3. Function Approximation

[1] D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017.

[2] R. Beals, Special Functions and Orthogonal Polynomials, Cambridge Studies in Advanced Mathematics, vol. 153, Cambridge University Press, 2016.

Function Approximation

- Function Approximation
  - Theorems for Orthogonal Basis (Sec. 3.1)
  - by Polynomials
    - Legendre Polynomials (Sec. 3.2)
    - Other Polynomials (Sec. 3.4，只教不考)
  - by Other Function Set (Sec. 3.3)

Sec. 3.1 Review for Orthogonal Basis Expansion

(1) inner product on an interval [a, b]

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx \qquad (f_1, f_2 \text{ 為 real 時})$$

or $$(f_1, f_2) = \int_a^b f_1(x) f_2^*(x) dx \qquad \text{(more standard definition)}$$

(2) orthogonal on an interval [a, b]

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0 \qquad (f_1, f_2 \text{ 為 real 時})$$

$$(f_1, f_2) = \int_a^b f_1(x) f_2^*(x) dx = 0 \qquad \text{(more standard definition)}$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Sections 11-1~11-3.

注意：任何 even function 和任何 odd function 在 [-*b*, *b*] 之間必為 orthogonal, (*a* = -*b*)

$$\int_{-b}^{b} f_1(x) f_2(x) dx = 0$$

even odd

(3) orthogonal set

For a set of functions $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ………., if

$$\int_{a}^{b} \phi_m(x) \phi_n^*(x) dx = 0 \quad \text{for any } m \neq n$$

then $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. is an orthogonal set on the interval [*a*, *b*]

(4) square norm

$$\|f(x)\|^2 = (f(x), f(x)) = \int_a^b f(x) f^*(x)\,dx = \int_a^b |f(x)|^2\,dx$$

(5) norm

$$\|f(x)\| = \sqrt{(f(x), f(x))} = \sqrt{\int_a^b f(x) f^*(x)\,dx}$$

(6) orthonormal set

對一個 orthogonal set, 若更進一步的滿足

$$\int_a^b \phi_n(x)\phi_n^*(x)\,dx = 1 \qquad \text{for all } n$$

則被稱為 orthonormal set

(7) normalize

$$\psi(x) \xrightarrow{\text{normalize}} v(x) = \frac{\psi(x)}{\|\psi(x)\|}$$

將 norm 變為 1

[Example 1] Suppose that $[a, b] = [-1, 1]$. Then

$$\left\{ \frac{1}{\sqrt{2}} e^{j\pi nx} \mid n = \cdots\cdots, -2, -1, 0, 1, 2, 3, \cdots\cdots \right\}$$

forms an orthonormal set.

(Proof): $$\left(\frac{1}{\sqrt{2}} e^{j\pi nx}, \frac{1}{\sqrt{2}} e^{j\pi mx} \right) = \frac{1}{2} \int_{-1}^{1} e^{j\pi nx} e^{-j\pi mx} dx$$

If $m \neq n$,

$$\left(\frac{1}{\sqrt{2}} e^{j\pi nx}, \frac{1}{\sqrt{2}} e^{j\pi mx} \right) = \left. \frac{e^{j\pi(n-m)x}}{j2\pi(n-m)} \right|_{-1}^{1} = \frac{e^{j\pi(n-m)} - e^{-j\pi(n-m)}}{j2\pi(n-m)} = \frac{\sin\left(\pi(n-m)\right)}{\pi(n-m)} = 0$$

If $m = n$,

$$\left(\frac{1}{\sqrt{2}} e^{j\pi nx}, \frac{1}{\sqrt{2}} e^{j\pi mx} \right) = \frac{1}{2} \int_{-1}^{1} 1 dx = 1$$

(8) complete

若在 interval $[a, b]$ 之間，任何一個 function $f(x)$ 都可以表示成 $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ...的 linear combination

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots\cdots\cdots = \sum_{n=0}^{\infty} c_n\phi_n(x)$$

則 $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ... 被稱作 complete

(9) orthogonal series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x) \qquad \text{where} \qquad c_n = \frac{\int_a^b f(x)\phi_n^*(x)\,dx}{\int_a^b \phi_n(x)\phi_n^*(x)\,dx}$$

Specially, if $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ... are orthonormal

$$c_n = \int_a^b f(x)\phi_n^*(x)\,dx$$

(10) weight

inner product with weight function

$$\left(f_1(x), f_2(x)\right) = \int_a^b w(x) f_1(x) f_2^*(x)\, dx$$

$w(x)$ is called the weight function

With the weight function

(10-1) orthogonal 的定義改成

$$\left(f_m, f_n\right) = \int_a^b w(x) f_m(x) f_n^*(x)\, dx = 0 \quad \text{for } m \neq n$$

(10-2) square norm 的定義改成 $\|f(x)\|^2 = \int_a^b w(x) f(x) f^*(x)\, dx$

(10-3) norm 的定義改成 $\|f(x)\| = \sqrt{\int_a^b w(x) f(x) f^*(x)\, dx}$

(10-4) orthonormal 的定義改成

$$\int_a^b w(x) f_m(x) f_n^*(x)\,dx = 0 \quad \text{for } m \neq n$$

$$\int_a^b w(x) f_n(x) f_n^*(x)\,dx = 1$$

(10-5) normalize 的算法改成

$$v(x) = \frac{\psi(x)}{\|\psi(x)\|} = \frac{\psi(x)}{\sqrt{\int_a^b w(x)\psi(x)\psi^*(x)\,dx}}$$

