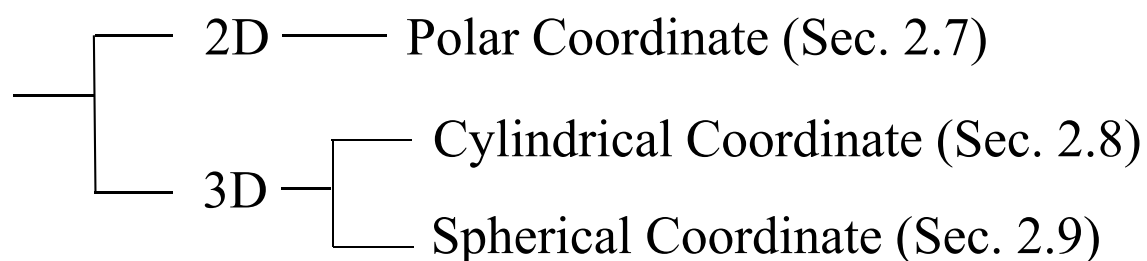
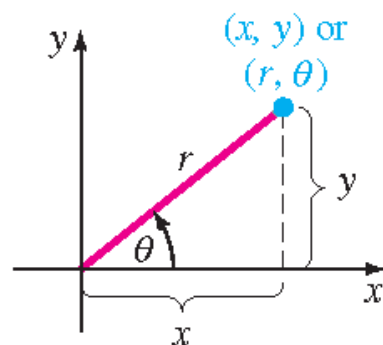


2.7 Polar Coordinates

Sections 2.7, 2.8, 2.9 are extended from Sec. 2.1, but the **polar, cylindrical, and spherical coordinates** are adopted.



D. G. Zill and Michael R. Cullen, *Differential Equations-with Boundary-Value Problem (metric version)*, 9th edition, Cengage Learning, 2017, Section 13.1



From D. G. Zill and Michael R. Cullen, *Differential Equations-with Boundary-Value Problem (metric version)*, 9th edition, Cengage Learning, 2017, Section 13.1.

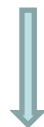
Fig. 2.7.1 Polar coordinates of a point (x, y) are (r, θ)

(x, y)	→	(r, θ)
original coordinate		polar coordinate

$$x = r \cos \theta, \quad y = r \sin \theta, \quad \text{and} \quad r^2 = x^2 + y^2$$

the Laplacian of u in
 x - y coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$



the Laplacian of u in
polar coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

In this section we focus only on boundary-value problems involving **Laplace's equation** $\nabla^2 u = 0$ in polar coordinates:

The key points
of this section.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0.$$

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}} \longrightarrow \boxed{\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}}$$

(Proof): Since

From page 184 to page 188

$$x = r \cos \theta, \quad y = r \sin \theta$$

we have

$$r = \sqrt{x^2 + y^2} \quad \theta = \arctan \frac{y}{x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}} \frac{\partial u}{\partial r} + \frac{1}{1 + (y/x)^2} \frac{-y}{x^2} \frac{\partial u}{\partial \theta}$$

$$\text{from } \frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$

$$= \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial u}{\partial r} + \frac{1}{\sqrt{x^2 + y^2}} \frac{-y}{\sqrt{x^2 + y^2}} \frac{\partial u}{\partial \theta} = \cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta}$$

$$r = \sqrt{x^2 + y^2} \quad \theta = \arctan \frac{y}{x}$$

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial r}{\partial y} \frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial u}{\partial \theta} = \frac{2y}{2\sqrt{x^2 + y^2}} \frac{\partial u}{\partial r} + \frac{1}{1 + (y/x)^2} \frac{1}{x} \frac{\partial u}{\partial \theta} \\ &= \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial u}{\partial r} + \frac{1}{\sqrt{x^2 + y^2}} \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial u}{\partial \theta} = \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \end{aligned}$$

$$\frac{\partial u}{\partial x} = \cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial u}{\partial y} = \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \frac{\partial u}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} \frac{\partial u}{\partial x} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} \frac{\partial u}{\partial x} \\ &= \cos \theta \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) \\ &= \cos^2 \theta \frac{\partial^2 u}{\partial r^2} + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial \theta \partial r} \\ &\quad + \frac{\sin^2 \theta}{r} \frac{\partial u}{\partial r} - \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial \theta \partial r} + \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} \end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos^2 \theta \frac{\partial^2 u}{\partial r^2} - \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial \theta \partial r} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \frac{\partial u}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial}{\partial r} \frac{\partial u}{\partial y} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} \frac{\partial u}{\partial y}$$

$$= \sin \theta \frac{\partial}{\partial r} \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right) + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right)$$

$$= \sin^2 \theta \frac{\partial^2 u}{\partial r^2} - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} \\ + \frac{\cos^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2 \theta \frac{\partial^2 u}{\partial r^2} + \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial \theta \partial r} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2 \theta}{r} \frac{\partial u}{\partial r} - \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos^2 \theta \frac{\partial^2 u}{\partial r^2} - \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial \theta \partial r} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2 \theta \frac{\partial^2 u}{\partial r^2} + \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial \theta \partial r} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2 \theta}{r} \frac{\partial u}{\partial r} - \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

The proof is completed.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

In this section, we focus on the Laplace's equation with steady temperature, i.e.,

$$\nabla^2 u = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

[Example 1] Steady Temperatures in a Circular Plate

Solve Laplace's equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to $u(c, \theta) = f(\theta), \quad 0 < \theta < 2\pi$

SOLUTION

(Step 1) Suppose that $u(r, \theta) = R(r)\Theta(\theta)$

$$R''(r)\Theta(\theta) + \frac{1}{r} R'(r)\Theta(\theta) + \frac{1}{r^2} R(r)\Theta''(\theta) = 0$$

$$r^2 \frac{R''(r)}{R(r)} + r \frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

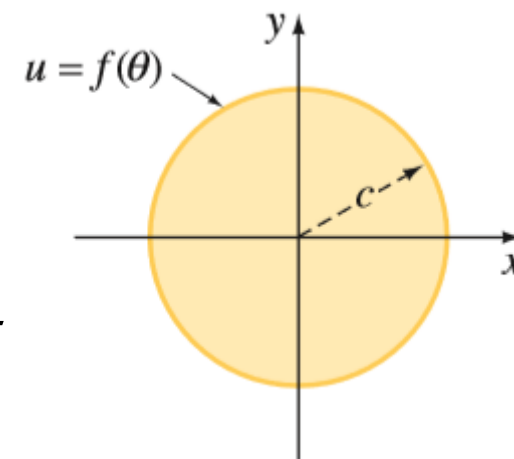


Fig. 2.8.2

From D. G. Zill and Michael R. Cullen, *Differential Equations-with Boundary-Value Problem (metric version)*, 9th edition, Cengage Learning, 2017, Section 13.2.

$$\text{(Step 2)} \quad \frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

$$\Downarrow$$

$$r^2 R'' + rR' - \lambda R = 0 \quad \Theta'' + \lambda \Theta = 0.$$

There is no zero boundary condition.

But note that $\Theta(\theta)$ should be periodic: $\Theta(\theta) = \Theta(\theta + 2\pi)$

(Step 3) Then, we try to solve

$$\Theta''(\theta) + \lambda \Theta(\theta) = 0, \quad \Theta(\theta) = \Theta(\theta + 2\pi)$$

$$\text{Case 1: } \lambda = 0, \implies \Theta''(\theta) = 0, \implies \Theta(\theta) = c_1 + c_2 \theta$$

$$\text{from } \Theta(\theta) = \Theta(\theta + 2\pi) \implies \Theta(\theta) = c_1$$

$$\text{Case 2: } \lambda < 0, \text{ set } \lambda = -\alpha^2 \implies \Theta''(\theta) - \alpha^2 \Theta(\theta) = 0,$$

$$\implies \Theta(\theta) = c_1 \cosh \alpha \theta + c_2 \sinh \alpha \theta \implies \Theta(\theta) = 0$$

(trivial)

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \quad \Theta(\theta) = \Theta(\theta + 2\pi)$$

Case 3: $\lambda > 0$, set $\lambda = \alpha^2 \implies \Theta''(\theta) + \alpha^2\Theta(\theta) = 0$,

$$\implies \Theta(\theta) = c_1 \cos \alpha\theta + c_2 \sin \alpha\theta$$

From $\Theta(\theta) = \Theta(\theta + 2\pi)$

$$c_1 \cos \alpha\theta + c_2 \sin \alpha\theta = c_1 \cos(\alpha\theta + \alpha 2\pi) + c_2 \sin(\alpha\theta + \alpha 2\pi)$$

$$\alpha 2\pi = n 2\pi \quad \text{where } n \text{ is a positive integer,}$$

$$\alpha = n, \quad (\lambda = n^2)$$

$$\Theta(\theta) = c_1 \cos n\theta + c_2 \sin n\theta \quad \text{where } n \text{ is a positive integer,}$$

Combine the results of Cases 1 and 3

$$\Theta(\theta) = c_1 \cos n\theta + c_2 \sin n\theta \quad (\lambda = n^2)$$

where n is a **nonnegative** integer

(Step 4)

$$r^2 R'' + rR' - \lambda R = 0$$

This is an important application of the
Cauchy-Euler equation on page 37

Since $\lambda_n = n^2$, $n = 0, 1, 2, \dots$

$$r^2 R'' + rR' - n^2 R = 0 \longleftarrow \text{Cauchy-Euler}$$

Auxiliary: $m(m-1) + m - n^2 = 0$, $m = \pm n$

the solutions are

$$R(r) = c_3 + c_4 \ln r, \quad n = 0$$

$$R(r) = c_5 r^n + c_6 r^{-n}, \quad n = 1, 2, \dots$$

Since $\ln 0 \rightarrow -\infty$ $0^{-n} \rightarrow \infty$ but $R(0)$ should not be infinite

$c_4 = c_6 = 0$ should be satisfied

$$\implies R(r) = c_3, \quad n = 0, \quad R(r) = c_5 r^n, \quad n = 1, 2, \dots$$

$\Theta(\theta) = c_1 \cos n\theta + c_2 \sin n\theta$ where n is a nonnegative integer

$$R(r) = c_3, \quad n = 0, \quad R(r) = c_5 r^n, \quad n = 1, 2, \dots$$

(Step 5) $u_n(r, \theta) = R(r)\Theta(\theta)$

$$u_0(r, \theta) = A_0 \text{ when } n = 0,$$

$$u_n(r, \theta) = r^n (A_n \cos n\theta + B_n \sin n\theta) \text{ when } n = 1, 2, \dots$$

(Step 6)

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

(Step 7)

By applying the boundary condition $u(c, \theta) = f(\theta)$, $0 < \theta < 2\pi$.

