19}{144}x^2+\cdots\cdots\right]$$

$$=y_1(x)\ln x+\left[-1+\frac{1}{2}x+\frac{1}{2}x^2+\cdots\cdots\right]$$

6-3-5 多項式的長除法 ☆15

計算 $\dfrac{1}{x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{72}x^5 + \cdots\cdots}$

$$
\begin{array}{cccccc}
 & & (x^{-2}) & (x^{-1}) & (1) & (x) \\
 & & 1 & 1 & \frac{7}{12} & \frac{19}{72} \quad \cdots \\
\hline
1 & -1 & \frac{5}{12} & -\frac{7}{72} & \cdots & \\
(x^2) & (x^3) & (x^4) & (x^5) & &
\end{array}
$$

$$
\begin{array}{cccccc}
1 & & & & & \\
1 & -1 & \frac{5}{12} & -\frac{7}{72} & \cdots & \\
\hline
(1) & (x) & (x^2) & (x^3) & &
\end{array}
$$

$$
\begin{array}{cccccc}
 & 1 & -\frac{5}{12} & \frac{7}{72} & \cdots & \\
 & 1 & -1 & \frac{5}{12} & -\frac{7}{72} & \cdots \\
\hline
 & & (x^2) & (x^3) & (x^4) &
\end{array}
$$

$$
\begin{array}{cccccc}
 & & \frac{7}{12} & -\frac{23}{72} & \cdots & \cdots \\
 & & \frac{7}{12} & -\frac{7}{12} & \frac{35}{144} & -\frac{49}{864} \quad \cdots \\
\hline
 & & & (x^3) & (x^4) &
\end{array}
$$

$$
\begin{array}{cccccc}
 & & & \frac{19}{72} & \cdots &
\end{array}
$$

名4

6-3-6 Indicial Equation

指示

2nd order case $y'' + P(x)y' + Q(x)y = 0$

If x_0 is a regular singular point

$$(x-x_0)^2 y'' + (x-x_0)p(x)y' + q(x)y = 0$$

where $p(x) = (x-x_0)P(x) \quad q(x) = (x-x_0)^2 Q(x)$

由於 $p(x)$ 和 $q(x)$ 皆為 analytic

$$\boxed{p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots\cdots\cdots}$$

$$\boxed{q(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \cdots\cdots\cdots}$$

$$\boxed{y(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r},} \qquad \boxed{y'(x) = \sum_{n=0}^{\infty} c_n (n+r)(x-x_0)^{n+r-1},}$$

$$\boxed{y''(x) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)(x-x_0)^{n+r-2},}$$

將 $y(x), y'(x), y''(x), p(x), q(x)$ 代入

$$(x-x_0)^2 y'' + (x-x_0)p(x)y' + q(x)y = 0$$

$$\boxed{\begin{aligned}&\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)(x-x_0)^{n+r}\\&+\left(a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots\cdots\cdots\right)\sum_{n=0}^{\infty} c_n (n+r)(x-x_0)^{n+r}\\&+\left(b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \cdots\cdots\cdots\right)\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r} = 0\end{aligned}}$$

