Chapter 12 Boundary-Value Problem in Rectangular Coordinates

PDE (partial differential equations)

- Role of Chapter 12:

Discuss the boundary-value problem for the case of **two independent variables**.

12-8 three or more indep. variable

(x-y 座標)　　(圓座標的問題在 Chapter 13 當中有討論
但不在這學期的上課範圍之中)

Use the methods of

(1) separation of variables　　Chapter 12

(2) the Laplace / Fourier transform　　Chapter 14
(不在這學期的上課範圍)

本章的架構

只有這兩節是這學期授課範圍

- 12.1 介紹解法 Separation of variables (很重要)
- 12.2 名詞和定義
- 12.3 Heat equation
- 12.4 Wave equation
- 12.5 Laplace's equation
- 12.6 Nonhomogeneous PDE
- 12.8 PDE with Three or More Dimensions

(可視為12.1 的應用題)

兩大重點：

(1) 熟悉 separation of variables 解 PDE 的方法

(2) 名詞和定義

縮寫: boundary value problem (BVP)

initial value problem (IVP)

例：
$$a^2\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{\partial^2 u(x,t)}{\partial t^2}$$

BVP: $u(0,t)=0$ $u(L,t)=0$

IVP: $u(x,0)=f(x)$ $\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)$

partial differential equation (PDE)

ordinary differential equation (ODE)

Section 12.1 Separable Partial Differential Equations

12.1.1 Section 12.1 綱要

(1) linear second order partial differential equation for two independent variables

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

7 terms

(i) 双曲 $x^2-y^2=k$
$A=1, B=0, C=-1$
$B^2-4AC>0$

☆4-1 双曲線 ☆4-2 拋物線

$B^2-4AC>0$: hyperbolic, $B^2-4AC=0$: parabolic

名3 名4

☆4-3 $B^2-4AC<0$: elliptic 橢圓

名5

(ii) 拋物
$x^2=y$
$A=1, B=0, C=0$
$B^2-4AC=0$

(2) Partial differential equation (PDE) 主要解法之一：

Separation of variables (see pages 422-424).

(iii) $x^2+y^2=k$
橢圓 $A=C=1, B=0$
$B^2-4AC<0$

名詞： real separation constant (page 422)

12.1.2 Linear Second Order Partial Differential Equation

$$\boxed{A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G}$$

independent variables: x, y dependent variables: $u(x, y)$, 簡寫成 u

<u>homogeneous</u> : $G(x, y) = 0$, <u>nonhomogeneous</u> : $G(x, y) \neq 0$

名 2 名 2

particular solution, general solution 的定義一如往昔

【Theorem 12.1.1】 Superposition Principle ☆5

If u_1, u_2, …., u_k are solutions of a homogeneous linear partial differential equation, then

$$u = c_1 u_1 + c_2 u_2 + \cdots\cdots + c_k u_k$$

is also a solution of the homogeneous linear partial differential equation.

If $\frac{\partial^2}{\partial x^2} u_1 = 0$, $\frac{\partial^2}{\partial x^2} u_2 = 0$, … $\frac{\partial^2}{\partial x^2} u_k = 0$

then $\frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2 + \cdots + c_n u_n) = c_1 \frac{\partial^2}{\partial x^2} u_1 + c_2 \frac{\partial^2}{\partial x^2} u_2 + \cdots + c_n \frac{\partial^2}{\partial x^2} u_n$

$= 0$

12.1.3 Method of Separation of Variables

解 PDE with BVP (or IVP) 的方法

(1) method of separation of variables

若 PDE 當中有對 x 及對 y 的偏微分，

假設解為 $u(x, y) = X(x)Y(y)$

(2) using the Laplace transform (or Fourier, Fourier cosine transform, Fourier sine transform) (see Chapter 14 ，期末考範圍外)

共通的精神： **PDE** $\longrightarrow$ **ODE**

Method of Separation of Variables 的流程

(Step 1) 假設解為 $u(x, y) = X(x)Y(y)$

☆☆₁

解法關鍵

(Step 2) 將 $u(x, y) = X(x)Y(y)$ 代入 PDE，把 PDE 變成

☆☆₂ “ function of X ” = “ function of Y ” = $-\lambda$

的型態, 並得出 $X(x)$ 的 ODE 和 $Y(y)$ 的 ODE

λ 被稱為 real separation constant 名₁

Steps 3, 4, 5 要分成不同的 Cases 來解 ODEs

除了trivial 的情形外，所有可能的 cases 都要考慮

(Pre-Step) 考慮等於 0 的 initial / boundary conditions

(Step 3) 將 function of $X = -\lambda$ 的解算出，即為 $X(x)$

註：有時，先解 $Y(y)$ 會比較容易

(視 boundary (initial) conditions 而定)