(10-6) orthogonal series expansion of $f(x)$ 的算法改成

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \qquad c_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x)\,dx}{\int_a^b w(x) \phi_n(x) \phi_n^*(x)\,dx}$$

orthonormal case: $c_n = \int_a^b w(x) f(x) \phi_n^*(x)\,dx$

trigonometric functions

$$\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \ldots\ldots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \sin\frac{3\pi}{p}x, \ldots\ldots\right\}$$

form a complete and orthogonal set on the interval of $[-p\ ,\ p]$

$\cos\frac{n\pi}{p}x,\quad \sin\frac{n\pi}{p}x$ 週期：$\frac{2p}{n}$ 頻率：$\frac{n}{2p}$

$$\cos\frac{n\pi}{p}x = \cos\left(\frac{n\pi}{p}x + 2\pi\right) = \cos\left(\frac{n\pi}{p}\left(x + \frac{2p}{n}\right)\right)$$

Fourier Series (expanded by trigonometric functions)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \qquad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

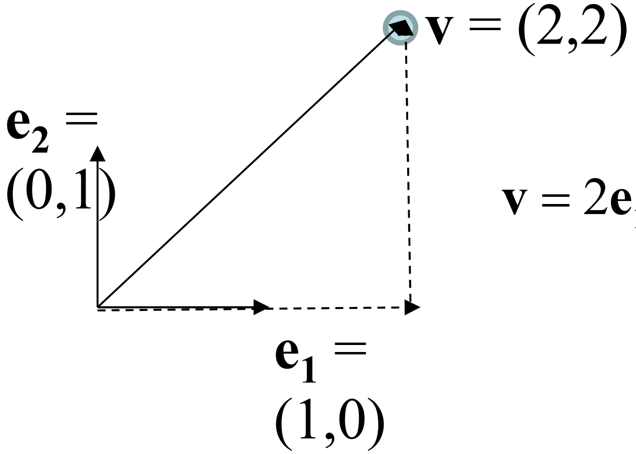
Q: Why should we use the orthogonal basis?

(1) The expansion coefficients are easier to determined,

(2) No Interference

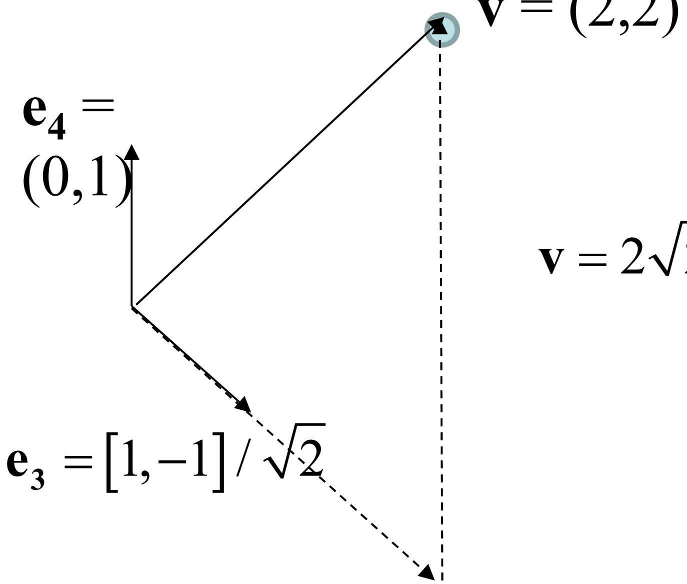
(3) More basis functions → Less error

$\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal



$$\mathbf{v} = 2\mathbf{e}_1 + 2\mathbf{e}_2$$

$\mathbf{e}_3$ and $\mathbf{e}_4$ are not orthogonal



$$\mathbf{v} = 2\sqrt{2}\mathbf{e}_3 + 4\mathbf{e}_4$$

Section 3-2 Polynomial Approximation Using Legendre Polynomials

- **Legendre's equation of order *n***

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

One of the solution: Legendre polynomials

$n=0$ $P_0(x) = 1$

$n=1$ $P_1(x) = x$

$n=2$ $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$n=3$ $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$n=4$ $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$n=5$ $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Sections 6-4, 11-5.

3.2.1 Legendre Polynomial

Legendre's Equation $(1-x^2)y''-2xy'+n(n+1)y=0$

$$y(x)=\sum_{k=0}^{\infty}c_k x^k$$ 代入，得出

Two linearly independent solutions are

$$y_1(x)=c_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6+\cdots\cdots\right]$$

$$y_2(x)=c_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5-\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7+\cdots\cdots\right]$$

(a) When n is not an integer, both the two solutions have infinite number of terms.

(b) When n is an even integer, $y_1(x)$ has finite number of terms.

In $y_1(x)$, the coefficient of x^k is zero when $k > n$.

(c) When n is an odd integer, $y_2(x)$ has finite number of terms.

In $y_2(x)$, the coefficient of x^k is zero when $k > n$.

$y_1(x)$ when n is an even integer and $y_2(x)$ when n is an odd integer are called the Legendre polynomials (denoted by $P_n(x)$).