其中 $(x-x_0)^r$ 的 coefficient 為

$$c_0 r(r-1) + c_0 a_0 r + c_0 b_0$$

$$\boxed{r(r-1) + a_0 r + b_0 = 0} \longrightarrow \textbf{indicial equation}$$

$$y(x)=\sum_{n=0}^{\infty}c_n\left(x-x_0\right)^{n+r}$$

當 linear DE 為 2nd order 時，r 可以由 $\boxed{r(r-1)+a_0r+b_0=0}$ 求出

其中

$$\boxed{a_0=p(x_0)} \qquad p(x)=(x-x_0)P(x) \qquad y''+P(x)y'+Q(x)y=0$$

$$\boxed{b_0=q(x_0)} \qquad q(x)=(x-x_0)^2Q(x)$$

For the 2nd order case

$$r(r-1)+a_0 r+b_0=0 \quad \text{two roots: } r_1, r_2$$

(Case 1) $r_1 \neq r_2$ and r_1, r_2 are real, $r_2 - r_1 \neq \text{integer}$

☆15-1

可以找出兩組 $y(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$ 的解

(Case 2) $r_1 \neq r_2$ and r_1, r_2 are real, $r_2 - r_1 = \text{integer}$

☆15-2

有時可以找出兩組 $y(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$ 的解

有時一個解是 $y_1(x)=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r_1}$

另一個解是 $y_2(x)=Cy_1(x)\ln x+\sum_{n=0}^{\infty} b_n (x-x_0)^{n+r_2}$

Sec 4-2 + long division

(Case 3) $r_1 = r_2$ 時　一定不能得出2個 Frobenius 解

☆15-3

$$y_1(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r_1}$$

$$y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}$$　C 一定不為 0

或寫成 $y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} \tilde{b}_n (x - x_0)^{n+r_2}$　　$\tilde{b}_n = b_n / C$

(Case 4) $r_1 \neq r_2$ and r_1, r_2 are complex

在此不予討論

6-3-7 Indicial Equation for Higher Order Case (補充)

當 linear DE 為 n^{th} order 時

$$y^{(n)} + P_{n-1}(x)\,y^{(n-1)} + \cdots\cdots + P_1(x)\,y' + P_0(x)\,y = 0$$

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$ 當中的 r 可以由

$$\boxed{\frac{r!}{(r-n)!} + a_{n-1,0}\frac{r!}{(r-n+1)!} + a_{n-2,0}\frac{r!}{(r-n+2)!} + \cdots\cdots + a_{1,0}\frac{r!}{(r-1)!} + a_{0,0} = 0}$$

求出

其中 $a_{k,0} = p_k(x_0)$, $\quad p_k(x_0) = (x - x_0)^{n-k} P_k(x_0)$

$k = 0, 1, 2, \ldots, n-1$

6-3-8 本節需要注意的地方

(1) Index「對齊」的計算要小心

(建議可以向 power 較小的對齊，否則會出現負的 k)

例如在 page 302 的 Step 2，一律對齊為 x^{k+r-1} 而非 x^{k+r}

(2) 若 $x = 0$ 為 regular singular point, 設 $x_0 = 0$ 即可

(3) 如果是 c_k 和 c_{k-1} (或 c_{k-1} 和 c_k) 的 recursive relation

其實有時可以立刻將 c_n 的式子觀察出來

(但是分母不可變為 0)

(4) Recursive relation 當中小心分母為 0 的情形 (如 page 308)

(5) 小心算出來的 $y_2(x)$ 和 $y_1(x)$ 相同的情形 (如 pages 308, 309)

(6)別忘了將最後的解寫出
最後的解易出錯的地方：別忘了乘上 x^r

(7) Interval of solution 依然要考慮，
且 interval 不包括任何singular point，
即使是 regular singular point

(8) 複習長除法

Section 6-4 Special Functions

(本節這學期不在考試範圍內)

Special cases of Sections 6-2 and 6-3

standard form $y''+\frac{y'}{x}+\left(1-\frac{v^2}{x^2}\right)y=0$

$xP_1=1,\ x^2P_0=x^2-v^2$

$x_0=0$ is a regular singular point

- **Bessel's equation of order v**

Sec 6-3

$$\boxed{x^2y''+xy'+(x^2-v^2)y=0}$$

Solution: $c_1J_v(x)+c_2Y_v(x)$

$$J_v(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+v+n)}\left(\frac{x}{2}\right)^{2n+v}$$: 1st kind Bessel function

$$Y_v(x)=\frac{\cos v\pi J_v(x)-J_{-v}(x)}{\sin v\pi}$$: 2nd kind Bessel function

- **Legendre's equation of order n**

$x_0=0$ is an ordinary point

Sec 6-2

$$\boxed{(1-x^2)y''-2xy'+n(n+1)y=0}$$

One of the solution: Legendre polynomials

(See page 338)