(Step 4) 將 function of $Y = -\lambda$ 的解算出，即為 $Y(y)$

需注意的地方和 Step 3 相同

(Step 5) $u(x, y) = X(x)Y(y)$

(Step 6) 將所有可能的解全部加起來

☆3

(Step 7) 用非零的 boundary (initial) conditions 將 coefficients 求出

註： 這一步經常會用到 Fourier series, Fourier cosine series 或 Fourier sine series

※ 若沒有 boundary (initial) conditions，Steps 6, 7 可以省略

Rules:

x 的 BVP (IVP) 簡單 ⟶ 先算 $X(x)$

y 的 BVP (IVP) 簡單 ⟶ 先算 $Y(y)$

沒有 BVP (IVP) ⟶ 先算 $X(x)$ 或 $Y(y)$ 皆可

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}=0$$

☆₁

$$u(0,y)=0 \quad - X(0)=0 \qquad u(L,y)=0 \quad - X(L)=0$$

$$u(x,0)=f(x) \qquad \left.\frac{\partial u}{\partial y}\right|_{y=0}=g(x)$$

此時先算 $X(x)$

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}$$

$$u(0,y)=f(y) \qquad u(L,y)=0$$

$$\left.\frac{\partial}{\partial y}u(x,y)\right|_{y=0}=0 \qquad \left.\frac{\partial}{\partial y}u(x,y)\right|_{y=H}=0$$

此時先算 $Y(y)$

$u(x,y)=X(x)Y(y)$ $\quad Y'(0)=0$ $\quad$ ☆₁ $\quad Y'(H)=0$

$u'(x,y)=X(x)Y'(y)$

$u'(x,0)=X(x)Y'(0)=0$

Example 2 (text page 462)

$$\boxed{\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}}$$

homogeneous, parabolic PDE

A=1, B=C=0

$B^2-4AC=0$

(Step 1) 假設解為 $u(x, y) = X(x)Y(y)$ (解法關鍵) ☆☆$_1$

(Step 2) 將 $u(x, y) = X(x)Y(y)$ 代入 $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$

$$X''(x)Y(y) = 4X(x)Y'(y)$$

$$\frac{X''(x)}{4X(x)} = \frac{Y'(y)}{Y(y)}$$ 各歸一邊

real separation constant

令 $\frac{X''(x)}{4X(x)} = \frac{Y'(y)}{Y(y)} = \underline{\underline{-\lambda}}$ (解法關鍵) ☆☆$_2$

$$X''(x) + 4\lambda X(x) = 0 \qquad Y'(y) + \lambda Y(y) = 0$$

1 PDE ⇒ 2 ODEs

$$X''(x)+4\lambda X(x)=0 \qquad Y'(y)+\lambda Y(y)=0$$

$x''+4\lambda x=0$

Sec 4-3 auxiliary $m^2+4\lambda=0$

$\boxed{\textbf{Case 1 for Steps 3, 4, 5} \quad \lambda=0}$

☆1 m有重根

(Step 3-1) $X''(x)=0$

auxiliary function $\quad m^2=0 \quad$ roots : 0, 0

$$X(x)=c_1+c_2x$$

(Step 4-1) $Y'(y)=0 \qquad Y(y)=c_3$

(Step 5-1) $u(x,y)=X(x)Y(y)=(c_1+c_2x)c_3=\underline{A_1+B_1x}$

$$A_1=c_1c_3 \qquad B_1=c_2c_3$$

Case 2 for Steps 3, 4, 5 $\lambda < 0$

m 相異根

☆1

☆2-1

為了方便起見，令 $\lambda = -\alpha^2$

any $\alpha > 0$

$m^2 - 4\alpha^2 = 0$ $m = \pm 2\alpha$

(Step 3-2) $X''(x) - 4\alpha^2 X(x) = 0$ roots of the auxiliary function: $2\alpha, -2\alpha$

$$X(x) = d_1 e^{2\alpha x} + d_2 e^{-2\alpha x}$$

$c_4 \frac{e^{2\alpha x} + e^{-2\alpha x}}{2} + c_5 \frac{e^{2\alpha x} - e^{-2\alpha x}}{2}$

$= \frac{c_4 + c_5}{2} e^{2\alpha x} + \frac{c_4 - c_5}{2} e^{-2\alpha x}$

☆2-2

通常將解改寫成 $X(x) = c_4 \cosh(2\alpha x) + c_5 \sinh(2\alpha x)$

(Step 4-2) $\frac{Y'(y)}{Y(y)} = \alpha^2$ $Y'(y) - \alpha^2 Y(y) = 0$

$m - \alpha^2 = 0$ $m = \alpha^2$

$$Y'(y) - \alpha^2 Y(y) = 0 \qquad Y(y) = c_6 e^{\alpha^2 y}$$

(Step 5-2) $u(x,y) = X(x)Y(y) = A_2 e^{\alpha^2 y} \cosh(2\alpha x) + B_2 e^{\alpha^2 y} \sinh(2\alpha x)$

$$A_2 = c_4 c_6 \qquad B_2 = c_5 c_6$$

Case 3 for Step 3 $\lambda > 0$ m複數根

☆1 為了方便起見，令 $\lambda = \alpha^2$ ☆2-3 any $\alpha > 0$ $m^2+4\alpha^2=0$

(Step 3-3) $X''(x) + 4\alpha^2 X(x) = 0$ roots of the auxiliary function: $j2\alpha, -j2\alpha$

$$X(x) = c_7 \cos(2\alpha x) + c_8 \sin(2\alpha x)$$

(Step 4-3) $\dfrac{Y'(y)}{Y(y)} = -\alpha^2$ $\quad Y'(y) + \alpha^2 Y(y) = 0$ $\quad Y(y) = c_9 e^{-\alpha^2 y}$