通常選

$$c_0 = (-1)^{n/2} \frac{1 \cdot 3 \cdots\cdots (n-1)}{2 \cdot 4 \cdots\cdots n} \qquad c_1 = (-1)^{(n-1)/2} \frac{1 \cdot 3 \cdots\cdots n}{2 \cdot 4 \cdots\cdots (n-1)}$$

(讓 $P_n(1)$ 一律等於 1)

由 $y_1(x)$

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

由 $y_2(x)$

$$P_1(x) = x$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

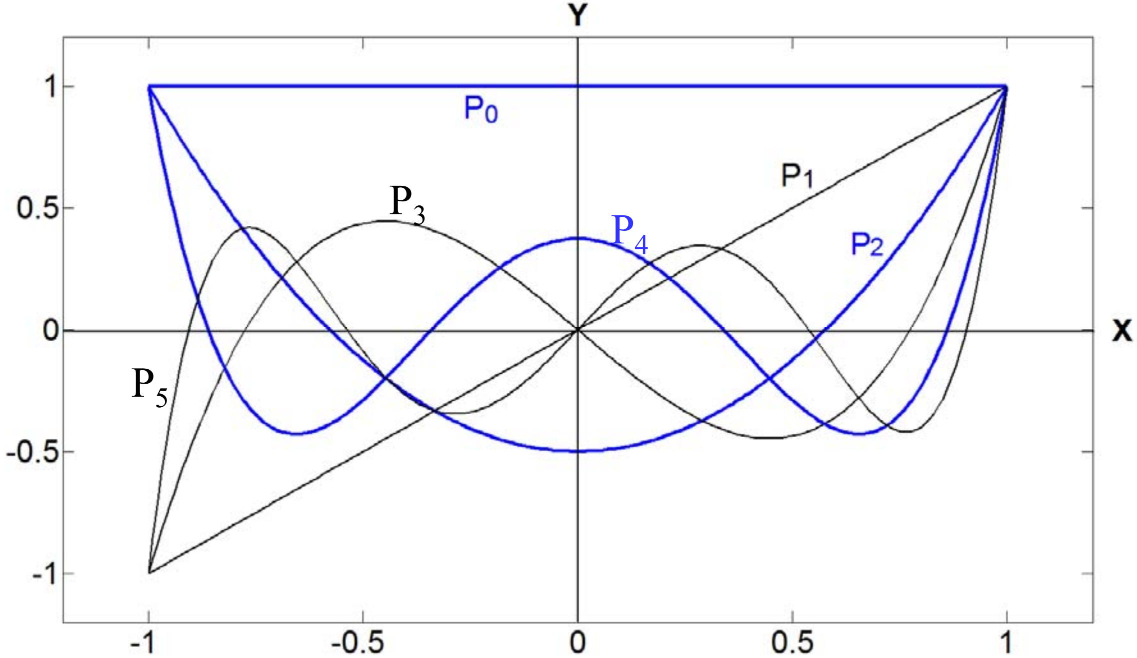
$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

In general,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(x^2 - 1\right)^n$$

Rodrigues' formula

Legendre polynomials



Interval:

$x \in [-1, 1]$

(1) $P_n(-x) = (-1)^n P_n(x)$ even / odd symmetry

(2) $P_n(1) = 1$ $P_n(-1) = (-1)^n$

(3) $P_n(0) = 0$ when n is odd

(4) $P_n'(0) = 0$ when n is even

(5) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$ recursive relation

(6) $\int_{-1}^{1} P_n(x) P_n(x) dx = \frac{2}{2n+1}$

(7) $\int_{-1}^{1} P_m(x) P_n(x) dx = 0$ If $m \neq n$ orthogonality property

Orthogonality property 才是 Legendre polynomials 最重要的性質

3.2.2 Expansion by Legendre Polynomials

若任何在 $x \in [-1, 1]$ 區間為 continuous 的函式 $f(x)$

皆可表示為

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

由於 $\int_{-1}^{1} f(x) P_m(x)\,dx = \underbrace{\sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_m(x) P_n(x)\,dx = a_m \int_{-1}^{1} P_m(x) P_m(x)\,dx}_{\text{根據 orthogonality property}}$

所以

$$a_n = \frac{\int_{-1}^{1} f(x) P_n(x)\,dx}{\int_{-1}^{1} P_n(x) P_n(x)\,dx} = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\,dx$$

Q: How do we use the N^{th} order polynomial to approximate a function $f(x)$ for $x \in [-1, 1]$?

Answer:

note

$$f(x) = \sum_{n=0}^{N} a_n P_n(x)$$

where $$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\, dx$$

The detail of proof can be seen from Section 3.3.

[Example 2] Suppose that

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \quad \text{when } -1 \le x \le 1.$$

Try to approximate $f(x)$ by a 2^{nd} order polynomial

$$f(x) \cong f_2(x) = a_0 + a_1 x + a_2 x^2$$

to minimize $\|f(x) - f_2(x)\|^2 = \int_{-1}^{1} \left(f(x) - f_2(x)\right)^2 dx$

(Solution):

Note that $1, x, x^2$ are not a orthonormal set with $x \in [-1, 1]$.
Therefore, instead, we may adopt the Legendre polynomials.