其他名詞

- Gamma function $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$
- modified Bessel equation of order v

$$x^2y''+xy'-(x^2+v^2)y=0$$

解：$c_1I_v(x)+c_2K_v(x)$

- modified Bessel equation of the 1st kind $I_v(x)=i^{-v}J_v(ix)$
- modified Bessel equation of the 2nd kind $K_v(x)=\frac{\pi}{2}\frac{I_{-v}(x)-I_v(x)}{\sin v\pi}$
- Bessel 的另一種變型

$$x^2y''+(1-2a)xy'+(b^2c^2x^{2c}+a^2-p^2c^2)y=0$$

解 $y=x^a\left[c_1J_p(bx^c)+c_2Y_p(bx^c)\right]$

6.4.1 Bessel's Equation

6.4.1.1 Solving for Bessel's equation of order v

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$

Steps 1~3 將 $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ 代入

經過一些計算 (See text pages 262, 263) 得出

$$c_0(r^2 - v^2)x^r + c_1((1+r)^2 - v^2)x^{r+1} + \sum_{k=2}^{\infty}[c_k((k+r)^2 - v^2) + c_{k-2}]x^{r+k} = 0$$

Step 4 $r^2 - v^2 = 0$ two roots: v and $-v$

Step 5 $c_1\left((1+r)^2 - v^2\right) = 0$ $\quad c_k = \dfrac{c_{k-2}}{v^2 - (k+r)^2}$

$c_1 = 0$

$$c_k = \frac{c_{k-2}}{v^2-(k+r)^2}$$

Step 6 當 $r=v$ $\quad c_k = -\dfrac{c_{k-2}}{k(k+2v)}$ $\qquad$ 當 $r=-v$ $\quad c_k = -\dfrac{c_{k-2}}{k(k-2v)}$

由於 $c_1 = 0,\ c_3 = c_5 = c_7 = c_9 = \ldots\ldots = 0$

$$c_{2n} = (-1)^n \frac{c_0}{2\cdot 4\cdot 6\cdots\cdots 2n\cdot(2+2v)(4+2v)(6+2v)\cdots\cdots(2n+2v)}$$

$$= \frac{(-1)^n c_0}{2^{2n} n!(1+v)(2+v)(3+v)\cdots\cdots(n+v)} \qquad \text{when } r=v$$

$$c_{2n} = (-1)^n \frac{c_0}{2\cdot 4\cdot 6\cdots\cdots 2n\cdot(2-2v)(4-2v)(6-2v)\cdots\cdots(2n-2v)}$$

$$= \frac{(-1)^n c_0}{2^{2n} n!(1-v)(2-v)(3-v)\cdots\cdots(n-v)} \qquad \text{when } r=-v$$

6.4.1.2 Gamma function: a generalization of $n!$

(Note 1)