(Step 5-3) $u(x, y) = A_3 e^{-\alpha^2 y} \cos(2\alpha x) + B_3 e^{-\alpha^2 y} \sin(2\alpha x)$

若要處理 boundary conditions，或著想得到 general solution，要將所有可能的解都加起來

☆3 (Step 6) Case1 Case2

$$u(x, y) = \boxed{A_1 + B_1 x} + \boxed{\sum_{\alpha>0} [A_{2,\alpha} e^{\alpha^2 y} \cosh(2\alpha x) + B_{2,\alpha} e^{\alpha^2 y} \sinh(2\alpha x)]}$$

$$\boxed{+\sum_{\alpha>0} [A_{3,\alpha} e^{-\alpha^2 y} \cos(2\alpha x) + B_{3,\alpha} e^{-\alpha^2 y} \sin(2\alpha x)]}$$ α 是任意實數

Case 3

(註：nonseparable 的解在這一步得到)

Exercise Problem 5

$$x\frac{\partial u}{\partial x} = y\frac{\partial u}{\partial y}$$

$X(x)Y(y)$

$u = XY$

$xX'Y = yXY'$

$\frac{xX'}{X} = \frac{yY'}{Y} = -\lambda$

$xX' + \lambda X = 0$

$yY' + \lambda Y = 0$

Cauchy Euler Sec(4.7)

$xX' + \lambda X = 0$

auxiliary

$m + \lambda = 0$

$m = -\lambda$

$X(x) = C_1 x^{-\lambda}$

$Y(y) = C_2 y^{-\lambda}$

$u_\lambda = XY = C_\lambda x^{-\lambda} y^{-\lambda}$

step 6

$u(x,y) = \sum_\lambda C_\lambda (xy)^{-\lambda}$

Exercise Problem 9

$$k\frac{\partial^2 u}{\partial x^2} - u = \frac{\partial u}{\partial t}$$

$k > 0$

$u = X(x)T(t)$

$kX''T - XT = XT'$

$(kX'' - X)T = XT'$

$\frac{kX'' - X}{X} = \frac{T'}{T} = -\lambda$

$kX'' + (\lambda - 1)X = 0$

$T' + \lambda T = 0$

Case 1 $\lambda = 1$

$kX'' = 0 \quad T' + T = 0$

$X = ax + b \quad T = e^{-t}$

$u = e^{-t}(ax + b)$

Case 2, $\lambda > 1$, $\lambda = 1 + \alpha^2$

$kX'' + \alpha^2 X = 0 \quad T' + (\alpha^2 + 1)T = 0$

$X(x) = C_1\cos\frac{\alpha}{\sqrt{k}}x + C_2\sin\frac{\alpha}{\sqrt{k}}x$

$T(t) = C_3 e^{-(\alpha^2+1)t}$

$u = e^{-(\alpha^2+1)t}\left(A_1\cos\left(\frac{\alpha}{\sqrt{k}}x\right) + A_2\sin\left(\frac{\alpha}{\sqrt{k}}x\right)\right)$

Case 3: $\lambda < 1$, $\lambda = 1 - \alpha^2$

$kX'' - \alpha^2 X = 0, \quad T' + (1 - \alpha^2)T = 0$

$X(x) = d_1\cosh\left(\frac{\alpha}{\sqrt{k}}x\right) + d_2\sinh\left(\frac{\alpha}{\sqrt{k}}x\right)$

$u = e^{(\alpha^2-1)t}\left(B_1\cos\left(\frac{\alpha}{\sqrt{k}}x\right) + B_2\sin\left(\frac{\alpha}{\sqrt{k}}x\right)\right)$

Step 6

$u(x,y) = e^{-t}(ax+b) + \sum_\alpha$ Case2解 $+ \sum_\alpha$ Case3 解

12.1.4 Classification

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0$$

$B^2 - 4AC > 0 \longrightarrow$ The PDE is said to be hyperbolic (雙曲線)

$B^2 - 4AC = 0 \longrightarrow$ The PDE is said to be parabolic (拋物線)

$B^2 - 4AC < 0 \longrightarrow$ The PDE is said to be elliptic (橢圓形)