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} c^n (A_n \cos n\theta + B_n \sin n\theta)$$

Next, solve the unknowns from the formula of the Fourier series

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} c^n (A_n \cos n\theta + B_n \sin n\theta)$$

From the formula of the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right) \quad -p < x < p$$

$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx$$

$$a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx$$

$$b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx$$



$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta$$

$$A_n = \frac{1}{c^n \pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta$$

$$B_n = \frac{1}{c^n \pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta.$$



Since $f(\theta) = f(\theta + 2\pi)$

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$$

$$A_n = \frac{1}{c^n \pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta$$

$$B_n = \frac{1}{c^n \pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta.$$

EXAMPLE 2 Steady Temperatures in a Semicircular Plate

Find the steady-state temperature $u(r, \theta)$ in

From D. G. Zill and Michael R. Cullen,
Differential Equations-with Boundary-
Value Problem (metric version), 9th
edition, Cengage Learning, 2017,
Section 13.2.

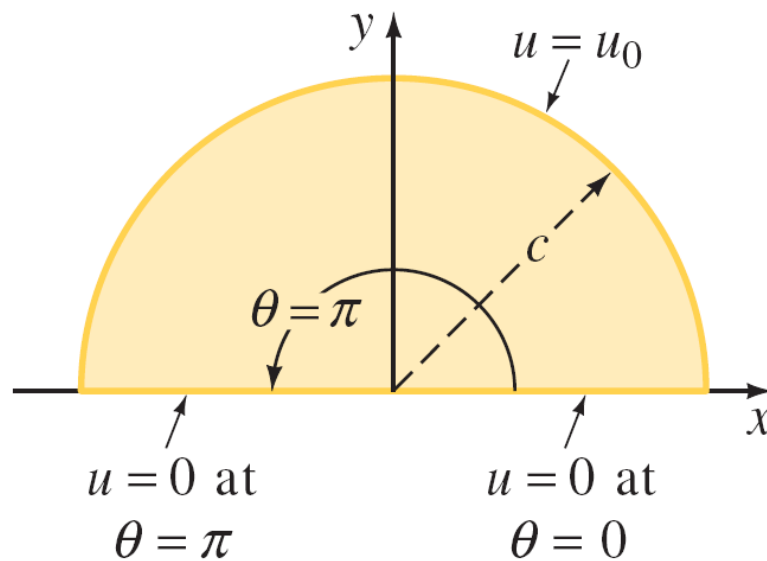


Fig. 2.8.3
in Example 2

Semicircular plate

SOLUTION The problem can be formulated as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

$$u(c, \theta) = u_0, \quad 0 < \theta < \pi$$

$$u(r, 0) = 0, \quad u(r, \pi) = 0, \quad 0 < r < c.$$

(Step 1) Suppose that $u(r, \theta) = R(r)\Theta(\theta)$

$$R''(r)\Theta(\theta) + \frac{1}{r} R'(r)\Theta(\theta) + \frac{1}{r^2} R(r)\Theta''(\theta) = 0$$

$$r^2 \frac{R''(r)}{R(r)} + r \frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

(Step 2)
$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$r^2 R'' + rR' - \lambda R = 0$$

$$\Theta'' + \lambda \Theta = 0.$$

(Step 3)

$$\begin{aligned} \text{From } u(r, 0) = 0, \quad u(r, \pi) = 0, \\ R(r)\Theta(0) = 0, \quad R(r)\Theta(\pi) = 0, \\ \Theta(0) = 0 \text{ and } \Theta(\pi) = 0. \end{aligned}$$

We then try to solve

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(0) = 0, \quad \Theta(\pi) = 0.$$

$$\text{Case 1: } \lambda = 0 \Rightarrow \Theta''(\theta) = 0 \Rightarrow \Theta(\theta) = c_1\theta + c_2$$

$$\text{From } \Theta(0) = 0, \quad \Theta(\pi) = 0 \Rightarrow \Theta(\theta) = 0$$

$$\text{Case 2: } \lambda < 0, \quad \text{set } \lambda = -\alpha^2 \Rightarrow \Theta''(\theta) - \alpha^2\Theta(\theta) = 0$$

$$\Rightarrow \Theta(\theta) = c_3 \cosh \alpha \theta + c_4 \sinh \alpha \theta$$

$$\text{From } \Theta(0) = 0, \quad \Theta(\pi) = 0 \Rightarrow \Theta(\theta) = 0$$

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(0) = 0, \quad \Theta(\pi) = 0.$$

Case 3: $\lambda > 0$, set $\lambda = \alpha^2 \Rightarrow \Theta''(\theta) + \alpha^2\Theta(\theta) = 0$

$$\Rightarrow \Theta(\theta) = c_5 \cos \alpha\theta + c_6 \sin \alpha\theta$$

From $\Theta(0) = 0$, $\Theta(\pi) = 0 \Rightarrow c_5 = 0$, $\alpha = n$

$$\Rightarrow \Theta(\theta) = c_6 \sin n\theta \quad n = 1, 2, 3, \dots$$

The only nontrivial solution for $\Theta(\theta)$ is

$$\Theta(\theta) = c_6 \sin n\theta \quad n = 1, 2, 3, \dots$$

In this case, $\lambda = n^2$

(Step 4) To solve $R(r)$

$$r^2 R'' + rR' - \lambda R = 0 \Rightarrow \underline{r^2 R'' + rR' - n^2 R = 0}$$

$$R(r) = c_7 r^n + c_8 r^{-n}$$

To be bounded at $r = 0$, c_8 must be 0

$$R(r) = c_7 r^n$$

$$n = 1, 2, 3, \dots$$

$$\text{(Step 5)} \quad u_n(r, \theta) = R(r)\Theta(\theta) = A_n r^n \sin n\theta,$$

$$\text{(Step 6)} \quad u(r, \theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

$$\text{(Step 7) From } u(c, \theta) = u_0$$

$$u_0 = \sum_{n=1}^{\infty} A_n c^n \sin n\theta$$

Using Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$0 < x < p$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

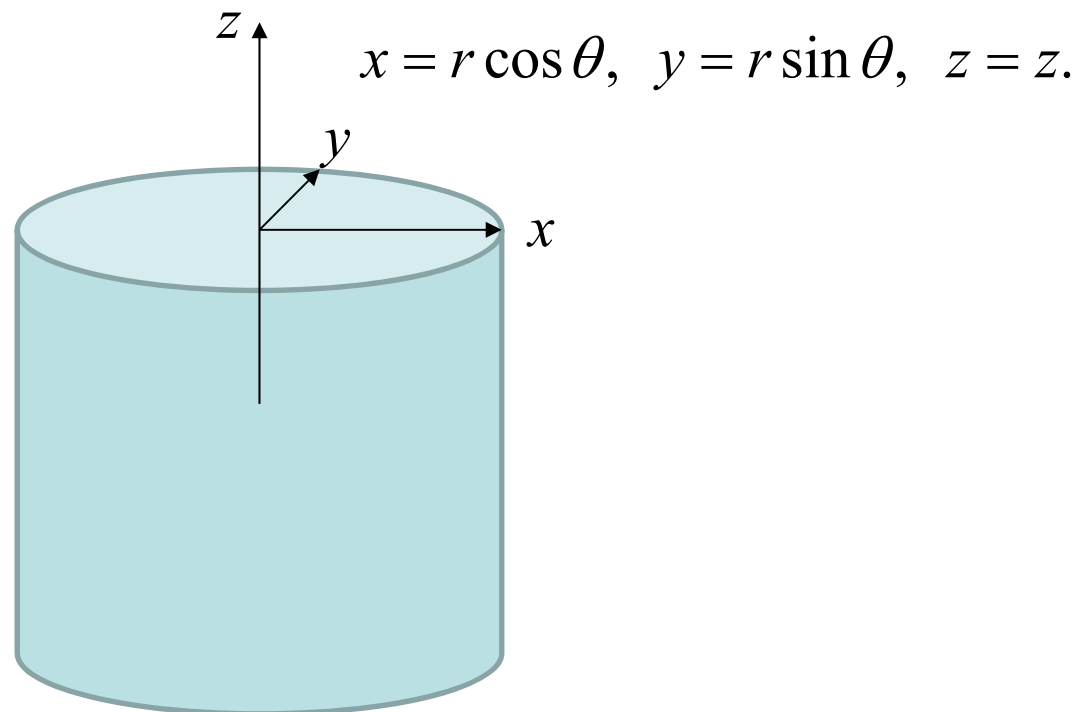
$$A_n c^n = \frac{2}{\pi} \int_0^{\pi} u_0 \sin n\theta d\theta$$

$$A_n = \frac{2u_0}{\pi c^n} \frac{1 - (-1)^n}{n}$$

Solution:

$$u(r, \theta) = \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \left(\frac{r}{c}\right)^n \sin n\theta$$

2.8 CYLINDRICAL COORDINATES



D. G. Zill and Michael R. Cullen, *Differential Equations-with Boundary-Value Problem (metric version)*, 9th edition, Cengage Learning, 2017, Section 13.2

2.8.1 Review for Special Functions

- Bessel's equation of order ν

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \quad \text{Solution: } c_1 J_\nu(x) + c_2 Y_\nu(x)$$

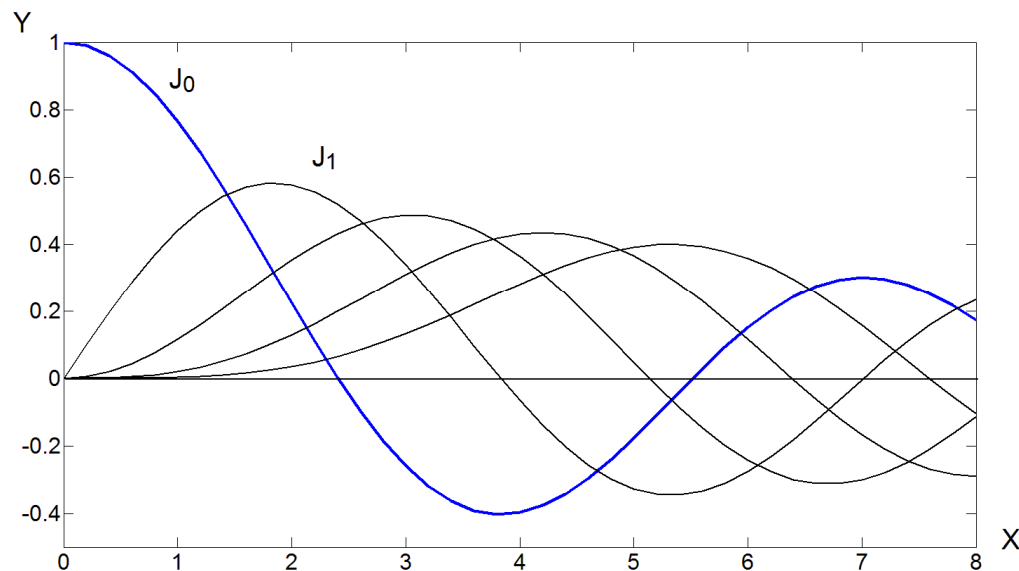
$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \nu + n)} \left(\frac{x}{2}\right)^{2n+\nu} \quad : \text{1}^{\text{st}} \text{ kind Bessel function}$$