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

Then $$f(x) \cong f_2(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)$$

where $$c_0 = \frac{1}{2}\int_{-1}^{1} f(x) P_0(x)\,dx = \frac{1}{2}\int_{-1}^{1} \cos\left(\frac{\pi}{2}x\right) dx = \frac{2}{\pi}$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} f(x) P_1(x)\,dx = \frac{3}{2}\int_{-1}^{1} x\cos\left(\frac{\pi}{2}x\right) dx = 0$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} f(x) P_2(x)\,dx = \frac{5}{2}\int_{-1}^{1} \frac{1}{2}\left(3x^2 - 1\right)\cos\left(\frac{\pi}{2}x\right) dx$$

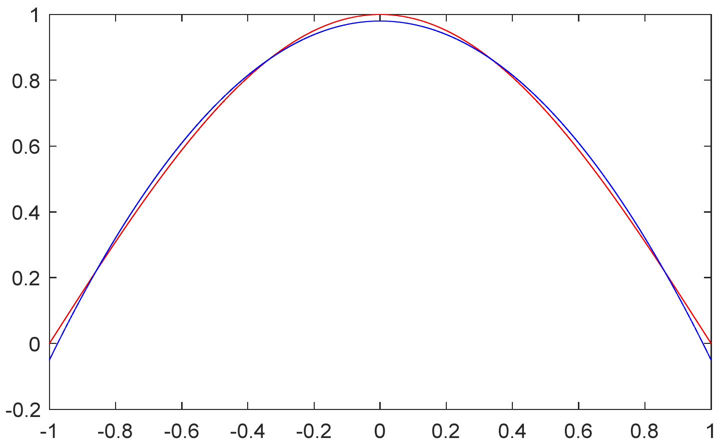
$$= \frac{1}{2}\frac{5}{2}\left(\frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)\left(3x^2 - 1\right)\Big|_{-1}^{1} - \int_{-1}^{1} \frac{12}{\pi} x\sin\left(\frac{\pi}{2}x\right) dx \right)$$

$$= \frac{5}{4}\left(\frac{8}{\pi} - \frac{96}{\pi^3}\right)$$

$$f_2(x) = \frac{2}{\pi}(1) + 0\cdot x + \frac{5}{4}\left(\frac{8}{\pi} - \frac{96}{\pi^3}\right)\frac{1}{2}\left(3x^2 - 1\right)$$

$$f_2(x) = \left(\frac{15}{\pi} - \frac{180}{\pi^3}\right)x^2 + \left(-\frac{3}{\pi} + \frac{60}{\pi^3}\right)$$

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \qquad f_2(x) = \left(\frac{15}{\pi} - \frac{180}{\pi^3}\right)x^2 + \left(-\frac{3}{\pi} + \frac{60}{\pi^3}\right)$$



red line: $f(x)$

blue line: $f_2(x)$

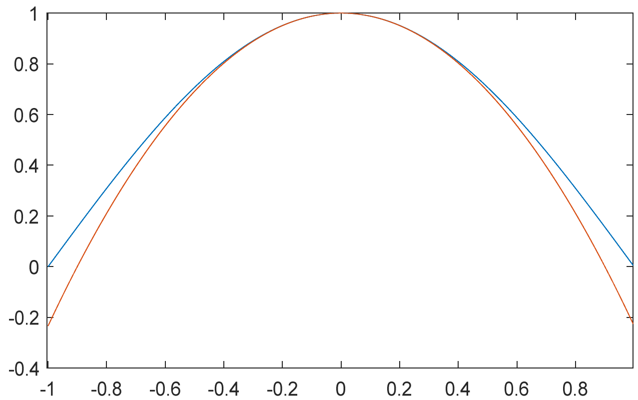
Approximation error is

$$\int_{-1}^{1}\left(f(x) - f_2(x)\right)^2 dx = 0.00059606$$

Comparison: When using the Taylor series,

$$f(x) \cong T(x) = 1 - \frac{\pi^2}{8}x^2$$

$$\int_{-1}^{1}\left(f(x) - T(x)\right)^2 dx = 0.0125$$



[Example 3] (Zill page 456)

$$f(x)=\begin{cases}0, & -1<x<0\\ 1, & 0\le x<1.\end{cases}$$

SOLUTION

Using $$c_n=\frac{\int_a^b f(x)P_n^*(x)\,dx}{\int_a^b P_n(x)P_n^*(x)\,dx}$$

$$c_0=\frac{1}{2}\int_{-1}^{1} f(x)P_0(x)dx=\frac{1}{2}\int_0^1 1\cdot 1dx=\frac{1}{2}$$

$$c_1=\frac{3}{2}\int_{-1}^{1} f(x)P_1(x)dx=\frac{3}{2}\int_0^1 1\cdot xdx=\frac{3}{4}$$

$$c_2=\frac{5}{2}\int_{-1}^{1} f(x)P_2(x)dx=\frac{5}{2}\int_0^1 1\cdot\frac{1}{2}(3x^2-1)dx=0$$

$$c_3=-\frac{7}{16},\quad c_4=0,\quad c_5=\frac{11}{32},\quad \cdots\cdots\cdots$$

$$f(x)=\frac{1}{2}P_0(x)+\frac{3}{4}P_1(x)-\frac{7}{16}P_3(x)+\frac{11}{32}P_5(x)+\cdots.$$

If we want to approximate $f(x)$ by the 1st order polynomial,

$$f(x) \cong c_0 P_0(x) + c_1 P_1(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) = \frac{1}{2} + \frac{3}{4} x$$

If we want to approximate $f(x)$ by the 2nd order polynomial,

$$f(x) \cong c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + 0 P_2(x) = \frac{1}{2} + \frac{3}{4} x$$

If we want to approximate $f(x)$ by the 3rd order polynomial,

$$f(x) \cong c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x)$$
$$= \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + 0 P_2(x) - \frac{7}{16} P_3(x) = \frac{1}{2} + \frac{45}{32} x - \frac{35}{32} x^3$$

3.2.3 Generalization for the Interval

Problem :

How do we expand $f(x)$ where the range of x is $[a, b]$ and $a \neq -1$, $b \neq 1$?