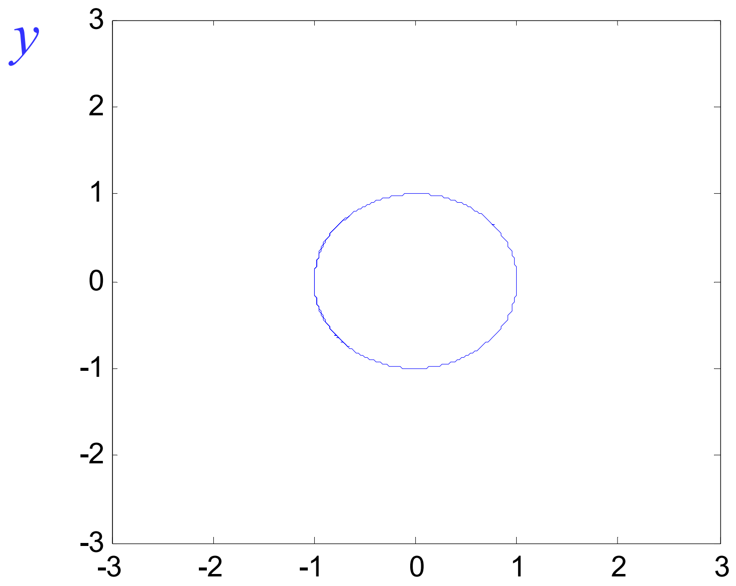
這些命名方式，是根據 2 次多項式在 *x*-*y* 平面上的軌跡

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

當 $x^2 + y^2 - 1 = 0$

$$x^2 + y^2 = 1$$

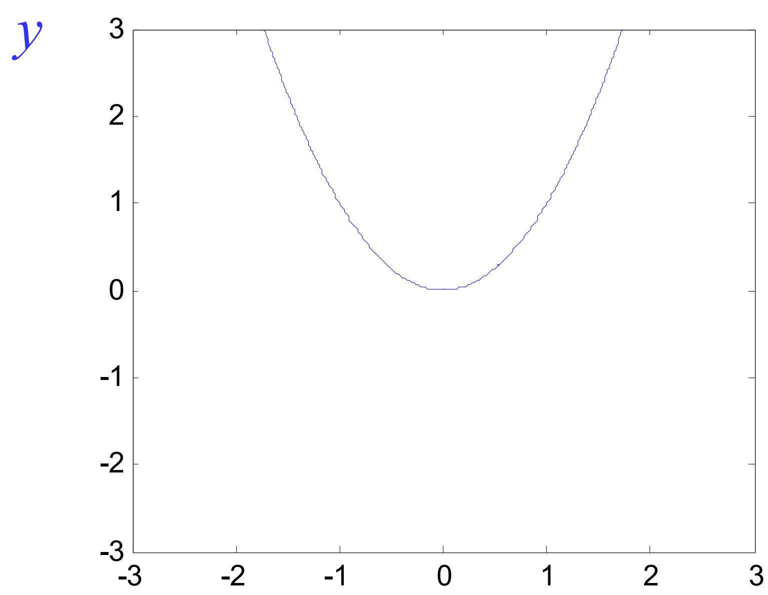
$$B^2 - 4AC = -4 < 0$$



當 $x^2 - y = 0$

$$y = x^2$$

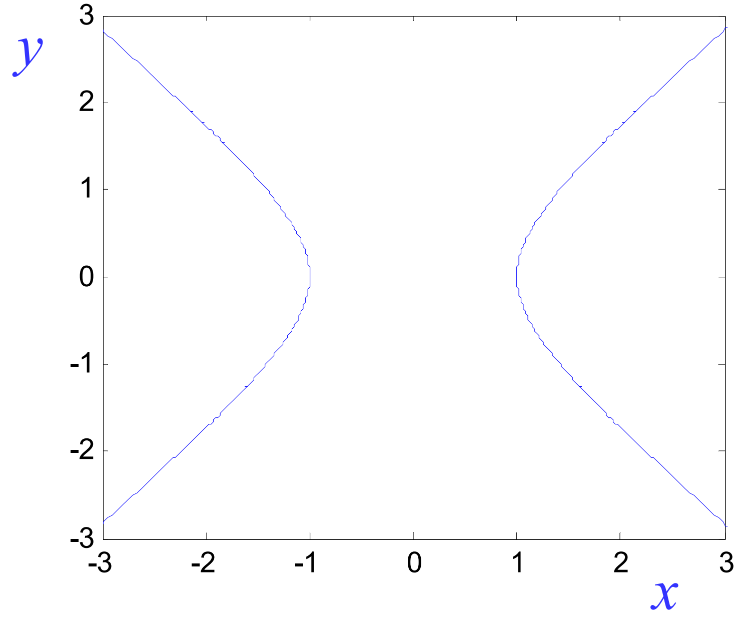
$$B^2 - 4AC = 0$$



當 $x^2 - y^2 - 1 = 0$

$x^2 - y^2 = 1$

$B^2 - 4AC = 4 > 0$



記憶秘訣：只要清楚幾個「特例」，就可以記住當

$$B^2 - 4AC < 0, \quad B^2 - 4AC = 0, \quad B^2 - 4AC > 0$$

的時候，應該是什麼圖形

Example 3 (text page 463)

$$3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$$

parabolic

$A=3,\ B=C=0$

$B^2-4AC=0$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

hyperbolic

$A=1,\ B=0,\ C=-1$

$B^2-4AC>0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

elliptic

$A=C=1,\ B=0$

$B^2-4AC<0$

12.1.5 本節需要注意的地方

(1) 本節除了定義以外，只有兩個重點： classification of equations 以及 method of separation of variables.

(2) 然而， method of separation of variables 解法的流程，稍有些複雜，需要熟悉 (Sections 12-4, 12-5 都將用這個方法)

關鍵：記住第一步 $u(x, y) = X(x)Y(y)$

第二步 function of X = function of $Y = -\lambda$

(3) Method of separation of variables 在計算時，會分成很多個 cases.