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} \quad : \text{2}^{\text{nd}} \text{ kind Bessel function}$$

where $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ (gamma function)

$$\Gamma(n+1) = n!$$

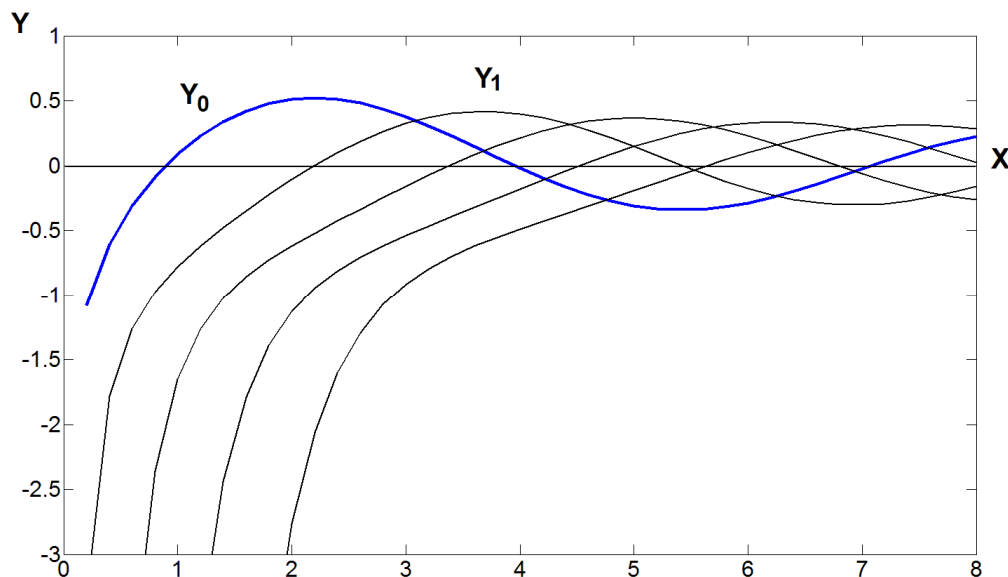
$$\Gamma(x+1) = x\Gamma(x)$$

1st kind Bessel function

$$J_0(0) = 1$$

$$J_\nu(0) = 0 \quad \text{for } \nu \neq 0$$

$$\lim_{x \rightarrow \infty} J_\nu(x) = 0$$

2nd kind Bessel function

$$Y_\nu(0) \rightarrow -\infty$$

From D. G. Zill and Michael R. Cullen,
Differential Equations-with Boundary-Value
Problem (metric version), 9th edition,
Cengage Learning, 2017, Section 6.4.

Zeros of J_0 , J_1 , Y_0 , and Y_1

	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
x_1	2.4048	0.0000	0.8936	2.1971
x_2	5.5201	3.8317	3.9577	5.4297
x_3	8.6537	7.0156	7.0861	8.5960
x_4	11.7915	10.1735	10.2223	11.7492
x_5	14.9309	13.3237	13.3611	14.8974

From D. G. Zill and Michael R. Cullen,
Differential Equations-with Boundary-Value
Problem (metric version), 9th edition,
Cengage Learning, 2017, Section 6.4.

• Generalization of Bessel's equation of order ν

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \quad \text{解： } c_1 J_\nu(x) + c_2 Y_\nu(x)$$

$$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2)y = 0 \quad \text{解： } c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$$

Proof: Set $t = \alpha x$

$$\frac{dy}{dx} = \frac{dt}{dx} \frac{dy}{dt} = \alpha \frac{dy}{dt}$$

Similarly,
$$\frac{d^2 y}{dx^2} = \frac{dt}{dx} \frac{d}{dt} \left(\frac{dy}{dx} \right) = \alpha \frac{d}{dt} \left(\alpha \frac{dy}{dt} \right) = \alpha^2 \frac{d^2 y}{dt^2}$$

$$\text{原式} = x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2)y = \frac{t^2}{\alpha^2} \alpha^2 \frac{d^2 y}{dt^2} + \frac{t}{\alpha} \alpha \frac{dy}{dt} + \left(\alpha^2 \frac{t^2}{\alpha^2} - \nu^2 \right) y$$

$$= \boxed{t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + (t^2 - \nu^2)y = 0} \rightarrow \text{對 } t \text{ 而言是 Bessel equation}$$

$$y = c_1 J_\nu(t) + c_2 Y_\nu(t) = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$$

• **Modified Bessel's equation of order ν**

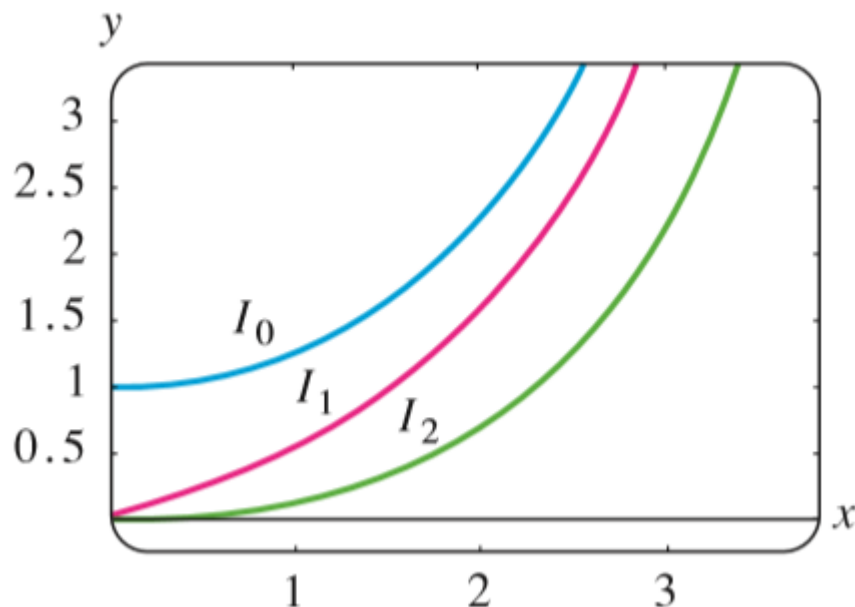
$$x^2 y'' + xy' + (-x^2 - \nu^2)y = 0 \quad \text{解： } c_1 I_\nu(x) + c_2 K_\nu(x)$$

$$x^2 y'' + xy' + (-\alpha^2 x^2 - \nu^2)y = 0 \quad \text{解： } c_1 I_\nu(\alpha x) + c_2 K_\nu(\alpha x)$$

其中 $I_\nu(x) = i^{-\nu} J_\nu(ix)$ 稱作是 **modified** Bessel function of the first kind of order ν

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi} \quad \text{稱作是 } \mathbf{modified} \text{ Bessel function of the second kind of order } \nu$$

當 ν 為整數時，也是取 limit

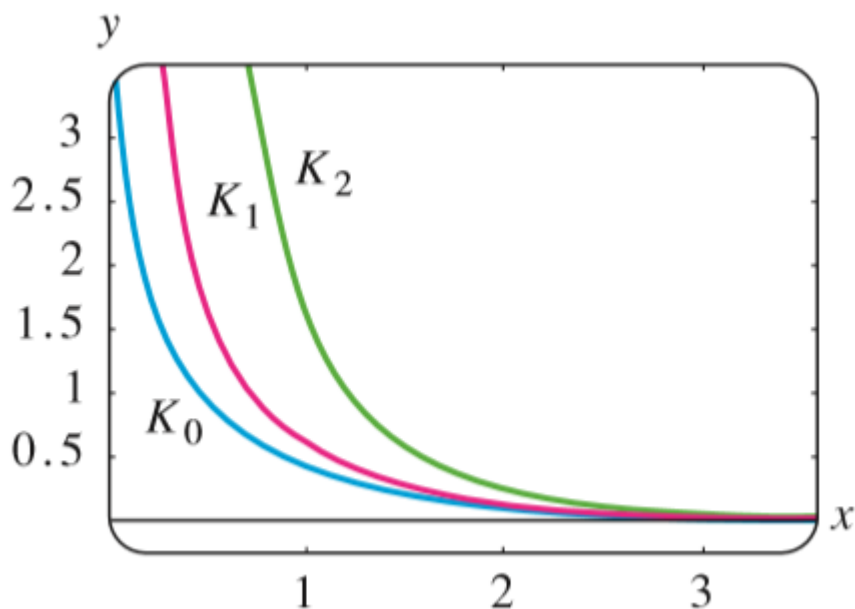


Modified Bessel functions of the first kind for $n = 0, 1, 2$

$$I_0(0) = 1$$

$$I_\nu(0) = 0 \quad \text{for } \nu \neq 0$$

$$I_\nu(x) \neq 0 \quad \text{for } x \neq 0$$



Modified Bessel functions of the second kind for $n = 0, 1, 2$

$$K_\nu(0) \rightarrow \infty$$

$$K_\nu(x) \neq 0 \quad \text{for } x \neq 0$$

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

- Legendre's equation of order n

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

One of the solution: Legendre polynomials $P_n(x)$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