(1) $g(x) = f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right)$ Note: The range of x is changed into [-1, 1]

(2) Expand $g(x)$ by Legendre polynomials

$$g(x) \cong \sum_{n=0}^{N} c_n P_n(x) \qquad \text{where} \quad c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} g(x) P_n(x)\, dx$$

(3) $f(x) = g\left(\frac{2}{b-a}\left(x - \frac{a+b}{2}\right)\right) \cong \sum_{n=0}^{N} c_n P_n\left(\frac{2}{b-a}\left(x - \frac{a+b}{2}\right)\right)$

Section 3.3 Generalization for Function Set Approximation

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ……… $\phi_N(x)$ are a set of independent functions where $x \in [a, b]$ and the weight function is $w(x)$.

Problem :

How do we expand $f(x)$ by $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$ …. $\phi_N(x)$

$$f_N(x) = \sum_{n=0}^{N} c_n \phi_n(x)$$

norm of $f(x) - f_N(x)$

where $$error = \left\| f(x) - f_N(x) \right\| = \sqrt{\int_a^b w(x)\left(f(x) - f_N(x)\right)^2 dx}$$

is minimized.

- Orthogonal
  - Complete (Case 1)
  - Incomplete
    - Orthonormal (Case 2)
    - Orthogonal (Case 3)
- Non-Orthogonal
  - Complete, Incomplete (Case 4)
  - Over-Complete (Compressive Sensing)

(Case 1): The function set is complete and orthogonal / orthonormal

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

$$c_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x)\, dx}{\int_a^b w(x) \left|\phi_n(x)\right|^2 dx}$$

(orthogonal case)

$$c_n = \int_a^b w(x) f(x) \phi_n^*(x)\, dx$$

(orthonormal case)

$$error = 0$$

(Case 2): The function set is incomplete and orthonormal

$$f(x) \cong f_N(x) = \sum_{n=0}^{N} d_n \phi_n(x) \qquad d_n = \int_a^b w(x) f(x) \phi_n^*(x)\, dx$$

$$\text{error} = \left\| f(x) - f_N(x) \right\| = \sqrt{\int_a^b w(x) f^2(x)\, dx - \sum_{n=0}^{N} |d_n|^2}$$

(Case 3): The function set is incomplete and orthogonal

$$f(x) \cong f_N(x) = \sum_{n=0}^{N} d_n \phi_n(x) \qquad d_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x)\, dx}{\int_a^b w(x) |\phi_n(x)|^2\, dx}$$

$$\text{error} = \left\| f(x) - f_N(x) \right\|$$

$$= \sqrt{\int_a^b w(x) f^2(x)\, dx - \sum_{n=0}^{N} |d_n|^2 \int_a^b w(x) |\phi_n(x)|^2\, dx}$$

(Case 4): The function set is not orthogonal

Using the Gram-Schmidt method to convert it into an orthonormal set.

(page 303)

3.3.1 Orthonormal / Orthogonal Function Set Expansion

(Case 1)

[Theorem 3.3.1] **Parseval's Theorem (Energy Preservation Theorem)**

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……….. are a complete and orthonormal function set for $x \in [a, b]$ and

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \qquad \text{where} \qquad c_n = (f(x), \phi_n(x)) = \int_a^b w(x) f(x) \phi_n^*(x)\, dx$$

Then

$$\|f(x)\|^2 = (f(x), f(x)) = \sum_{n=0}^{\infty} |c_n|^2 \qquad \int_a^b w(x) |f(x)|^2\, dx = \sum_{n=0}^{\infty} |c_n|^2$$

(Proof):

$$\int_a^b w(x)|f(x)|^2\,dx = \int_a^b w(x)f(x)f^*(x)\,dx$$

$$= \int_a^b w(x)\sum_{n=0}^{\infty} c_n\phi_n(x)\sum_{m=0}^{\infty} c_m^*\phi_m^*(x)\,dx$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n c_m^* \int_a^b w(x)\phi_n(x)\phi_m^*(x)\,dx$$

Since $\int_a^b w(x)\phi_n(x)\phi_m^*(x)\,dx = 0$ if $n \neq m$

$$\int_a^b w(x)\phi_n(x)\phi_n^*(x)\,dx = 1$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_n c_m^* \int_a^b w(x)\phi_n(x)\phi_m^*(x)\,dx = \sum_{n=0}^{\infty} c_n c_n^* = \sum_{n=0}^{\infty} |c_n|^2$$

$$\int_a^b w(x)|f(x)|^2\,dx = \sum_{n=0}^{\infty} |c_n|^2$$

(Case 2)

[Theorem 3.3.2] Error When Expanded by an Incomplete Orthonormal Set

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ………., $\phi_N(x)$ are an incomplete and orthonormal function set for $x \in [a, b]$. Now, we want to approximate $f(x)$ by a linear combination of $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ………., $\phi_N(x)$:

$$f(x) \cong f_N(x) = \sum_{n=0}^{N} d_n \phi_n(x)$$

To minimize the approximation error, d_n should be calculated from

$$d_n = \int_a^b w(x) f(x) \phi_n^*(x) dx$$

Then, the approximation error is:

$$\text{approximation error} = \|f(x) - f_N(x)\| = \sqrt{\int_a^b w(x)\left(f(x) - f_N(x)\right)^2 dx}$$

$$= \sqrt{\int_a^b w(x) f^2(x) dx - \sum_{n=0}^{N} |d_n|^2}$$

(Proof):