(4) Separation of variables要解 BVP 和 IVP時，需要將每個 cases 得出來的解都加起來 (Step 6)

(5) 為了方便解決 BVP 或 IVP，經常將 $d_1 e^{2\alpha x} + d_2 e^{-2\alpha x}$

改寫成 $c_4 \cosh(2\alpha x) + c_5 \sinh(2\alpha x)$

(6) Hyperbolic, parabolic, elliptic 的條件，可以用幾個 special cases 來記

Section 12.4 Wave Equation

12.4.1 本節綱要

要解決的問題 (one-dimensional wave equation)

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0 < x < L \qquad t > 0$$

BVP and IVP

$$u(0,t) = 0 \qquad u(L,t) = 0 \qquad \text{for } t > 0$$

$$u(x,0) = f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \qquad \text{for } 0 < x < L$$

例子見 page 439

解法見 page 440-448

實際上，Sections 12.4 可看成是 Section 12.1 的 method of separation of variables 的練習題

(可見得 method of separation of variables 有多重要)

名詞：

standing waves (page 449)	normal modes (page 449)
first standing wave (page 450)	fundamental frequency (page 450)
nodes (page 452)	overtones (page 452)

12.4.2 物理意義

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

$A=a^2,\ B=0,\ C=-1$
$B^2-4AC>0,$
hyperbolic PDE

u ：高度

$\frac{\partial u}{\partial t}$ ：速度

$\frac{\partial^2 u}{\partial t^2}$ ：加速度

$$u(0,t)=0 \qquad u(L,t)=0$$

$$u(x,0)=f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)$$

$\frac{\partial u^2}{\partial x^2}<0$

$\frac{\partial^2 u}{\partial x^2}>0$

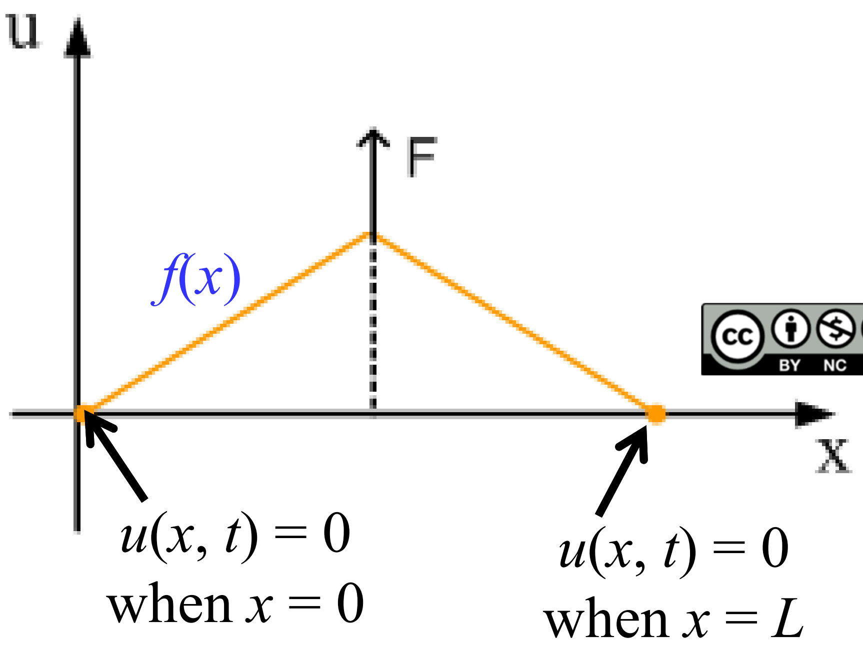


Fig. 12.2.4

12.4.3 Solutions for Wave Equations (自己挑戰解解看)

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}\qquad 0<x<L\qquad t>0$$

四大條件 $u(0,t)=0$ $\quad u(L,t)=0\quad$ for $t>0$

$u(x,0)=f(x)\quad \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)\quad$ for $0<x<L$

求解 (使用 method of separation of variables)

(Step 1) 假設解為 $u(x,t)=X(x)T(t)$ ☆☆,

(Step 2) 將 $u(x,y)=X(x)T(t)$ 代入 $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$

$$a^2X''(x)T(t)=X(x)T''(t)\qquad \frac{X''(x)}{X(x)}=\frac{T''(t)}{a^2T(t)}$$

令 $\dfrac{X''(x)}{X(x)} = \dfrac{T''(t)}{a^2 T(t)} = -\lambda$ ☆☆₂

得出2 個 ODEs $\underbrace{X''(x) + \lambda X(x) = 0}_{3\ \text{cases}}$ $\underline{T''(t) + a^2 \lambda T(t) = 0}$

☆₇

(Steps 3, 4, 5 的前處理)

(1) 因為 x 的 boundary condition 較簡單，所以先解 $X(x)$

(2) 分成 $\lambda = 0$, $\lambda < 0$, $\lambda > 0$ 三個 cases $u(x,t) = X(x)T(t)$

(3) 由於 $u(0,t) = 0$ for all $t > 0$ $u(0, t) = X(0)T(t) = 0$

$T(t)$ 不可為 0 (否則 $u(x, t) = X(x)T(t) = 0$ for any x, t)

trivial solution

所以 $\boxed{X(0) = 0}$

$X(L)T(t) = 0 \Rightarrow X(L) = 0$

同理，由 $u(L,t) = 0$ 可以立即判斷 $\boxed{X(L) = 0}$

$\boxed{X''(x) + \lambda X(x) = 0}$ subject to $\boxed{X(0) = 0}$ and $\boxed{X(L) = 0}$