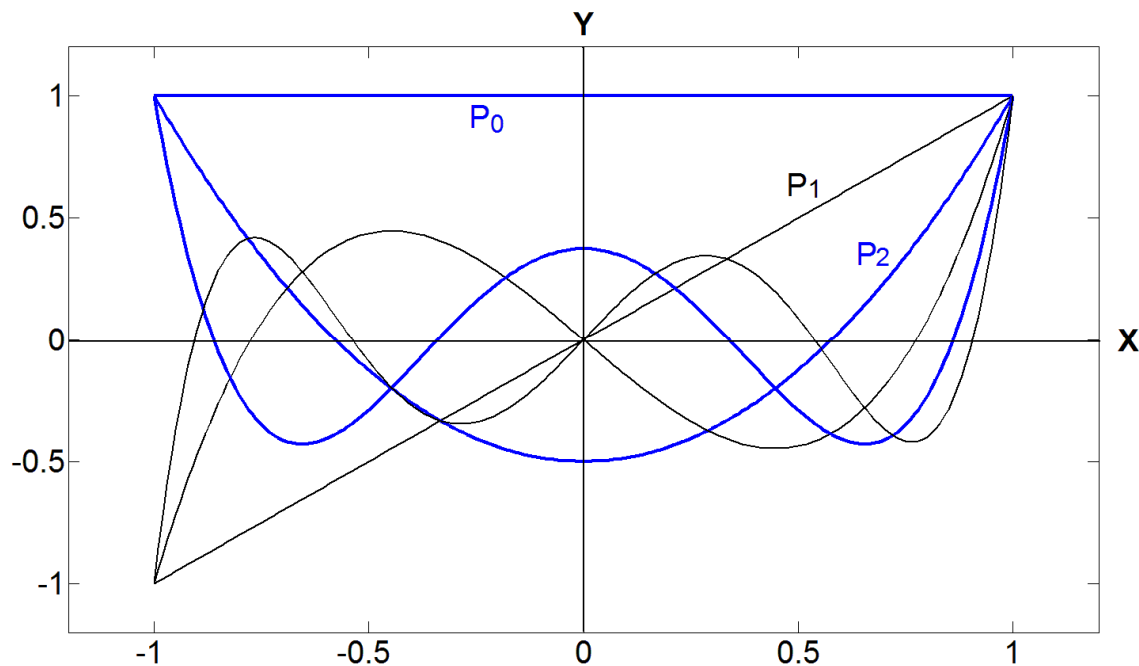
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$\int_{-1}^1 P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{n,m}$$

Legendre polynomials



Interval:

$$x \in [-1, 1]$$



From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

2.8.2 PDE for Cylindrical Coordinates

Laplacian in Cylindrical Coordinates

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

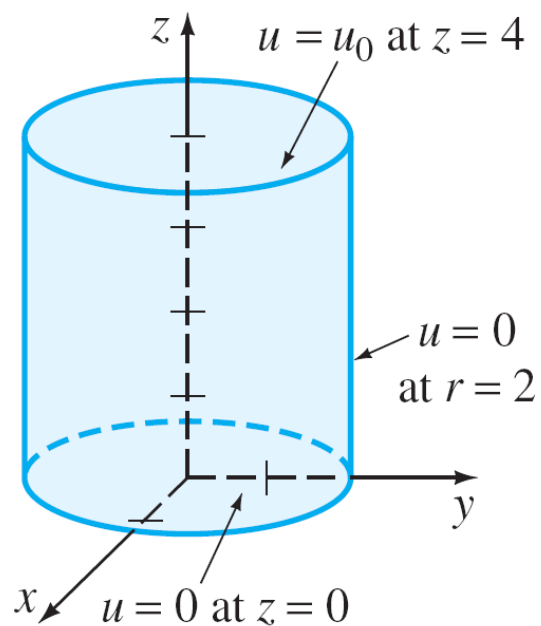
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

For the case of **radial symmetry** $\frac{\partial u}{\partial \theta} = 0$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}.$$

[Example 1] Steady Temperatures in a Circular Cylinder



From D. G. Zill and Michael R. Cullen, *Differential Equations-with Boundary-Value Problem (metric version)*, 9th edition, Cengage Learning, 2017, Section 13.2.

FIGURE 13.2.5 Circular cylinder in Example 2

$$u(2, z) = 0, \quad 0 < z < 4$$

$$u(r, 0) = 0, \quad u(r, 4) = u_0, \quad 0 < r < 2.$$

It is a problem of the 2D heat equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} = 0$$

steady temperatures



$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

radial symmetry

(since $u = 0$ when $r = 2$)



$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$u(2, z) = 0, \quad 0 < z < 4$$

$$u(r, 0) = 0, \quad u(r, 4) = u_0, \quad 0 < r < 2$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

(Step 1) $u(r, z) = R(r)Z(z)$

$$R''(r)Z(z) + \frac{1}{r}R'(r)Z(z) + R(r)Z''(z) = 0$$

$$\frac{R''(r)}{R(r)} + \frac{1}{r} \frac{R'(r)}{R(r)} + \frac{Z''(z)}{Z(z)} = 0$$

(Step 2) $\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -\lambda$

$$rR'' + R' + \lambda rR = 0$$

$$Z'' - \lambda Z = 0$$

From $u(2, z) = 0, \quad u(r, 0) = 0,$

$$R(2) = 0 \quad Z(0) = 0$$

$$rR'' + R' + \lambda rR = 0 \quad R(2) = 0$$

$$Z'' - \lambda Z = 0 \quad Z(0) = 0$$

(Step 3)

Case 1: $\lambda = 0$ $rR'' + R' = 0$

auxiliary: $m(m-1) + m = 0, \quad m = 0, 0$

$$R(r) = c_1 + c_2 \ln r$$

Since $\ln 0 \rightarrow -\infty, c_2 = 0$

Since $R(2) = 0 \rightarrow c_1 + c_2 \ln 2 = 0 \rightarrow c_1 = 0$

$$R(r) = 0 \quad (\text{trivial})$$

Case 2: $\lambda = -\alpha^2 < 0$ $rR'' + R' - \alpha^2 rR = 0$

$$r^2 R'' + rR' - \alpha^2 r^2 R = 0$$

compared to modified Bessel function (page 205)

$$x^2 y'' + xy' - (x^2 + \nu^2)y = 0 \quad \text{solution : } c_1 I_\nu(x) + c_2 K_\nu(x)$$

$$x^2 y'' + xy' - (\alpha^2 x^2 + \nu^2)y = 0 \quad \text{solution : } c_1 I_\nu(\alpha x) + c_2 K_\nu(\alpha x)$$

$I_\nu(x)$: modified Bessel function of the 1st kind

$K_\nu(x)$: modified Bessel function of the 2nd kind

The solution of $rR'' + R' - \alpha^2 rR = 0$ is

$$R(r) = c_1 I_0(\alpha r) + c_2 K_0(\alpha r)$$

(page 206) Since $K_0(0) \rightarrow \infty$, $c_2 = 0$

Since $R(2) = 0$, $c_1 I_0(2\alpha) = 0$

$$\implies R(r) = 0$$

(trivial)

(page 206) From $I_0(x) \neq 0$ for all $x \neq 0 \rightarrow c_1 = 0$

Case 3: $\lambda = \alpha^2 > 0$

$$rR'' + R' + \alpha^2 rR = 0$$

$$r^2 R'' + rR' + \alpha^2 r^2 R = 0$$

compared to the Bessel function on page 204

$$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2)y = 0$$

solution : $c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$

$$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$$

(page 202)

$$\text{Since } Y_0(0) \rightarrow -\infty, \quad c_2 = 0 \quad R(r) = c_1 J_0(\alpha r)$$

$$\text{Since } R(2) = 0 \quad c_1 J_0(2\alpha) = 0$$

Therefore,

$$R(r) = c_1 J_0(\alpha_n r) \quad \text{where} \quad \alpha_n = x_n / 2$$

x_n are the zeros of $J_0(x)$, i.e., $J_0(x_n) = 0$

(defined on page 203) $\lambda_n = \alpha_n^2, \quad \lambda_n = x_n^2 / 4$

(Step 4) Try to solve

$$Z'' - \lambda Z = 0 \quad Z(0) = 0$$

Since $\lambda_n = \alpha_n^2$,

$$Z'' - \alpha_n^2 Z = 0$$

$$Z(z) = c_3 \cosh(\alpha_n z) + c_4 \sinh(\alpha_n z)$$

From $Z(0) = 0$, $c_3 = 0$

$$Z(z) = c_4 \sinh(\alpha_n z) \quad \text{where } \alpha_n = x_n / 2 \quad J_0(x_n) = 0$$

(Step 5) $u(r, z) = R(r)Z(z) = A_n J_0(\alpha_n r) \sinh(\alpha_n z)$

(Step 6) $u(r, z) = \sum_{n=1}^{\infty} A_n \sinh \alpha_n z J_0(\alpha_n r)$.

where $\alpha_n = x_n / 2 \quad J_0(x_n) = 0$

(Step 7) From $u(r, 4) = u_0$,

$$u_0 = \sum_{n=1}^{\infty} A_n \sinh(4\alpha_n) J_0(\alpha_n r).$$

From Eq. (9) in Section 11.5 of D. G. Zill and Michael R. Cullen, *Differential Equations-with Boundary-Value Problem (metric version)*, 9th edition, Cengage Learning, 2017

$$\int_0^2 r J_0(\alpha_n r) J_0(\alpha_m r) dr = \begin{cases} 0 & \text{if } m \neq n \\ 2J_1^2(2\alpha_n) & \text{if } m = n \end{cases}$$

$$\begin{aligned} \int_0^2 u_0 r J_0(\alpha_m r) dr &= \sum_{n=1}^{\infty} A_n \sinh(4\alpha_n) \int_0^2 J_0(\alpha_n r) r J_0(\alpha_m r) dr \\ &= A_m \sinh(4\alpha_m) 2J_1^2(2\alpha_m) \end{aligned}$$

$$A_n = \frac{u_0}{2 \sinh(4\alpha_n) J_1^2(2\alpha_n)} \int_0^2 r J_0(\alpha_n r) dr = \frac{u_0}{\alpha_n \sinh(4\alpha_n) J_1(2\alpha_n)}$$

(Solution):

$$u(r, z) = u_0 \sum_{n=1}^{\infty} \frac{1}{\alpha_n \sinh(4\alpha_n) J_1(2\alpha_n)} \sinh(\alpha_n z) J_0(\alpha_n r)$$

where $\alpha_n = x_n / 2$ $J_0(x_n) = 0$

[Example 2] Steady Temperatures in a Circular Cylinder

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$u(1, z) = 1 - z, \quad 0 < z < 1$$

$$u(r, 0) = 0, \quad u(r, 1) = 0 \quad 0 < r < 1$$

(Solution):

$$\text{(Step 1)} \quad u(r, z) = R(r)Z(z)$$

$$R''(r)Z(z) + \frac{1}{r}R'(r)Z(z) + R(r)Z''(z) = 0$$

$$\text{(Step 2)} \quad \frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -\lambda$$

$$rR'' + R' + \lambda rR = 0 \quad Z'' - \lambda Z = 0$$

$$\text{From } u(r, 0) = 0, \quad u(r, 1) = 0$$

$$Z(0) = 0, \quad Z(1) = 0$$

(Step 3): Solve

$$Z'' - \lambda Z = 0 \quad Z(0) = 0, \quad Z(1) = 0$$

There is no non-trivial solution for $\lambda = 0$ and $\lambda > 0$

When $\lambda < 0$, set $\lambda = -\alpha^2$,

$$Z'' + \alpha^2 Z = 0$$

$$Z(z) = c_1 \cos \alpha z + c_2 \sin \alpha z$$

From $Z(0) = 0, \quad Z(1) = 0$

$$Z(z) = c_2 \sin \alpha_n z \quad \text{where} \quad \alpha_n = n\pi \quad \lambda_n = -\alpha_n^2 = -n^2 \pi^2$$