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ……….., $\phi_N(x)$ is a subset of the complete and orthonormal function set $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ……..., $\phi_N(x)$, $\phi_{N+1}(x)$, …….……. Then, $f(x)$ can be expressed as

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \quad \text{where} \quad c_n = \int_a^b w(x) f(x) \phi_n^*(x) dx$$

Therefore,

$$f(x) - f_N(x) = \sum_{n=0}^{N} (c_n - d_n) \phi_n(x) + \sum_{n=N+1}^{\infty} c_n \phi_n(x)$$

Then, from Theorem 3.3.1,

$$\left\| f(x) - f_N(x) \right\|^2 = \sum_{n=0}^{N} \left| c_n - d_n \right|^2 + \sum_{n=N+1}^{\infty} \left| c_n \right|^2$$

To minimize $\left\| f(x) - f_N(x) \right\|^2$, we should set

$$d_n = c_n = \int_a^b w(x) f(x) \phi_n^*(x) dx \qquad \text{for } n = 0, 1, \ldots, N$$

Note that, when $d_n = c_n$, then

$$\left\| f(x) - f_N(x) \right\|^2 = \sum_{n=N+1}^{\infty} \left| c_n \right|^2 \qquad \text{from } \left\| f(x) \right\|^2 = \sum_{n=0}^{\infty} \left| c_n \right|^2$$

$$= \left\| f(x) \right\|^2 - \sum_{n=0}^{N} \left| c_n \right|^2$$

$$= \int_a^b w(x) \left| f(x) \right|^2 dx - \sum_{n=0}^{N} \left| c_n \right|^2$$

Theorem 3.3.2 is hence proved.

(Case 3)

[Theorem 3.3.3] Error When Expanded by an Incomplete Orthogonal Set

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_N(x)$ are an incomplete and orthogonal function set for $x \in [a, b]$. Now, we want to approximate $f(x)$ by a linear combination of $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ..., $\phi_N(x)$:

$$f(x) \cong f_N(x) = \sum_{n=0}^{N} d_n \phi_n(x)$$

To minimize the approximation error, d_n should be calculated from

$$d_n = \int_a^b w(x) f(x) \phi_n^*(x) dx \Big/ \int_a^b w(x) |\phi_n(x)|^2 dx$$

Then, the approximation error is:

$$\text{approximation error} = \| f(x) - f_N(x) \| = \sqrt{\int_a^b w(x) \left(f(x) - f_N(x) \right)^2 dx}$$

$$= \sqrt{\int_a^b w(x) f^2(x) dx - \sum_{n=0}^{N} |d_n|^2 \int_a^b w(x) |\phi_n(x)|^2 dx}$$

(Proof): The formula of d_n is directly from page 273.

If $\psi_n(x) = \dfrac{\phi_n(x)}{\left\|\phi_n(x)\right\|}$

then $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, ……….., $\psi_N(x)$ form an orthonormal set.

From (Case 2),

$$f(x) \cong f_N(x) = \sum_{n=0}^{N} \hat{d}_n \psi_n(x) \quad \text{where} \quad \hat{d}_n = \int_a^b w(x) f(x) \psi_n^*(x)\,dx$$

$$\text{error} = \left\| f(x) - f_N(x) \right\| = \sqrt{\int_a^b w(x) f^2(x)\,dx - \sum_{n=0}^{N} \left|\hat{d}_n\right|^2}$$

Since

$$d_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x)\,dx}{\left\|\phi_n(x)\right\|^2} = \frac{\int_a^b w(x) f(x) \psi_n^*(x)\,dx}{\left\|\phi_n(x)\right\|} = \frac{\hat{d}_n}{\left\|\phi_n(x)\right\|}$$

$$\text{error} = \sqrt{\int_a^b w(x) f^2(x)\,dx - \sum_{n=0}^{N} \left|\hat{d}_n\right|^2} = \sqrt{\int_a^b w(x) f^2(x)\,dx - \sum_{n=0}^{N} \left|d_n\right|^2 \left\|\phi_n(x)\right\|^2}$$

3.3.2 Non-Orthogonal Function Set Expansion

(Case 4)

How do we generate an orthonormal function set $\{\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \ldots\ldots\ldots\}$ if the input $\{p_0(x), p_1(x), p_2(x), p_3(x), \ldots\ldots\ldots\}$ is independent but not an orthogonal function set?

Gram-Schmidt process

$$\phi_0(x) = \frac{p_0(x)}{\|p_0(x)\|} \qquad n = 0$$

$$g_n(x) = p_n(x) - \sum_{m=0}^{n-1} \left(p_n(x), \phi_m(x)\right) \phi_m(x)$$

$$\phi_n(x) = \frac{g_n(x)}{\|g_n(x)\|}$$

$$n = n+1$$

Proof for orthogonality of the function set generated from the Gram-Schmidt process

Suppose that we have known that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ………..., $\phi_{n-1}(x)$ are orthonormal.