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

properties of Gamma function

(1) $\Gamma(n+1) = n!$ when n is a positive integer

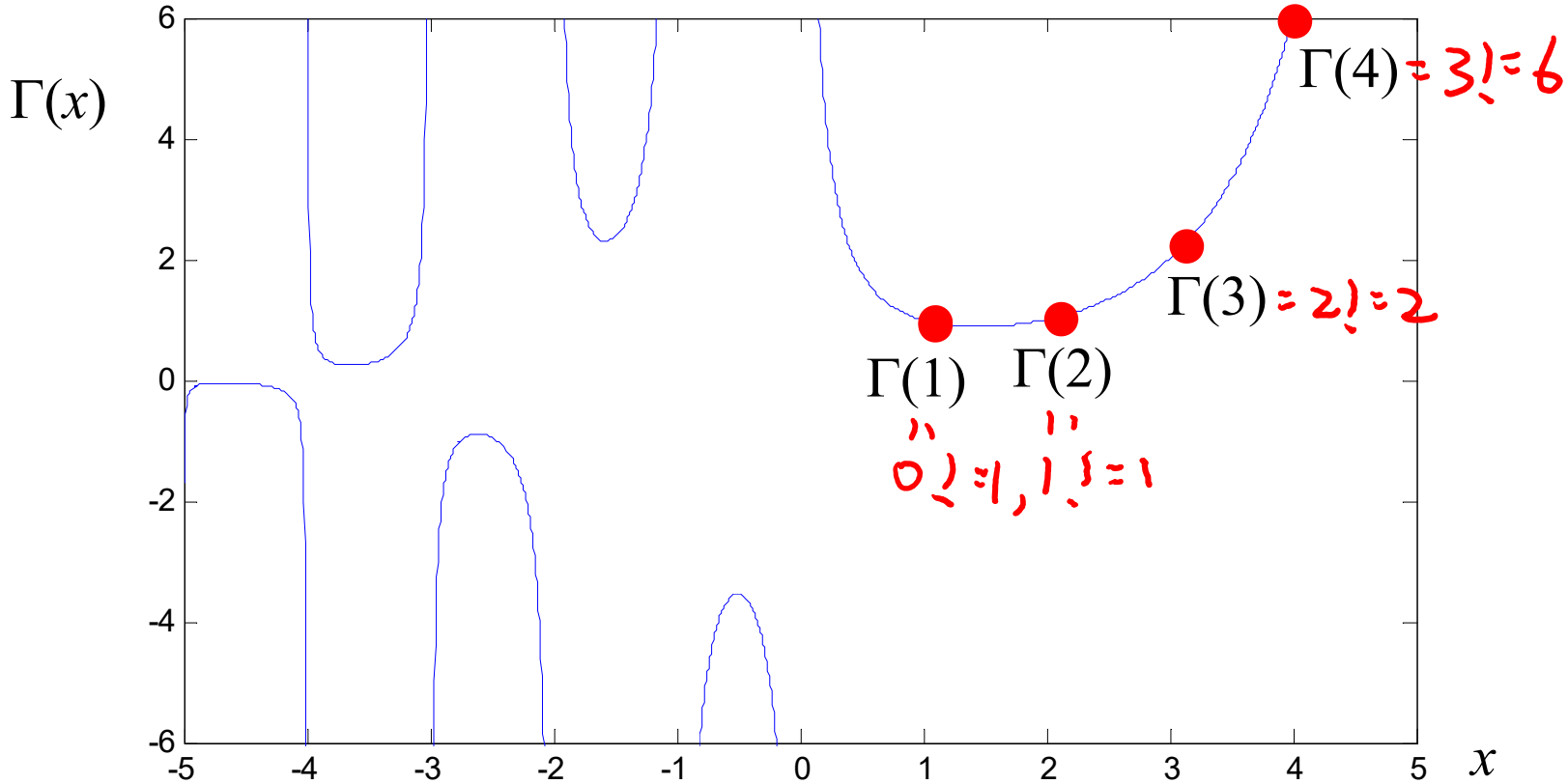
$\Gamma(1) = 0! = 1$

(2) $\Gamma(x+1) = x\Gamma(x)$

參照課本 Appendix 1

(3) $\Gamma(n) \to \infty$ when n is a negative integer or $n = 0$

(4) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ $\quad \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$



6.4.1.1 回到 Solving for Bessel function

$$c_{2n} = \frac{(-1)^n c_0}{2^{2n} n!(1+v)(2+v)(3+v)\cdots\cdots(n+v)}$$ when $r = v$

Set $c_0 = \dfrac{1}{2^v \Gamma(1+v)}$

$$c_{2n} = \frac{(-1)^n}{2^{2n+v} n!(1+v)(2+v)(3+v)\cdots\cdots(n+v)\Gamma(1+v)}$$

$$= \frac{(-1)^n}{2^{2n+v} n!(2+v)(3+v)\cdots\cdots(n+v)\Gamma(2+v)}$$ $\Gamma(2+v) = (1+v)\Gamma(1+v)$

$$= \frac{(-1)^n}{2^{2n+v} n!(3+v)\cdots\cdots(n+v)\Gamma(3+v)}$$ $\Gamma(3+v) = (2+v)\Gamma(2+v)$

$$\vdots$$

$$= \frac{(-1)^n}{2^{2n+v} n!\Gamma(n+v+1)}$$

同理，當 $r=-v$ set $c_0=\dfrac{1}{2^{-v}\Gamma(1-v)}$

$$c_{2n}=\frac{(-1)^n}{2^{2n-v}n!\Gamma(n-v+1)}$$

Two independent solutions of the Bessel's equation

代入 $\displaystyle\sum_{n=0}^{\infty}c_n x^{n+r}$

When $r=v$ $\boxed{J_v(x)=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1+v+n)}\left(\frac{x}{2}\right)^{2n+v}}$