(Step 4): Solve

$$rR'' + R' + \lambda rR = 0$$

$$rR'' + R' - n^2 \pi^2 rR = 0$$

$$rR'' + R' - n^2\pi^2 rR = 0 \quad r^2R'' + rR' - n^2\pi^2 r^2R = 0$$

Since $x^2y'' + xy' - (\alpha^2x^2 + \nu^2)y = 0$ solution : $c_1I_\nu(\alpha x) + c_2K_\nu(\alpha x)$

the solution of $r^2R'' + rR' - n^2\pi^2 r^2R = 0$ is

$$R(r) = c_3I_0(n\pi r) + c_4K_0(n\pi r) \quad (\text{page 205})$$

$I_\nu(x)$: modified Bessel function of the 1st kind

$K_\nu(x)$: modified Bessel function of the 2nd kind

Since $K_0(0) \rightarrow \infty$

$$R(r) = c_3I_0(n\pi r)$$

(Step 5): $u(r, z) = R(r)Z(z) \quad u_n(r, z) = A_nI_0(n\pi r)\sin(n\pi z)$

(Step 6): $u(r, z) = \sum_{n=1}^{\infty} A_nI_0(n\pi r)\sin(n\pi z)$

$$u(r, z) = \sum_{n=1}^{\infty} A_n I_0(n\pi r) \sin(n\pi z)$$

(Step 7): From $u(1, z) = 1 - z$

$$\sum_{n=1}^{\infty} A_n I_0(n\pi) \sin(n\pi z) = 1 - z$$

From the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x \quad b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

we have

$$A_n I_0(n\pi) = 2 \int_0^1 (1 - z) \sin(n\pi z) dz = \frac{2}{n\pi} (z - 1) \cos(n\pi z) \Big|_0^1 - \int_0^1 \frac{2}{n\pi} \cos(n\pi z) dz$$

$$A_n = \frac{2}{n\pi I_0(n\pi)}$$

$$u(r, z) = 2 \sum_{n=1}^{\infty} \frac{I_0(n\pi r)}{n\pi I_0(n\pi)} \sin(n\pi z)$$

2.9 SPHERICAL COORDINATES

Spherical Coordinates

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

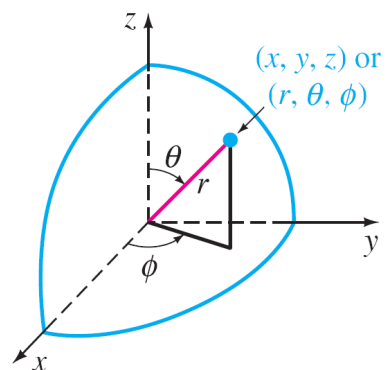


FIGURE 13.3.1 Spherical coordinates of a point (x, y, z) are (r, θ, ϕ) .

From D. G. Zill and Michael R. Cullen, *Differential Equations-with Boundary-Value Problem (metric version)*, 9th edition, Cengage Learning, 2017, Section 13.3.

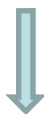
D. G. Zill and Michael R. Cullen, *Differential Equations-with Boundary-Value Problem (metric version)*, 9th edition, Cengage Learning, 2017, Section 13.3

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

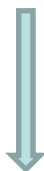
Laplacian in Spherical Coordinates

The 3D Laplacian is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$



$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta}.$$

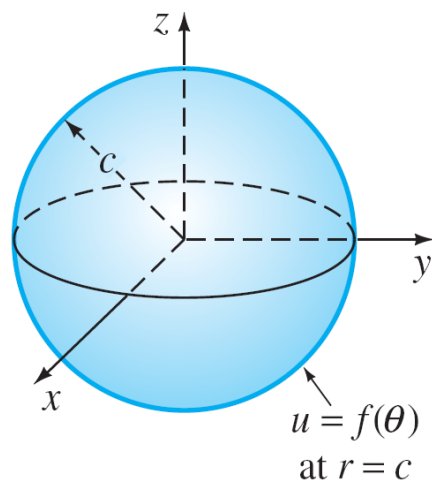


Suppose that u is independent of ϕ .

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta}.$$

[Example 1] Steady Temperatures in a Sphere

Find the steady-state temperature $u(r, \theta)$ within the sphere shown in Figure 13.3.2.



From D. G. Zill and Michael R. Cullen, *Differential Equations-with Boundary-Value Problem (metric version)*, 9th edition, Cengage Learning, 2017, Section 13.3.

FIGURE 13.3.2 Dirichlet problem for a sphere in Example 1

SOLUTION

Steady Temperatures

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

↓ Analogous to page 184-188

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0$$

With $u(c, \theta) = f(\theta), \quad 0 < \theta < \pi$

(Step 1): $u = R(r)\Theta(\theta),$

$$R''(r)\Theta(\theta) + \frac{2}{r} R'(r)\Theta(\theta) + \frac{1}{r^2} R(r)\Theta''(\theta) + \frac{\cot \theta}{r^2} R(r)\Theta'(\theta) = 0$$

$$\frac{R''(r)}{R(r)} + \frac{2}{r} \frac{R'(r)}{R(r)} + \frac{1}{r^2} \frac{\Theta''(\theta)}{\Theta(\theta)} + \frac{\cot \theta}{r^2} \frac{\Theta'(\theta)}{\Theta(\theta)} = 0$$

$$r^2 \frac{R''(r)}{R(r)} + 2r \frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} + \cot \theta \frac{\Theta'(\theta)}{\Theta(\theta)} = 0$$

$$\text{(Step 2): } \frac{r^2 R'' + 2rR'}{R} = -\frac{\Theta'' + \cot \theta \Theta'}{\Theta} = \lambda.$$

$$r^2 R'' + 2rR' - \lambda R = 0 \quad \sin \theta \Theta'' + \cos \theta \Theta' + \lambda \sin \theta \Theta = 0.$$

(Step 3): Try to solve

$$\sin \theta \Theta''(\theta) + \cos \theta \Theta'(\theta) + \lambda \sin \theta \Theta(\theta) = 0, \quad 0 \leq \theta \leq \pi$$

Set $x = \cos(\theta)$ $-1 \leq x \leq 1$

$$\frac{d}{d\theta} \Theta = \frac{dx}{d\theta} \frac{d}{dx} \Theta = -\sin \theta \frac{d}{dx} \Theta$$

$$\begin{aligned} \frac{d^2}{d\theta^2} \Theta &= \frac{d}{d\theta} \frac{d}{d\theta} \Theta = -\frac{d}{d\theta} \left(\sin \theta \frac{d}{dx} \Theta \right) = -\cos \theta \frac{d}{dx} \Theta - \sin \theta \frac{d}{d\theta} \frac{d}{dx} \Theta \\ &= -\cos \theta \frac{d}{dx} \Theta - \sin \theta \frac{dx}{d\theta} \frac{d^2}{dx^2} \Theta = -\cos \theta \frac{d}{dx} \Theta + \sin^2 \theta \frac{d^2}{dx^2} \Theta \end{aligned}$$

$$\sin^3 \theta \frac{d^2 \Theta}{dx^2} - 2 \sin \theta \cos \theta \frac{d\Theta}{dx} + \lambda \sin \theta \Theta = 0$$

$$\sin^3 \theta \frac{d^2 \Theta}{dx^2} - 2 \sin \theta \cos \theta \frac{d\Theta}{dx} + \lambda \sin \theta \Theta = 0 \quad x = \cos(\theta)$$

$$\sin^2 \theta \frac{d^2 \Theta}{dx^2} - 2 \cos \theta \frac{d\Theta}{dx} + \lambda \Theta = 0$$

$$(1-x^2) \frac{d^2 \Theta}{dx^2} - 2x \frac{d\Theta}{dx} + \lambda \Theta = 0 \quad -1 \leq x \leq 1$$

page 207

When $\lambda = n(n+1)$, it becomes Legendre's equation of order n

This is one of the applications of the Legendre equation

The solution is

$$\Theta(x) = cP_n(x) \quad P_n(x): \text{the Legendre polynomial of order } n$$

$$\Theta(\theta) = cP_n(\cos \theta) \quad n = 0, 1, 2, \dots$$

Note: The other linearly independent solution may not have finite derivatives at $x = \pm 1$.

(Step 4): Try to solve

$$r^2 R'' + 2rR' - \lambda R = 0$$

Since $\lambda = n(n+1)$,

$$r^2 R'' + 2rR' - n(n+1)R = 0$$

Auxiliary: $m(m-1) + 2m - n(n+1) = 0$

$$m = n, \quad -(n+1)$$

$$R(r) = c_1 r^n + c_2 r^{-(n+1)}$$

Since $0^{-(n+1)} \rightarrow \infty$ $c_2 = 0$

$$R(r) = c_1 r^n$$

(Step 5): $u_n(r, \theta) = A_n r^n P_n(\cos \theta) \quad n = 0, 1, 2, \dots$

(Step 6): $u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos \theta)$

$$u(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos \theta)$$

(Step 7): From $u(c, \theta) = f(\theta)$

$$\sum_{n=0}^{\infty} A_n c^n P_n(\cos \theta) = f(\theta)$$

We have known that

$$\int_{-1}^1 P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{n,m} \quad \delta_{n,n} = 1$$

$$\delta_{n,m} = 0 \quad \text{if } n \neq m$$

Set $x = \cos(\theta) \quad \frac{dx}{d\theta} = -\sin \theta$

$$\int_{\pi}^0 P_n(\cos \theta) P_m(\cos \theta) (-\sin \theta) d\theta = \frac{2}{2n+1} \delta_{n,m}$$

$$\int_0^{\pi} P_n(\cos \theta) P_m(\cos \theta) \sin \theta d\theta = \frac{2}{2n+1} \delta_{n,m}$$

(orthogonal with the weight function $\sin \theta$)

$$f(\theta) = \sum_{m=0}^{\infty} A_m c^m P_m(\cos \theta) \quad \int_0^{\pi} P_n(\cos \theta) P_m(\cos \theta) \sin \theta d\theta = \frac{2}{2n+1} \delta_{n,m}$$

$$\int_0^{\pi} f(\theta) P_n(\cos \theta) \sin \theta d\theta = \sum_{m=0}^{\infty} A_m c^m \int_0^{\pi} P_m(\cos \theta) P_n(\cos \theta) \sin \theta d\theta$$

$$\int_0^{\pi} f(\theta) P_n(\cos \theta) \sin \theta d\theta = A_n c^n \frac{2}{2n+1}$$

$$A_n = \frac{2n+1}{2c^n} \int_0^{\pi} f(\theta) P_n(\cos \theta) \sin \theta d\theta$$

$$u(r, \theta) = \sum_{n=0}^{\infty} \left(\frac{2n+1}{2} \int_0^{\pi} f(\theta) P_n(\cos \theta) \sin \theta d\theta \right) \left(\frac{r}{c} \right)^n P_n(\cos \theta)$$

Another Method for Solving PDEs: Method of Characteristics (只教不考)

Method of Characteristics

The method is suitable for the **1st Order PDE**.