If $k < n$,

$$\left(\phi_n(x),\phi_k(x)\right)=\left(\frac{g_n(x)}{\left\|g_n(x)\right\|},\phi_k(x)\right)$$

$$=\frac{1}{\left\|g_n(x)\right\|}\left(p_n(x)-\sum_{m=0}^{n-1}\left(p_n(x),\phi_m(x)\right)\phi_m(x),\quad \phi_k(x)\right)$$

$$=\frac{1}{\left\|g_n(x)\right\|}\left[\left(p_n(x),\phi_k(x)\right)-\sum_{m=0}^{n-1}\left(p_n(x),\phi_m(x)\right)\left(\phi_m(x),\phi_k(x)\right)\right]$$

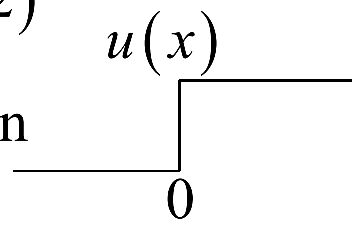
$$=\frac{1}{\left\|g_n(x)\right\|}\left[\left(p_n(x),\phi_k(x)\right)-\left(p_n(x),\phi_k(x)\right)\right]=0$$

[Example 4]

Derive $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ……….. such that

$$\phi_n(x) = \sum_{k=0}^{n} \tau_{n,k} p_k(x), \qquad p_k(x) = u(x-k) - u(x-k-2)$$

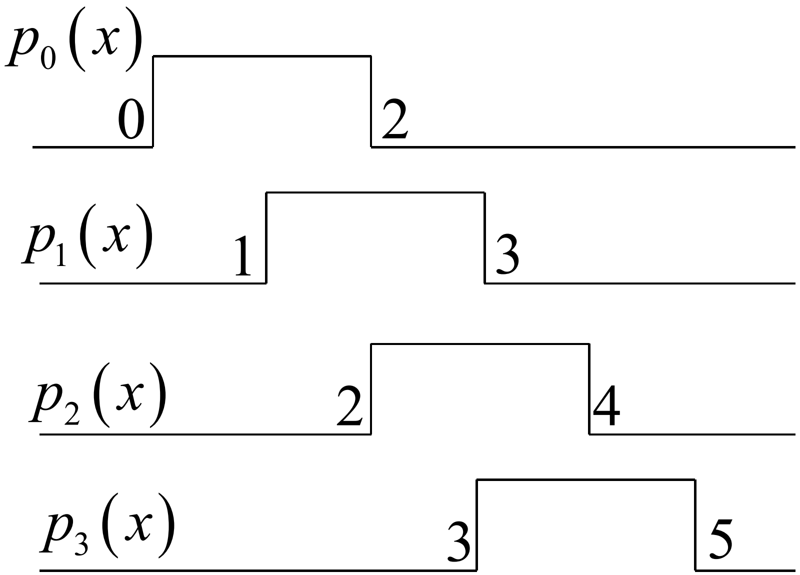
$u(x)$: unit step function



and

$$\int_0^\infty \phi_n(x)\phi_m(x)\,dx = 0 \qquad \int_0^\infty \phi_n(x)\phi_n(x)\,dx = 1$$

when $m \neq n$



⟹ orthonormalization

Answer:

Set $p_k(x) = u(x-k) - u(x-k-2)$

$$\phi_0(x) = \frac{p_0(x)}{\|p_0(x)\|} = \frac{u(x) - u(x-2)}{\sqrt{2}}$$

$$g_1(x) = p_1(x) - (p_1(x), \phi_0(x))\phi_0(x)$$

$$= u(x-1) - u(x-3) - \frac{1}{\sqrt{2}}\left(\frac{u(x) - u(x-2)}{\sqrt{2}}\right)$$

$$\phi_1(x) = \frac{g_1(x)}{\|g_1(x)\|} = \frac{u(x-1) - u(x-3) - \frac{1}{2}(u(x) - u(x-2))}{\sqrt{6}/2}$$

$$g_2(x) = p_2(x) - (p_2(x), \phi_0(x))\phi_0(x) - (p_2(x), \phi_1(x))\phi_1(x)$$

$$= u(x-2) - u(x-4) - \frac{2}{\sqrt{6}} \frac{u(x-1) - u(x-3) - \frac{1}{2}(u(x) - u(x-2))}{\sqrt{6}/2}$$

$$= u(x-2) - u(x-4) - \frac{2}{3}u(x-1) + \frac{2}{3}u(x-3) + \frac{1}{3}(u(x) - u(x-2))$$

$$\phi_2(x) = \frac{g_2(x)}{\|g_2(x)\|}$$

$$= \frac{u(x-2) - u(x-4) - \frac{2}{3}(u(x-1) - u(x-3)) + \frac{1}{3}(u(x) - u(x-2))}{2/\sqrt{3}}$$

....

....

附錄七 Compressive Sensing

The problems that compressive sensing deals with:

Suppose that $b_0(t)$, $b_1(t)$, $b_2(t)$, $b_3(t)$ ………….. form an over-complete and non-orthogonal spanning set.

(Problem 1) We want to minimize $||c||_0$ （$||\ ||_0$ 是 zero-order norm， $||c||_0$ 意指 c_m 的值不為 0 的個數） such that

$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $||c||_0$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When $||c||_0$ is limited to M, we want to minimize

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$

Examples for the spanning set used for compressive sensing (Since they are not linearly independent, they are not basis).

3-atom form

$$b_m(t) = \exp(j2\pi f_m t)\exp\left(-\frac{\pi(t-t_m)^2}{\sigma_m^2}\right)$$

4-atom form

$$b_m(t) = \exp(j2\pi(f_m t + \eta_m t^2))\exp\left(-\frac{\pi(t-t_m)^2}{\sigma_m^2}\right)$$

Problems: over-complete, non-orthogonal, too many functions in the sets

Section 3.4 Other Orthogonal Polynomials

(只教不考)

In addition to Legendre Polynomials, there are many other orthogonal polynomials. However, their weight functions and intervals are different.