When $r=-v$ $\boxed{J_{-v}(x)=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(1-v+n)}\left(\frac{x}{2}\right)^{2n-v}}$

稱作 Bessel functions of the first kind of order v and $-v$

6.4.1.3 Bessel function of the second kind

注意，兩個 roots 的差為 $2v$

(1) 當 $2v$ 不為整數時，Bessel's equation 的解即為

$$c_1 J_v(x) + c_2 J_{-v}(x) \quad (\text{也可表示成 } c_1 J_v(x) + c_2 Y_v(x))$$

(2) 當 $2v$ 為整數，但 $v = m + 1/2$ (m 是一個整數) 時，Bessel's equation 的解亦為 $c_1 J_v(x) + c_2 J_{-v}(x)$ (也可表示成 $c_1 J_v(x) + c_2 Y_v(x)$)

(3) 當 $2v$ 為整數，且 v 是一個整數時， Bessel's equation 的解為

$$c_1 J_v(x) + c_2 Y_v(x)$$

$Y_v(x)$: Bessel function of the second kind of order v

(見後頁） (Note 2)

$Y_v(x)$: Bessel function of the second kind of order v

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

當 m 為整數時， $Y_m(x)$ 定義成

$$Y_m(x) = \lim_{v \to m} \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$

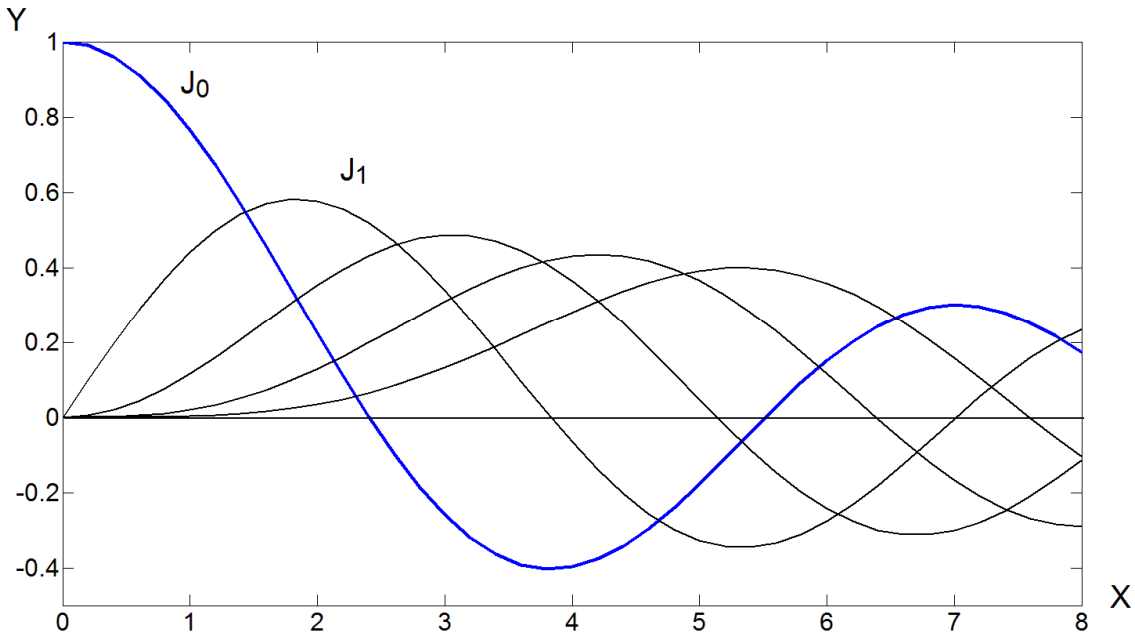
用 L'Hopital's rule 來算

$$Y_m(x) = \lim_{v \to m} \frac{-\pi \sin v\pi J_v(x) + \cos v\pi \frac{\partial}{\partial v} J_v(x) - \frac{\partial}{\partial v} J_{-v}(x)}{\pi \cos v\pi}$$