$\boxed{X''(x)+\lambda X(x)=0}$ subject to $\boxed{X(0)=0}$ and $\boxed{X(L)=0}$

$\boxed{T''(t)+a^2\lambda T(t)=0}$

~~Case~~ 1 for Steps 3, 4, 5 $\lambda = 0$ ☆1

(Step 3-1) $X''(x)=0$ $X(x)=d_1x+d_0$

根據 boundary conditions

$$\begin{aligned} d_0 &= 0 \\ d_1L+d_0 &= 0 \end{aligned} \Longrightarrow \begin{aligned} d_0 &= 0 \\ d_1 &= 0 \end{aligned} \qquad X(x)=0$$

這個 case 得出 trivial solution $u(x, t) = X(x)T(t) = 0$

$u(x,0)=f(x)$ 將無法滿足 $\lambda=0$ 時無解

無需再解 Step 4-1, Step 5-1

~~Case~~ 2 of Steps 3, 4, 5: $\lambda < 0$ ☆1

(Step 3-2) 令 $\lambda = -\alpha^2$ ☆2-1

$$X''(x) - \alpha^2 X(x) = 0$$

Solution: $X(x) = d_2 e^{\alpha x} + d_3 e^{-\alpha x}$ ☆2-2

$\frac{d_4+d_5}{2} e^{\alpha x} + \frac{d_4-d_5}{2} e^{-\alpha x}$

可改寫成 $\boxed{X(x) = d_4 \cosh(\alpha x) + d_5 \sinh(\alpha x)}$ 較易處理 boundary conditions

根據 boundary conditions $X(0) = 0$ and $X(L) = 0$

$\sinh(0) = 0$ $\cosh(0) = 1$

$$d_4 = 0$$

$$d_4 \cosh(\alpha L) + d_5 \sinh(\alpha L) = 0$$

$$\Longrightarrow \quad d_4 = 0, \quad d_5 = 0 \qquad X(x) = 0$$

from page 175, $\sinh(x) \neq 0$ if $x \neq 0$

這個 case 得出 trivial solution $u(x, t) = X(x)T(t) = 0$

$u(x,0) = f(x)$ 將無法滿足 $\lambda < 0$ 時無解

無需再解 Step 4-2, Step 5-2

Case 3 of Steps 3, 4, 5: $\lambda > 0$ ☆1

(Step 3-3) 令 $\lambda = \alpha^2$ α is any positive constant

$$X''(x) + \alpha^2 X(x) = 0$$

Solution: $X(x) = \cancel{c_1 \cos \alpha x} + c_2 \sin \alpha x$

根據 boundary conditions $X(0) = 0$ and $X(L) = 0$

$$\begin{aligned} &c_1 = 0 \\ &\cancel{c_1 \cos \alpha L} + c_2 \sin \alpha L = 0 \end{aligned} \Longrightarrow \begin{aligned} &c_1 = 0 \\ &\alpha = \frac{n\pi}{L} \\ &c_2 = \text{any nonzero constant} \end{aligned}$$

$\sin(n\pi) = 0$

n 是任意整數

特別注意：☆8

不可直接由 $\begin{cases} c_1 = 0 \\ c_1 \cos \alpha L + c_2 \sin \alpha L = 0 \end{cases}$ 就斷言 $\begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$

應該看看是否有適當的 α, 讓第二個式子等於零

$$\boxed{X(x) = c_2 \sin\frac{n\pi}{L}x}$$ n 是任意正整數，c_2 是任意數

$$\alpha = \frac{n\pi}{L} \qquad \lambda = \alpha^2 = \frac{n^2\pi^2}{L^2}$$

(Step 4-3) $T''(t) + a^2\lambda T(t) = 0$

$$T''(t) + \frac{a^2n^2\pi^2}{L^2}T(t) = 0$$

Solution: $\boxed{T(t) = c_3\cos\left(\frac{na\pi}{L}t\right) + c_4\sin\left(\frac{na\pi}{L}t\right)}$ n 是任意整數

(Step 5-3)

$$u_n(x,t) = X(x)T(t) = c_2\sin\left(\frac{n\pi}{L}x\right)\left[c_3\cos\left(\frac{na\pi}{L}t\right) + c_4\sin\left(\frac{na\pi}{L}t\right)\right]$$

$$= \sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right) + B_n\sin\left(\frac{na\pi}{L}t\right)\right]$$ n 是任意正整數

$$A_n = c_2c_3, \quad B_n = c_2c_4,$$

注意：$u_n(x,t)=\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]$

只是其中一個解，因為 n 是任意整數

要將這些解加起來

(Step 6)