[1] R. Beals, Special Functions and Orthogonal Polynomials, Cambridge Studies in Advanced Mathematics, vol. 153, Cambridge University Press, 2016.

[2] M. R. Spiegel, Mathematical Handbook, Schaum, 1990.

Summary of Weights and Supports of Orthogonal Polynomials

Name	Weight Function $w(x)$	Support
Legendre	1	$x \in [-1, 1]$
Jacobi	$(1+x)^n(1-x)^m$	$x \in [-1, 1]$
Associated Legendre (Ultraspherical, Gegenbauer)	$(1-x^2)^m$	$x \in [-1, 1]$
Chebyshev (1st Kind)	$(1-x^2)^{-1/2}$	$x \in [-1, 1]$
Chebyshev (2st Kind)	$(1-x^2)^{1/2}$	$x \in [-1, 1]$
Laguerre	e^{-x}	$x \in [0, \infty)$
Associated Laguerre	$x^m e^{-x}$	$x \in [0, \infty)$
Hermite	e^{-x^2}	$x \in (-\infty, \infty)$

- Associated Legendre Functions

$$P_{n,m}(x) = \frac{d^m}{dx^m} P_n(x)$$

$P_n(x)$: the Legendre polynomial of order n, in fact, $P_n(x) = P_{n,0}(x)$

They are orthogonal for $x \in [-1, 1]$, $w(x) = (1-x^2)^m$

$$\int_{-1}^{1} (1-x^2)^m P_{n,m}(x) P_{k,m}(x) dx = 0 \qquad \text{if } n \neq k$$

$$\int_{-1}^{1} (1-x^2)^m (P_{n,m}(x))^2 dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$$

Ex: When $m = 1$,

$P_{1,1}(x) = 1$ $\qquad$ $P_{2,1}(x) = 2x$

$P_{3,1}(x) = \frac{15}{2}x^2 - \frac{3}{2}$ $\qquad$ $P_{4,1}(x) = \frac{35}{2}x^3 - \frac{15}{2}x$

- Chebychev polynomials 電子學和 filter design 常用

$$P_n(\cos\theta) = \cos n\theta$$

$$P_n(x) = \sum_{k=0}^{n/2} \binom{n}{2k} x^{n-2k} \left(1-x^2\right)^k$$

They are orthogonal for $x \in [-1, 1]$ $\qquad w(x) = \frac{1}{\sqrt{1-x^2}}$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} P_n(x) P_k(x)\, dx = 0 \quad \text{if } n \neq k$$

$$\int_{-1}^{1} \frac{P_n^2(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi & \text{if } n = 0, \\ \pi/2 & \text{otherwise} \end{cases}$$

They are the solutions of $(1-x^2)P_n''(x) - xP_n'(x) + n^2 P_n(x) = 0$

$P_0(x) = 1$ $\qquad P_1(x) = x$

$P_2(x) = 2x^2 - 1$ $\qquad P_3(x) = 4x^3 - 3x$

- Laguerre polynomials

$$P_n(x) = e^x \frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

They are orthogonal for $x \in [0, \infty)$ $w(x) = e^{-x}$

$$\int_0^\infty e^{-x} P_n(x) P_k(x)\, dx = 0 \quad \text{if } n \neq k$$

$$\int_0^\infty e^{-x} \left(P_n(x)\right)^2 dx = (n!)^2$$

They are the solutions of $xP_n''(x) + (1-x)P_n'(x) + nP_n(x) = 0$

$P_0(x) = 1$

$P_1(x) = -x + 1$

$P_2(x) = x^2 - 4x + 2$

$P_3(x) = -x^3 + 9x^2 - 18x + 6$

- Associated Laguerre polynomials

$$P_{n.m}(x) = \frac{d^m}{dx^m}\left(P_n(x)\right)$$

where $P_n(x)$ is the n^{th} order Laguarre polynomial

They are orthogonal for $x \in [0, \infty)$ $w(x) = x^m e^{-x}$

$$\int_0^\infty x^m e^{-x} P_{n,m}(x) P_{k,m}(x)\, dx = 0 \quad \text{if } n \neq k \qquad \int_0^\infty x^m e^{-x}\left(P_{n,m}(x)\right)^2 dx = \frac{(n!)^2}{(n-m)!}$$

They are the solutions of $xP_n''(x) + (m+1-x)P_n'(x) + (n-m)P_n(x) = 0$

$P_{1,1}(x) = -1$

$P_{2,1}(x) = 2x - 4$

$P_{3,1}(x) = -3x^2 + 18x - 18$

$P_{4,1}(x) = 4x^3 - 48x^2 + 144x - 96$

- Hermite polynomials 電磁波、光學、頻譜分析常用

$$P_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

They are orthogonal for $x \in (-\infty, \infty)$ $w(x) = e^{-x^2}$

$$\int_{-\infty}^{\infty} e^{-x^2} P_n(x) P_k(x)\, dx = 0 \quad \text{if } n \neq k$$

$$\int_{-\infty}^{\infty} e^{-x^2} \left(P_n(x)\right)^2 dx = 2^n n! \sqrt{\pi}$$

They are the solutions of $P_n''(x) - xP_n'(x) + nP_n(x) = 0$

$P_0(x) = 1$

$P_1(x) = 2x$

$P_2(x) = 4x^2 - 2$

$P_3(x) = 8x^3 - 12x$

$P_4(x) = 16x^4 - 48x^2 + 12$

$P_5(x) = 32x^5 - 160x^3 + 120x$