6.4.1.4 Bessel function of the 1st kind (order m 為整數時)的性質

(1) $J_0(0) = 1, \qquad J_m(0) = 0 \;\; \text{for } m \neq 0$

(2) Zero crossing 的位置，隨著 m 增加而越來越遠 (見 Table 6.4.1)



(3) $J_m(-x) = (-1)^m J_m(x)$ when m is an integer

(4) $J_{-m}(x) = (-1)^m J_m(x)$ when m is an integer

(5) $\frac{d}{dx}\left[x^{-v} J_v(x)\right] = -x^{-v} J_{v+1}(x)$

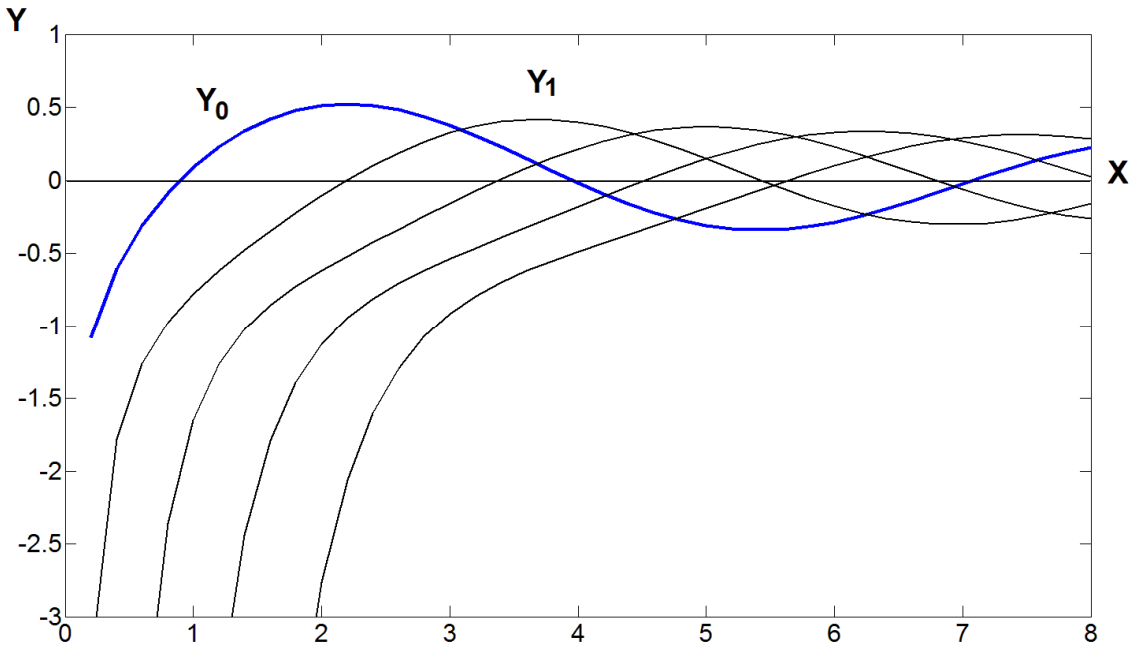
(6) $\frac{d}{dx}\left[x^{v} J_v(x)\right] = x^{v} J_{v-1}(x)$

見 Example 6, text pages 268, 269

6.4.1.5 Bessel function of the 2nd kind (order m 為整數時)的性質

(1) $\lim_{x\to 0} Y_m(x) = -\infty$

(2) Zero crossing 的位置，隨著 m 增加而越來越遠



6.4.1.6 Bessel's equation 的變型

$$x^2y''+xy'+(x^2-v^2)y=0 \qquad \text{解：} c_1J_v(x)+c_2Y_v(x)$$

(A) $$x^2y''+xy'+(\alpha^2x^2-v^2)y=0 \qquad \text{解：} c_1J_v(\alpha x)+c_2Y_v(\alpha x)$$