Suppose that there is a 1st Order PDE as follows:

$$a_1(x_1, x_2, \dots, x_k, u) \frac{\partial u(x_1, x_2, \dots, x_k)}{\partial x_1} + a_2(x_1, x_2, \dots, x_k, u) \frac{\partial u(x_1, x_2, \dots, x_k)}{\partial x_2} \\ + \dots + a_k(x_1, x_2, \dots, x_k, u) \frac{\partial u(x_1, x_2, \dots, x_k)}{\partial x_k} = g(x_1, x_2, \dots, x_k, u)$$

Method of Characteristics

$$a_1(x_1, x_2, \dots, x_k, u) \frac{\partial u(x_1, x_2, \dots, x_k)}{\partial x_1} + a_2(x_1, x_2, \dots, x_k, u) \frac{\partial u(x_1, x_2, \dots, x_k)}{\partial x_2} \\ + \dots + a_k(x_1, x_2, \dots, x_k, u) \frac{\partial u(x_1, x_2, \dots, x_k)}{\partial x_k} = g(x_1, x_2, \dots, x_k, u)$$

We set a variable s and suppose that

$$\frac{\partial x_1}{\partial s} = a_1(x_1, x_2, \dots, x_k, u), \quad \frac{\partial x_2}{\partial s} = a_2(x_1, x_2, \dots, x_k, u), \\ \dots, \quad \frac{\partial x_k}{\partial s} = a_k(x_1, x_2, \dots, x_k, u).$$

Then, the original equation can be expressed as

$$\frac{\partial x_1}{\partial s} \frac{\partial u}{\partial x_1} + \frac{\partial x_2}{\partial s} \frac{\partial u}{\partial x_2} + \dots + \frac{\partial x_k}{\partial s} \frac{\partial u}{\partial x_k} = g(x_1, x_2, \dots, x_k, u)$$

$$\frac{du}{ds} = g(x_1, x_2, \dots, x_k, u) \implies \text{Solve the 1}^{\text{st}} \text{ order ODE for } u$$

Then, we apply the process as follows to solve u :

(Step 1) Solve the relation between x_1, x_2, \dots, x_k and s from each of the following ODEs:

$$\begin{array}{ccc}
 \frac{\partial x_1}{\partial s} = a_1(x_1, x_2, \dots, x_k, u) & \longrightarrow & x_1 = a_1(x_1, x_2, \dots, x_k, u)s + c_1 \\
 \frac{\partial x_2}{\partial s} = a_2(x_1, x_2, \dots, x_k, u), & & x_2 = a_2(x_1, x_2, \dots, x_k, u)s + c_2 \\
 \vdots & & \vdots \\
 \frac{\partial x_k}{\partial s} = a_k(x_1, x_2, \dots, x_k, u). & & x_k = a_k(x_1, x_2, \dots, x_k, u)s + c_k
 \end{array}$$

(Step 2) Try to find the general equation b such that

$$b(x_1, x_2, \dots, x_k) = c_b \quad \text{where } c_b \text{ is a constant}$$

(Step 3) Solve the following ODE

$$\frac{du}{ds} = g(x_1(s), x_2(s), \dots, x_k(s), u) \quad \text{Solve the 1}^{\text{st}} \text{ order ODE for } u$$

The solution has the form of $u = u_1(s, c_g)$

where c_g is some unknown constant.

(Specially, for case where $g = 0$, $u = c_g$)

(Step 4) Replace c_g by

$$c_g = f(c_b) = f(b(x_1, x_2, \dots, x_k, u))$$

where f is any function, then the solution is

$$u = u_1(s(x_1), f(c_b)) \quad \text{Here, we express } s \text{ as a function of } x_1.$$

(Specially, for case where $g = 0$, $u = f(c_b)$)

[Example 1] Solve

$$\frac{\partial u(x, y)}{\partial x} = 2 \frac{\partial u(x, y)}{\partial y}$$

(Solution):
$$\frac{\partial u(x, y)}{\partial x} - 2 \frac{\partial u(x, y)}{\partial y} = 0$$

(1)
$$\frac{\partial x}{\partial s} = 1, \quad x = s + c_x$$

$$\frac{\partial y}{\partial s} = -2, \quad y = -2s + c_y$$

(2)
$$2x + y = c_b$$

(3)
$$\frac{du}{ds} = 0, \quad u = c_g = f(c_b)$$

$$u = f(2x + y) \quad \text{where } f \text{ is any function.}$$

[Example 2] Solve

$$\frac{\partial u(x, y, z)}{\partial x} = \frac{\partial u(x, y, z)}{\partial y} - \frac{\partial u(x, y, z)}{\partial z} + u(x, y, z)$$

(Solution):

$$\frac{\partial u(x, y, z)}{\partial x} - \frac{\partial u(x, y, z)}{\partial y} + \frac{\partial u(x, y, z)}{\partial z} = u(x, y, z)$$

$$(1) \quad \frac{\partial x}{\partial s} = 1, \quad \frac{\partial y}{\partial s} = -1, \quad \frac{\partial z}{\partial s} = 1$$

$$x = s + c_x, \quad y = -s + c_y, \quad z = s + c_z$$

(2) Note that

$$x + y = c_x + c_y \quad \text{and} \quad y + z = c_y + c_z$$

$$b_1(x, y, z) = x + y$$

$$b_2(x, y, z) = y + z$$

can all generate a constant. Although $x-z$ can also generate a constant, it is dependent on $x+y$ and $y+z$. Therefore, **a general way to obtain a constant** is

$$f(x + y, y + z) = c$$

where f is any function with two independent variables.

$$(3) \quad \frac{\partial x}{\partial s} \frac{\partial u(x, y, z)}{\partial x} + \frac{\partial y}{\partial s} \frac{\partial u(x, y, z)}{\partial y} + \frac{\partial z}{\partial s} \frac{\partial u(x, y, z)}{\partial z} = u(x, y, z)$$

$$\frac{du}{ds} = u,$$

Since $x = s + c_x$

$$u = c_1 e^s = c_1 e^{x-c_x} = c e^x$$

Here we set $c = c_1 e^{-c_x}$

$$u = f(x + y, y + z) e^x$$

附錄五 Review for Laplace Transforms

Laplace transform (1-sided form) $F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$

Differentiation Property for the Laplace transform

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\uparrow$$

$$s^a f^{(b)}(0), \quad a + b = n - 1$$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$\exp(at)$	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2 + k^2}$
$\cos(kt)$	$\frac{s}{s^2 + k^2}$
$\sinh(kt)$	$\frac{k}{s^2 - k^2}$
$\cosh(kt)$	$\frac{s}{s^2 - k^2}$

Seven Important Properties of the Laplace Transform

input	Laplace transform
<p>(1) Differentiation</p> $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$
<p>(2) Multiplication by t</p> $t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} F(s)$
<p>(3) Integration</p> $\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$

(續)

input	Laplace transform
(4) Multiplication by exp $e^{at} f(t)$	$F(s - a)$
(5.1) Translation $u(t)$: unit step $f(t - a)u(t - a)$	$e^{-as} F(s)$
(5.2) Translation $g(t)u(t - a)$	$e^{-as} \mathcal{L}\{g(t + a)\}$
(6) Convolution $y(t) = \int_0^t f(\tau)g(t - \tau)d\tau$	$Y(s) = F(s)G(s)$
(7) Periodic Input $f(t) = f(t + T)$	$\frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$

2.10 Solving PDEs by Laplace Transforms

(只教不考)

In this section, we see that a linear PDE with constant coefficients is transformed into an ODE **using the 1-sided Laplace transform.**

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 14.2

Transform of a Function of Two Variables

$$U(x, s) = \mathcal{L}\{u(x, t)\} = \int_0^{\infty} e^{-st} u(x, t) dt$$

To avoid confusing, we denote it as

$$U(x, s) = \mathcal{L}_{t \rightarrow s}\{u(x, t)\} = \int_0^{\infty} e^{-st} u(x, t) dt$$

Transform of Partial Derivatives

$$\mathcal{L}_{t \rightarrow s} \left\{ \frac{\partial u}{\partial t} \right\} = sU(x, s) - u(x, 0),$$

$$\mathcal{L}_{t \rightarrow s} \left\{ \frac{\partial^2 u}{\partial t^2} \right\} = s^2 U(x, s) - su(x, 0) - u_t(x, 0).$$

where

$$u_t(x, t) = \frac{\partial}{\partial t} u(x, t)$$

Because we are transforming with **respect to t** ,
(not to x)

Be careful

$$\mathcal{L}_{t \rightarrow s} \left\{ \frac{\partial^2 u}{\partial x^2} \right\} = \frac{d^2 U}{dx^2}.$$

[Example 1] Laplace Transform of a PDE

Find the Laplace transform of the wave equation $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $t > 0$.
(respect to t)

SOLUTION

$$\mathcal{L}_{t \rightarrow s} \left\{ a^2 \frac{\partial^2 u}{\partial x^2} \right\} = \mathcal{L}_{t \rightarrow s} \left\{ \frac{\partial^2 u}{\partial t^2} \right\} \quad \text{where} \quad u_t = \frac{\partial}{\partial t} u$$

$$a^2 \frac{d^2}{dx^2} \mathcal{L}_{t \rightarrow s} \{u(x, t)\} = s^2 \mathcal{L}_{t \rightarrow s} \{u(x, t)\} - su(x, 0) - u_t(x, 0)$$

$$a^2 \frac{d^2 U(x, s)}{dx^2} - s^2 U(x, s) = -su(x, 0) - u_t(x, 0).$$

The process for solving the BVP or IVP of a partial differential equation (PDE) by the Laplace transform

The range of the independent variable should be $[0, \infty)$ 

(Step 1) Apply the **Laplace transform** for one independent variable to change the PDE into an **ordinary differential equation (ODE)** with **another independent variable**.