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]}$$

(Step 6)

$$\boxed{u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]}$$

☆9

討論：既然 n 是任意整數，那為什麼 n 是從1 加到 ∞，

而非由 $-\infty$ 加到 ∞ ？

因為 $\sin\left(\frac{n\pi}{L}x\right)=-\sin\left(\frac{-n\pi}{L}x\right)$, $\cos\left(\frac{na\pi}{L}t\right)=\cos\left(\frac{-na\pi}{L}t\right)$,

$$\sin(0)=0$$

可證明 $$\sum_{n=-\infty}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[C_n\cos\left(\frac{na\pi}{L}t\right)+D_n\sin\left(\frac{na\pi}{L}t\right)\right]$$

$$=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]$$

$$A_n=C_n-C_{-n}\qquad\qquad B_n=D_n+D_{-n}$$

(Step 7) $$\boxed{u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]}$$

☆10

由 initial conditions

$$u(x,0)=f(x) \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)$$

$$\left.\frac{\partial}{\partial t}\sin\left(\frac{na\pi}{L}t\right)\right|_{t=0}=\frac{na\pi}{L}$$

$$f(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right) \qquad g(x)=\sum_{n=1}^{\infty}B_n\frac{na\pi}{L}\sin\left(\frac{n\pi}{L}x\right)$$

☆10

也就是說，A_n 是 $f(x)$ 的 Fourier sine series (Sec. 11-3, page 389),

$p \to L$

$B_n\dfrac{na\pi}{L}$ 是 $g(x)$ 的 Fourier sine series

$$\boxed{A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx}$$

$$B_n\frac{na\pi}{L}=\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx \qquad \boxed{B_n=\frac{2}{na\pi}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx}$$

$b_n=B_n\dfrac{na\pi}{L}$, $p \to L$

12.4.4 名詞

period: $\frac{2\pi}{na\pi/L} = \frac{2L}{na}$

$$u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\underline{\frac{na\pi}{L}}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]$$

frequency $= \frac{1}{\text{period}} = \frac{a}{2L}n$

$$u(x,t)=u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots\cdots\cdots$$

其中 $$u_n(x,t)=\sin\left(\frac{n\pi}{L}x\right)\left[A_n\cos\left(\frac{na\pi}{L}t\right)+B_n\sin\left(\frac{na\pi}{L}t\right)\right]$$

$$=C_n\sin\left(\frac{n\pi}{L}x\right)\left[\sin\left(\frac{na\pi}{L}t+\phi_n\right)\right]$$

$$C_n=\sqrt{A_n^2+B_n^2}\qquad \cos\phi_n=\frac{B_n}{C_n}\qquad \sin\phi_n=\frac{A_n}{C_n}$$

$u_n(x, t)$ 被稱作 standing waves (駐波) 或 normal modes

$n=1$ 時， $u_1(x, t)$ 被稱作 first standing wave 或 first normal mode 或 fundamental mode of vibration

$$u_1(x,t)=C_1\sin\left(\frac{\pi}{L}x\right)\left[\sin\left(\frac{a\pi}{L}t+\phi_1\right)\right]$$

$$u_1\left(x,t+\frac{2L}{a}\right)=C_1\sin\left(\frac{\pi}{L}x\right)\left[\sin\left(\frac{a\pi}{L}t+2\pi+\phi_1\right)\right]=u_1(x,t)$$