Proof: Set $t=\alpha x$

$$\frac{dy}{dx}=\frac{dt}{dx}\frac{dy}{dt}=\alpha\frac{dy}{dt}$$

Similarly, $$\frac{d^2y}{dx^2}=\frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right)=\alpha\frac{d}{dt}\left(\alpha\frac{dy}{dt}\right)=\alpha^2\frac{d^2y}{dt^2}$$

$$\text{原式}=x^2y''+xy'+(\alpha^2x^2-v^2)y=\frac{t^2}{\alpha^2}\alpha^2\frac{d^2y}{dt^2}+\frac{t}{\alpha}\alpha\frac{dy}{dt}+(\alpha^2\frac{t^2}{\alpha^2}-v^2)y$$

$$=\boxed{t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t^2-v^2)y=0} \longrightarrow \text{對 } t \text{ 而言是 Bessel equation}$$

$$y=c_1J_v(t)+c_2Y_v(t)=c_1J_v(\alpha x)+c_2Y_v(\alpha x)$$

(B) modified Bessel equation of order v

$$x^2y'' + xy' - (x^2 + v^2)y = 0$$

解：$c_1 I_v(x) + c_2 K_v(x)$

其中 $I_v(x) = i^{-v} J_v(ix)$ 稱作是 modified Bessel function of the first kind of order v

$$K_v(x) = \frac{\pi}{2}\frac{I_{-v}(x) - I_v(x)}{\sin v\pi}$$

稱作是 modified Bessel function of the second kind of order v

當 v 為整數時，也是取 limit

(C) $\boxed{x^2y'' + (1-2a)xy' + (b^2c^2x^{2c} + a^2 - p^2c^2)y = 0}$

解： $y = x^a\left[c_1J_p(bx^c) + c_2Y_p(bx^c)\right]$

式子有點複雜，但可以用來解許多物理上的問題

Example 4 (text page 266)

$$xy'' + 3y' + 9y = 0$$

$$\downarrow$$

$$x^2y'' + 3xy' + 9xy = 0$$

6.4.2 Legendre's Equation

6.4.2.1 Legendre's Equation

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

$y(x)=\sum_{k=0}^{\infty}c_k x^k$ 代入，得出 (過程見text pages 270, 271)

Two linearly independent solutions are

$$y_1(x)=c_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6+\cdots\cdots\right]$$

$$y_2(x)=c_0\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5-\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7+\cdots\cdots\right]$$

(a) When n is not an integer, both the two solutions have infinite number of terms.

(b) When n is an even integer, $y_1(x)$ has finite number of terms.

In $y_1(x)$, the coefficient of x^k is zero when $k > n$.

(c) When n is an odd integer, $y_2(x)$ has finite number of terms.

In $y_2(x)$, the coefficient of x^k is zero when $k > n$.

$y_1(x)$ when n is an even integer and $y_2(x)$ when n is an odd integer are called the Legendre polynomials (denoted by $P_n(x)$).

通常選

$$c_0 = (-1)^{n/2}\frac{1\cdot 3\cdot\cdots\cdots(n-1)}{2\cdot 4\cdot\cdots\cdots n} \qquad c_1 = (-1)^{(n-1)/2}\frac{1\cdot 3\cdot\cdots\cdots n}{2\cdot 4\cdot\cdots\cdots(n-1)}$$

(讓 $P_n(1)$ 一律等於 1)

When n is a non-negative integer, one of the solutions is the nth order polynomial.

(Note 3)