(Step 2) Solve the ODE obtain in Step 1.

(Step 3) Solution in Step 2 contains some constants. Find these constants by transforming the initial conditions.

(Step 4) Inverse transform.

[Example 2] Using the Laplace Transform to Solve a BVP

Solve
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0$$

Subject to
$$u(0, t) = 0, \quad u(1, t) = 0, \quad t > 0$$

$$u(x, 0) = 0, \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \sin \pi x, \quad 0 < x < 1.$$

SOLUTION

(Step 1) From Example 1 and the given initial conditions,

$$\frac{d^2 U(x, s)}{dx^2} - s^2 U(x, s) = -\sin \pi x,$$

where $U(x, s) = \mathcal{L}_{t \rightarrow s} \{u(x, t)\}$.

$$\mathcal{L}_{t \rightarrow s} \{u(0, t)\} = U(0, s) = 0 \quad \mathcal{L}_{t \rightarrow s} \{u(1, t)\} = U(1, s) = 0.$$

$$\text{(Step 2)} \quad \frac{d^2U(x,s)}{dx^2} - s^2U(x,s) = -\sin \pi x,$$

Complementary Function

$$U_c(x,s) = c_1 \cosh sx + c_2 \sinh sx.$$

Particular Solution

$$U_p(x,s) = a \sin \pi x + b \cos \pi x.$$

$$a = \frac{1}{s^2 + \pi^2}, \quad b = 0$$

$$U(x,s) = U_c(x,s) + U_p(x,s)$$

$$= c_1 \cosh sx + c_2 \sinh sx + \frac{1}{s^2 + \pi^2} \sin \pi x.$$

$$U(x, s) = c_1 \cosh sx + c_2 \sinh sx + \frac{1}{s^2 + \pi^2} \sin \pi x.$$

(Step 3) From

$$\begin{aligned} \mathcal{L}_{t \rightarrow s} \{u(0, t)\} = U(0, s) = 0 & & \mathcal{L}_{t \rightarrow s} \{u(1, t)\} = U(1, s) = 0. \\ c_1 = 0. & & c_1 \cosh s + c_2 \sinh s = 0 \end{aligned}$$

We have $c_1 = 0$ and $c_2 = 0$.

$$U(x, s) = \frac{1}{s^2 + \pi^2} \sin \pi x$$

$$\text{(Step 4)} \quad u(x, t) = \mathcal{L}_{s \rightarrow t}^{-1} \left\{ \frac{1}{s^2 + \pi^2} \sin \pi x \right\} = \frac{1}{\pi} \sin \pi x \mathcal{L}_{s \rightarrow t}^{-1} \left\{ \frac{\pi}{s^2 + \pi^2} \right\}.$$

$$u(x, t) = \frac{1}{\pi} \sin \pi x \sin \pi t.$$

[Example 3] The Wave Equation with Gravity

Solve
$$a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \quad x > 0, \quad t > 0$$

Subject to
$$u(0, t) = 0, \quad \lim_{x \rightarrow \infty} \frac{\partial u}{\partial x} = 0, \quad t > 0$$

$$u(x, 0) = 0, \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0, \quad x > 0$$

Solution:

(Step 1)
$$\mathcal{L}_{t \rightarrow s} \left\{ a^2 \frac{\partial^2 u}{\partial x^2} \right\} - \mathcal{L}_{t \rightarrow s} \{ g \} = \mathcal{L}_{t \rightarrow s} \left\{ \frac{\partial^2 u}{\partial t^2} \right\}$$

$$a^2 \frac{\partial^2 U(x, s)}{\partial x^2} - \frac{g}{s} = s^2 U(x, s) - su(x, 0) - u_t(x, 0) = s^2 U(x, s)$$

$$\frac{\partial^2 U(x, s)}{\partial x^2} - \frac{s^2}{a^2} U(x, s) = \frac{g}{a^2 s}$$

(Step 2)

$$\frac{\partial^2 U(x, s)}{\partial x^2} - \frac{s^2}{a^2} U(x, s) = \frac{g}{a^2 s}$$

Complementary Function

$$U_c(x, s) = c_1 \cosh\left(\frac{s}{a} x\right) + c_2 \sinh\left(\frac{s}{a} x\right)$$

Particular Solution

$$U_p(x, y) = c_3$$

$$c_3 = -\frac{g}{s^3}$$

$$U(x, s) = c_1 \cosh\left(\frac{s}{a} x\right) + c_2 \sinh\left(\frac{s}{a} x\right) - \frac{g}{s^3}$$

$$U(x, s) = c_1 \cosh\left(\frac{s}{a} x\right) + c_2 \sinh\left(\frac{s}{a} x\right) - \frac{g}{s^3}$$

(Step 3) From initial conditions

$$\mathcal{L}_{t \rightarrow s} \{u(0, t)\} = U(0, s) = 0 \quad \mathcal{L}_{t \rightarrow s} \left\{ \lim_{x \rightarrow \infty} \frac{\partial u(x, t)}{\partial x} \right\} = \lim_{x \rightarrow \infty} \frac{\partial}{\partial x} U(x, s) = 0.$$

$$c_1 - \frac{g}{s^3} = 0$$

$$\lim_{x \rightarrow \infty} c_1 \frac{s}{a} \sinh\left(\frac{s}{a} x\right) + c_2 \frac{s}{a} \cosh\left(\frac{s}{a} x\right) = 0$$

$$c_1 = \frac{g}{s^3}$$

$$\lim_{x \rightarrow \infty} c_1 \frac{s}{2a} \exp\left(\frac{s}{a} x\right) + c_2 \frac{s}{2a} \exp\left(\frac{s}{a} x\right) = 0$$

$$c_2 = -c_1 = -\frac{g}{s^3}$$

Note:

$$\lim_{x \rightarrow \infty} \sinh(x) = \lim_{x \rightarrow \infty} \cosh(x) = \exp(x) / 2$$

$$U(x, s) = \frac{g}{s^3} \left(\cosh\left(\frac{s}{a} x\right) - \sinh\left(\frac{s}{a} x\right) \right) - \frac{g}{s^3} = \frac{g}{s^3} \exp\left(-\frac{s}{a} x\right) - \frac{g}{s^3}$$

$$U(x, s) = \frac{g}{s^3} \exp\left(-\frac{s}{a} x\right) - \frac{g}{s^3}$$

(Step 4)

$$u(x, t) = \mathcal{L}_{s \rightarrow t}^{-1} \left\{ \frac{g}{s^3} \exp\left(-\frac{s}{a} x\right) - \frac{g}{s^3} \right\}$$

$$u(x, t) = \frac{1}{2} g \left(t - \frac{x}{a}\right)^2 \mathbf{u}\left(t - \frac{x}{a}\right) - \frac{1}{2} g t^2$$

where $\mathbf{u}(t)$ is the unit step function

We have applied the translation property:

$$\mathcal{L}\{f(t-a)\mathbf{u}(t-a)\} = \exp(-as)F(s)$$

附錄六 Review for Fourier Transforms

Fourier transform

$$\mathfrak{F}[g(x)] = \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx = G(f)$$

\mathfrak{F} 代表 Fourier transform

inverse Fourier transform

$$\mathfrak{F}^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f) e^{j2\pi fx} df = g(x)$$

文獻上其他 Fourier transform 的定義

$$\mathfrak{F}[g(x)] = \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx = G(\omega)$$

$$\mathfrak{F}^{-1}[G(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega x} d\omega = g(x)$$

或者

$$\mathfrak{F}[g(x)] = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx = G(\omega)$$

$$\mathfrak{F}^{-1}[G(\omega)] = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G(\omega) e^{j\omega x} d\omega = g(x)$$

或者

$$\mathfrak{F}[g(x)] = \int_{-\infty}^{\infty} g(x) e^{j\alpha x} dx = G(\alpha)$$

$$\mathfrak{F}^{-1}[G(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\alpha) e^{-j\alpha x} d\alpha = g(x)$$

When $g(x)$ is even ($g(x) = g(-x)$)

Fourier transform \longrightarrow Fourier cosine transform

$$\mathfrak{F}_c [g(x)] = \int_0^{\infty} g(x) \cos(2\pi fx) dx = G_c(f)$$

$$\mathfrak{F}_c^{-1} [G_c(f)] = 4 \int_0^{\infty} G_c(f) \cos(2\pi fx) df = g(x)$$

Remember that if $k(x)$ is even

$$\int_{-\infty}^{\infty} k(x) dx = 2 \int_0^{\infty} k(x) dx$$

If $k(x)$ is odd

$$\int_{-\infty}^{\infty} k(x) dx = 0$$

When $g(x)$ is odd ($g(x) = -g(-x)$)

Fourier transform \longrightarrow Fourier sine transform

$$\mathfrak{F}_s [g(x)] = \int_0^{\infty} g(x) \sin(2\pi fx) dx = G_s(f)$$

$$\mathfrak{F}_s^{-1} [G_s(f)] = 4 \int_0^{\infty} G_s(f) \sin(2\pi fx) df = g(x)$$

Fourier, Fourier Cosine / Sine Transforms 的微分性質

(1) Fourier transform 的微分性質

$$\begin{aligned}\mathfrak{T}[g'(x)] &= \int_{-\infty}^{\infty} g'(x) e^{-j2\pi fx} dx = g(x) e^{-j2\pi fx} \Big|_{-\infty}^{\infty} + j2\pi f \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx \\ &= j2\pi f \mathfrak{T}[g(x)]\end{aligned}$$

微分性質做了一些假設： $g(x) = 0$ when $x \rightarrow \infty$ and $x \rightarrow -\infty$

以此類推 $\mathfrak{T}[g^{(n)}(x)] = (j2\pi f)^{(n)} G(f)$

比較：對 Laplace transform

$$L\{f'(x)\} = sL\{f(x)\} - f(0) \quad \int_0^{\infty} f(x) e^{-sx} dx$$

對 Fourier transform

$$s \rightarrow j2\pi f, \text{ without initial conditions}$$

(2) Fourier cosine transform 的微分性質

$$\begin{aligned}
 \mathfrak{F}_c[g'(x)] &= \int_0^{\infty} g'(x) \cos(2\pi fx) dx \\
 &= g(x) \cos(2\pi fx) \Big|_0^{\infty} + 2\pi f \int_0^{\infty} f(x) \sin(2\pi fx) dx \\
 &= 2\pi f \mathfrak{F}_s[g(x)] - g(0)
 \end{aligned}$$