對於 t 而言，週期 $=\dfrac{2L}{a}$ 頻率 =1/週期 $=\dfrac{a}{2L}$

$f_1=\dfrac{a}{2L}$ 被稱作 fundamental frequency (基頻) 或 first harmonic

$n=1$ 名1

以此類推， $u_2(x, t)$ 被稱作 second standing wave

$u_3(x, t)$ 被稱作 third standing wave

First standing wave

Second standing wave

Third standing wave

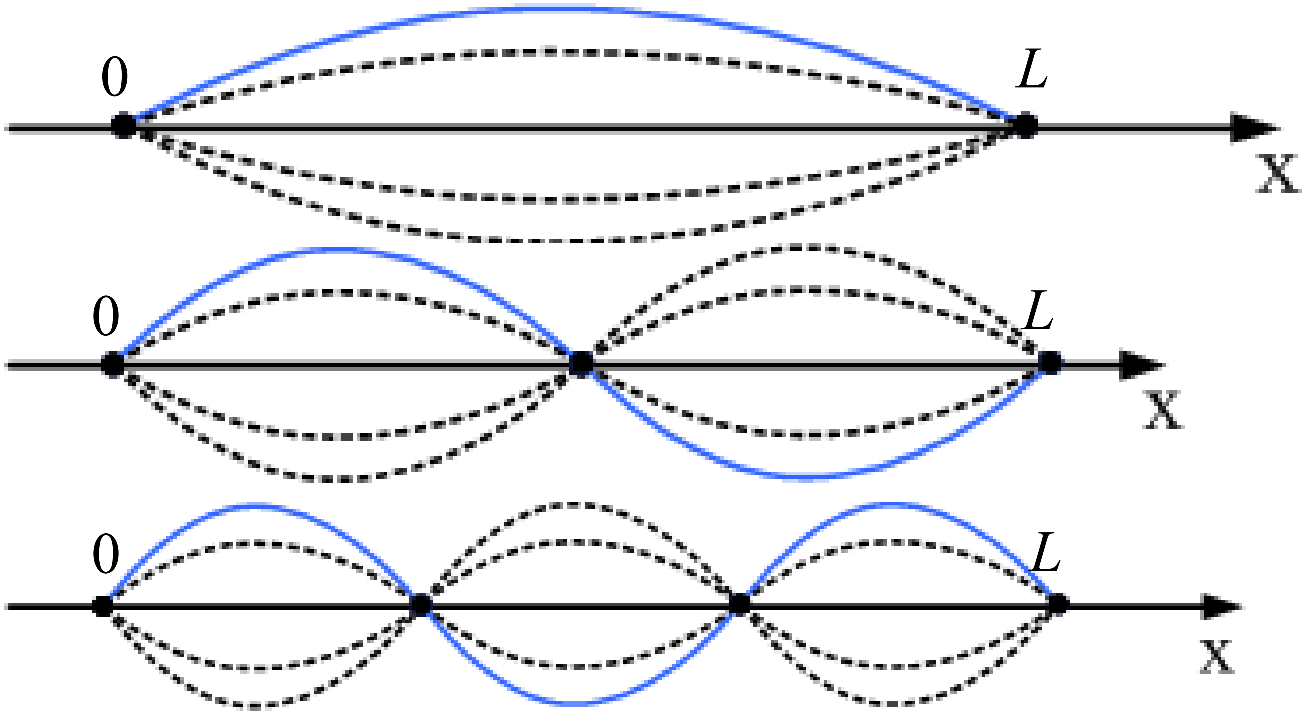


Fig. 12.4.2

$$u_n(x,t) = C_n \sin\left(\frac{n\pi}{L}x\right)\left[\sin\left(\frac{na\pi}{L}t + \phi_n\right)\right]$$

$x = \dfrac{L}{n}$ 時，無論 t 等於多少，$u_n\left(\dfrac{L}{n}, t\right) = 0$

$x = \dfrac{L}{n}$ 是 n^{th} standing wave 的 node (節點)

$$u_n(x,t) = u_n\left(x, t + \frac{2L}{an}\right)$$

$u_n(x, t)$ 的頻率 =1/週期 = $n\dfrac{a}{2L}$

$f_n = n\dfrac{a}{2L} = nf_1$ 被稱作 overtones (泛音)

12.4.5 Sections 12.4 需要注意的地方

(1) Method of separation of variables 解 PDE 的過程雖然長，但是把握住講義 pages 422-424 的 7 個 steps，並練習幾次，就可以熟悉。

(這些對大二下和大三上的電磁學很重要)

(2) 雖然概念不難，但是計算過程很長且繁雜

所以一定要多研究簡化運算、快速判斷的方法

(3) 有沒有注意到，

若 boundary conditions 出現 $u(0, y) = 0,\ u(L, y) = 0,$

最後的解總是和 sine 有關 $X(x) = c_2 \sin\frac{n\pi}{L}x$ 週期為 $2L/n$

$X'(0)=0$ $X'(L)=0$

若 boundary conditions 出現 $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$ $\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$

Case 3 Case 1

最後的解總是和 cosine 或 constant 有關 Case 2 無解

$X(x) = c_1$ or $X_n(x) = c_1 \cos\frac{n\pi}{L}x$ 週期也為 $2L/n$

(4) 經驗足夠後，看到 $u(x, y)$ 的 boundary conditions

出現 $u(a, y) = 0 \longrightarrow$ 就知道 $X(a) = 0$，

看到 $u(x, b) = 0 \longrightarrow$ 就知道 $Y(b) = 0$。

看到 $\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0 \longrightarrow$ 就知道 $X'(a) = 0$，

看到 $\left.\frac{\partial u}{\partial y}\right|_{y=b} = 0 \longrightarrow$ 就知道 $Y'(b) = 0$

☆7

(5) 對於 wave equations 而言， $X(x)$ 和 $T(t)$ 的解有相同的型態

如果 $X(x)$ 為 sine & cosine, $T(t)$ 也為 sine & cosine

對於 Laplace's equations而言， $X(x)$ 和 $Y(y)$ 的解型態不同

如果$X(x)$ 為 sine & cosine, $Y(y)$ 為 sinh & cosh

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2 T(t)} = -\lambda \qquad \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

(6) 要熟悉 $\cosh(x)$, $\sinh(x)$ 的性質

(7) Method of separation of variables 在計算上容易出錯的地方

(以 講義 pages 440-448 wave equations 為例)

(a) $$\frac{X''(x)}{X(x)}=\frac{T''(t)}{a^2T(t)}=-\lambda$$

(b) Steps 3, 4, 5 要考慮所有 cases

(c) 不可直接由 $c_1=0$ 及 $c_1\cos\alpha L+c_2\sin\alpha L=0$ 判斷 $c_1=c_2=0$

因為 α 可以是 $\pi n/L$, 如講義 page 444 所述

(d) 在 Step 6，要將所有可能的解加起來，才是 $u(x, t)$ 的一般解

如講義 pages 429, 446, 447 所述

Exercise for Practice

Section 12-1	3, 6, 9, 10, 12, 14, 16, 18, 22, 23, 30, 32
Section 12-4	1, 4, 7, 10, 11, 15, 17, 21, 23
Review 12	1, 2, 5, 13

Happy New Year!

祝各位期末考順利，寒假愉快！