由 $y_1(x)$

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x + 3\right)$$

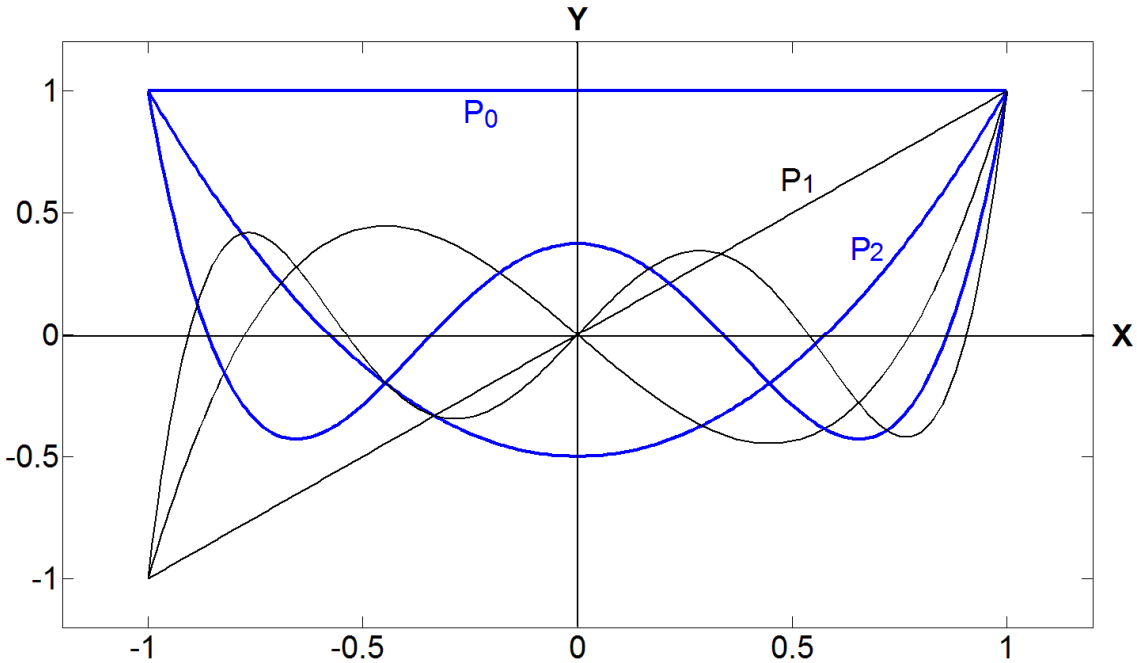
由 $y_2(x)$

$$P_1(x) = x$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

Legendre polynomials



Interval:

$x \in [-1, 1]$

6.4.2.2 Properties of Legendre Polynomials

(1) $P_n(-x) = (-1)^n P_n(x)$ even / odd symmetry

(2) $P_n(1) = 1$ $P_n(-1) = (-1)^n$

(3) $P_n(0) = 0$ when n is odd

(4) $P_n'(0) = 0$ when n is even

(5) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$ recursive relation

(6) $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n$ Rodrigues' formula

(7) $\int_{-1}^{1} P_m(x)P_n(x)dx = 0$ If $m \neq n$ orthogonality property (Note 4)

(8) 若任何在 $x \in [-1, 1]$ 區間為 continuous 的函式 $f(x)$

皆可表示為

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

由於 $\int f(x)P_m(x)dx = \underbrace{\sum_{n=0}^{\infty} a_n \int P_n(x)P_m(x)dx = a_n \int P_m(x)P_m(x)dx}_{\text{根據 orthogonality property}}$

所以 $a_n = \dfrac{\int f(x)P_m(x)dx}{\int P_m(x)P_m(x)dx}$

Orthogonality property 才是 Legendre polynomials 最重要的性質

Review of Chapter 6

解法適用範圍： Linear DE，且 coefficients 最好為 polynomials

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots\cdots + P_1(x)y' + P_0(x)y = 0$$

- 當 $P_m(x)$ 在 $x = x_0$ 時為 analytic

  x_0 為 ordinary point $\quad y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ 代入求解

- 當 $P_m(x)$ 在 $x = x_0$ 時不為 analytic

  但是 $(x - x_0)^{n-m} P_m(x)$ 在 $x = x_0$ 時為 analytic

  x_0 為 regular singular point $\quad y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ 代入求解

  有時，另一個解為 $\quad C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}$

Exercise for practice

Section 6-1: 4, 9, 12, 20, 22, 24, 30, 32, 34

Section 6-2: 2, 10, 13, 18, 20, 22, 24, 26, 27

Section 6-3: 4, 9, 13, 16, 22, 24, 26, 28, 30, 32, 33, 36

Review 6: 6, 7, 10, 14, 19, 20