(3) Fourier sine transform 的微分性質

$$\begin{aligned}
 \mathfrak{F}_s[g'(x)] &= \int_0^{\infty} g'(x) \sin(2\pi fx) dx \\
 &= g(x) \sin(2\pi fx) \Big|_0^{\infty} - 2\pi f \int_0^{\infty} g(x) \cos(2\pi fx) dx \\
 &= -2\pi f \mathfrak{F}_c[g(x)]
 \end{aligned}$$

注意：(1) Fourier sine, cosine transforms 互換

(2) α 正負號不同

(3) Fourier cosine transform 要考慮 initial condition

$$\mathfrak{I}_c [g'(x)] = 2\pi f \mathfrak{I}_s [g(x)] - g(0)$$

$$\mathfrak{I}_s [g'(x)] = -2\pi f \mathfrak{I}_c [g(x)]$$

$$\mathfrak{I}_c [g''(x)] = 2\pi f \mathfrak{I}_s [g'(x)] - g'(0) = -4\pi^2 f^2 \mathfrak{I}_c [g(x)] - g'(0)$$

$$\mathfrak{I}_s [g''(x)] = -2\pi f \mathfrak{I}_c [g'(x)] = -4\pi^2 f^2 \mathfrak{I}_s [g(x)] + 2\pi f g(0)$$

2.11 Solving PDEs by Fourier Transforms

(只教不考)

Differentiation \longrightarrow Multiplication

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 14.4

Method

(Condition 1) interval 為 $-\infty < \nu < \infty$ 時:

用 Fourier transform

(Condition 2) interval 為 $0 < \nu < \infty$,

有 “ $u(\nu, \dots) = 0$ or a constant when $\nu = 0$ ” 的 boundary condition 時:

用 Fourier sine transform

(Condition 3) interval 為 $0 < \nu < \infty$,

有 “ $\frac{\partial}{\partial \nu} u(\nu, \dots) = 0$ or a constant when $\nu = 0$ ” 的 boundary condition 時:

用 Fourier cosine transform

使用 Fourier transform, Fourier cosine transform, Fourier sine transform 來解 partial differential equation (PDE) 的 BVP 或 IVP 的解法流程

(Step 1) 以 page 262 的規則，來決定要針對 **哪一個 independent variable**，做**什麼 transform** (Fourier, Fourier cosine, 或 Fourier sine transform)

(Step 2) 對 PDE 做 Step 1 所決定的 transform, 則原本的 PDE 變成針對另外一個 independent variable 的 ordinary differential equation (ODE)

(Step 3) 將 Step 2 所得出的 ODE 的解算出來

(Step 4) Step 3 所得出來的解會有一些 constants，可以對 initial conditions (或 boundary conditions) 做 transform 將 constants 解出

(※ 和 Step 1 所做的 transform 一樣，只是 transform 的對象變成是 initial 或 boundary conditions，見 pages 265, 268 的例子)

(Step 5) 最後，別忘了做 inverse transform (畫龍點睛)

[Example 1]

heat equation: $k \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad -\infty < x < \infty \quad t > 0$

subject to $u(x, 0) = g(x)$ where $g(x) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

Step 1 決定針對 x 做 Fourier transform

$$\mathfrak{F}_{x \rightarrow f} \{u(x, t)\} = \int_{-\infty}^{\infty} u(x, t) e^{-i2\pi fx} dx = U(f, t)$$

Step 2 $\mathfrak{F}_{x \rightarrow f} \left\{ k \frac{\partial^2 u}{\partial x^2} \right\} = \mathfrak{F}_{x \rightarrow f} \left\{ \frac{\partial u}{\partial t} \right\}$

$$-k4\pi^2 f^2 U(f, t) = \frac{\partial U(f, t)}{\partial t}$$

原本對 x, t 兩個變數做偏微分

經過 Fourier transform 之後，
只剩下對 t 做偏微分

$$\frac{dU(f,t)}{dt} + 4k\pi^2 f^2 U(f,t) = 0$$

對於 t 而言，是 1st order ODE

Step 3 $U(f,t) = c e^{-4k\pi^2 f^2 t}$ 這邊的 c 值，對 t 而言是 constant，
但是可能會 **dependent on f** (特別注意)

Step 4 根據 $u(x, 0) = g(x)$ 將 c 解出

和 Step 1 一樣，也是針對 x 做 Fourier transform

只是對象改成 initial condition

$$\begin{aligned} \mathfrak{F}_{x \rightarrow f} \{u(x, 0)\} &= \int_{-\infty}^{\infty} g(x) e^{-i2\pi fx} dx = \int_{-1}^1 u_0 e^{-i2\pi fx} dx \\ &= u_0 \frac{e^{-i2\pi f} - e^{i2\pi f}}{-i2\pi f} = u_0 \frac{\sin(2\pi f)}{\pi f} \end{aligned}$$

因為 $\mathfrak{F}_{x \rightarrow f} \{u(x, 0)\} = U(f, 0)$

$$U(f, 0) = u_0 \frac{\sin(2\pi f)}{\pi f}$$

$$U(f, t) = c e^{-4k\pi^2 f^2 t} \xleftrightarrow{\text{比較係數}} U(f, 0) = u_0 \frac{\sin(2\pi f)}{\pi f}$$

$$\text{解出 } c = u_0 \frac{\sin(2\pi f)}{\pi f}$$

$$U(f, t) = u_0 \frac{\sin(2\pi f)}{\pi f} e^{-4k\pi^2 f^2 t}$$

Step 5 未完待續，別忘了最後要做 inverse Fourier transform

$$u(x, t) = \mathfrak{F}_{f \rightarrow x}^{-1} [U(f, t)] = \int_{-\infty}^{\infty} u_0 \frac{\sin(2\pi f)}{\pi f} e^{-4k\pi^2 f^2 t} e^{j2\pi fx} df$$

不易化簡，課本僅依據 $\frac{\sin(2\pi f)}{\pi f} e^{-4k\pi^2 f^2 t}$ 對 f 而言是 even function 將 $u(x, t)$ 化簡為

$$\begin{aligned} u(x, t) &= u_0 \int_{-\infty}^{\infty} \frac{\sin(2\pi f)}{\pi f} e^{-4k\pi^2 f^2 t} (\cos(2\pi fx) + j \sin(2\pi fx)) df \\ &= u_0 \int_{-\infty}^{\infty} \frac{\sin(2\pi f) \cos(2\pi fx)}{\pi f} e^{-4k\pi^2 f^2 t} df \end{aligned}$$

[Example 2] Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad 0 < x < 1 \quad y > 0$$

$$u(0, y) = 0 \quad u(1, y) = e^{-y} \quad y > 0$$

$$\left. \frac{\partial u}{\partial y} \right|_{y=0} = 0 \quad 0 < x < 1$$

Step 1 決定針對 y 做 Fourier cosine transform

$$\mathfrak{S}_{c, y \rightarrow f} \{u(x, y)\} = \int_0^{\infty} u(x, y) \cos(2\pi fy) dy = U(x, f)$$

Step 2
$$\mathfrak{S}_{c, y \rightarrow f} \left\{ \frac{\partial^2 u}{\partial x^2} \right\} + \mathfrak{S}_{c, y \rightarrow f} \left\{ \frac{\partial^2 u}{\partial y^2} \right\} = \mathfrak{S}_{c, y \rightarrow f} \{0\}$$

$$\text{from } \mathfrak{S}_c [g''(y)] = -4\pi^2 f^2 \mathfrak{S}_c [g(y)] - g'(0)$$

$$\frac{d^2 U(x, f)}{dx^2} - 4\pi^2 f^2 U(x, f) = 0 \quad \text{對於 } x \text{ 的 2}^{\text{nd}} \text{ order ODE}$$

Step 3
$$\frac{d^2U(x, f)}{dx^2} - 4\pi^2 f^2 U(x, f) = 0$$

→
$$U(x, f) = c_1 \cosh(2\pi fx) + c_2 \sinh(2\pi fx)$$

Step 4 由 $u(0, y) = 0$ $u(\pi, y) = e^{-y}$ 來解 c_1, c_2 (may be dependent on f)

和 Step 1 一樣，也是針對 y 做 Fourier cosine transform

只是對象改成 boundary conditions

$$(1) U(0, f) = \mathfrak{F}_{c, y \rightarrow f} \{u(0, y)\} = \int_0^\infty 0 \cdot \cos(2\pi fy) dy = 0$$

$$(2) U(1, f) = \mathfrak{F}_{c, y \rightarrow f} \{u(1, y)\} = \int_0^\infty e^{-y} \cdot \cos(2\pi fy) dy = \frac{1}{1 + (2\pi f)^2}$$

↑
(可以用 Laplace transform 的「取巧法」)

$$U(x, f) = c_1 \cosh(2\pi fx) + c_2 \sinh(2\pi fx)$$

分別代入 $U(0, f) = 0$ $U(1, f) = \frac{1}{1 + (2\pi f)^2}$

$$c_1 = 0 \quad c_1 \cosh(2\pi f) + c_2 \sinh(2\pi f) = \frac{1}{1 + (2\pi f)^2}$$

$$\Rightarrow c_1 = 0 \quad c_2 = \frac{1}{(1 + 4\pi^2 f^2) \sinh(2\pi f)}$$

$$U(x, f) = \frac{\sinh(2\pi fx)}{(1 + 4\pi^2 f^2) \sinh(2\pi f)}$$

Step 5 inverse cosine transform

$$u(x, y) = \mathfrak{F}_{c, f \rightarrow y}^{-1} [U(x, f)] = 4 \int_0^{\infty} \frac{\sinh(2\pi fx)}{(1 + 4\pi^2 f^2) \sinh(2\pi f)} \cos(2\pi fy) df$$

(算到這裡即可，難以繼續化簡)

- (1) 微分公式當中，Fourier cosine transform 和 Fourier sine transform 會有互換的情形。(See page 259)
- (2) 在解 boundary value problem 時，要了解
何時用 Fourier transform，
何時用 Fourier cosine transform,
何時用 Fourier sine transform (see page 262)
- (3) 在解 partial differential equation 時，往往只針對一個 independent variable 做 Fourier transform, 另一個 independent variable 不受影響，如 Examples 1 and 2 的例子

計算過程中，自己要清楚是對哪一個 independent variable 做 Fourier transform

※ 本講義中習慣用 下標 做記號