

## 3. Function Approximation

Section 3.1 Review for Orthogonal Basis Expansion

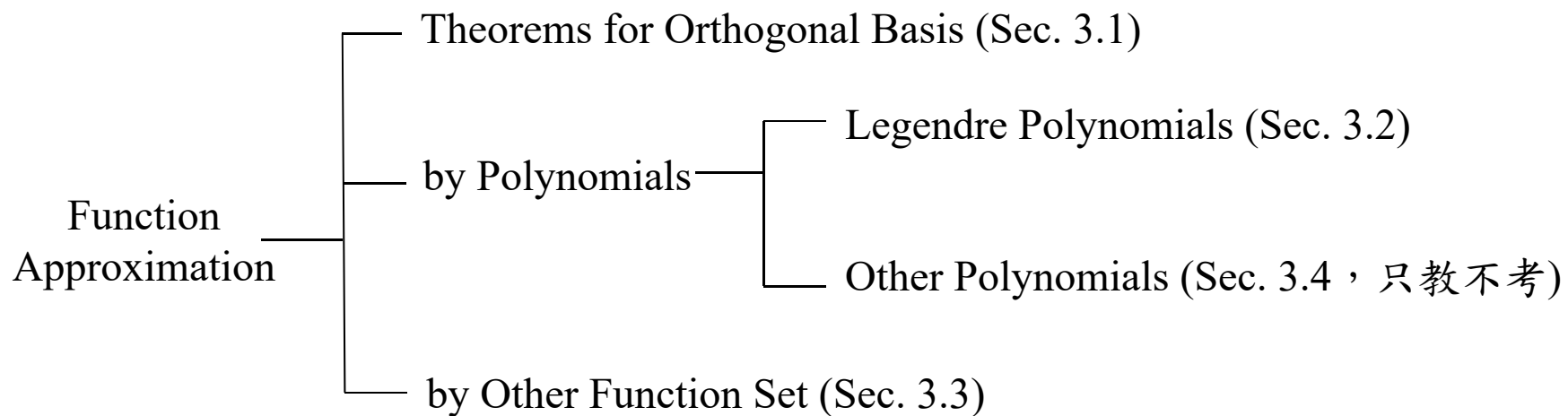
Section 3.2 Polynomial Approximation Using Legendre Polynomials

Section 3.3 Generalization for Function Set Approximation

Section 3.4 Other Orthogonal Polynomials (只教不考)

[1] D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9<sup>th</sup> edition, Cengage Learning, 2017.

[2] R. Beals, Special Functions and Orthogonal Polynomials, Cambridge Studies in Advanced Mathematics, vol. 153, Cambridge University Press, 2016.

**Function Approximation**

## Sec. 3.1 Review for Orthogonal Basis Expansion

(1) inner product on an interval  $[a, b]$

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx \quad (f_1, f_2 \text{ 為 real 時})$$

or  $(f_1, f_2) = \int_a^b f_1(x) f_2^*(x) dx$  (more standard definition)

(2) orthogonal on an interval  $[a, b]$

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0 \quad (f_1, f_2 \text{ 為 real 時})$$

$$(f_1, f_2) = \int_a^b f_1(x) f_2^*(x) dx = 0 \quad (\text{more standard definition})$$

注意：任何 even function 和任何 odd function 在  $[-b, b]$  之間必為 orthogonal,  $(a = -b)$

$$\int_{-b}^b f_1(x) f_2(x) dx = 0$$

↑
↑  
 even      odd

### (3) orthogonal set

For a set of functions  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$ , if

$$\int_a^b \phi_m(x) \phi_n^*(x) dx = 0 \quad \text{for any } m \neq n$$

then  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  is an orthogonal set on the interval  $[a, b]$

(4) square norm

$$\|f(x)\|^2 = (f(x), f(x)) = \int_a^b f(x) f^*(x) dx = \int_a^b |f(x)|^2 dx$$

(5) norm

$$\|f(x)\| = \sqrt{(f(x), f(x))} = \sqrt{\int_a^b f(x) f^*(x) dx}$$

(6) orthonormal set

對一個 orthogonal set, 若更進一步的滿足

$$\int_a^b \phi_n(x) \phi_n^*(x) dx = 1 \quad \text{for all } n$$

則被稱為 orthonormal set

(7) normalize

$$\psi(x) \xrightarrow{\text{normalize}} v(x) = \frac{\psi(x)}{\|\psi(x)\|} \quad \text{將 norm 變為 1}$$

**[Example 1]** Suppose that  $[a, b] = [-1, 1]$ . Then

$$\left\{ \frac{1}{\sqrt{2}} e^{j\pi nx} \mid n = \dots, -2, -1, 0, 1, 2, 3, \dots \right\}$$

forms an orthonormal set.

$$\text{(Proof): } \left( \frac{1}{\sqrt{2}} e^{j\pi nx}, \frac{1}{\sqrt{2}} e^{j\pi mx} \right) = \frac{1}{2} \int_{-1}^1 e^{j\pi nx} e^{-j\pi mx} dx$$

If  $m \neq n$ ,

$$\left( \frac{1}{\sqrt{2}} e^{j\pi nx}, \frac{1}{\sqrt{2}} e^{j\pi mx} \right) = \frac{e^{j\pi(n-m)x}}{j2\pi(n-m)} \Big|_{-1}^1 = \frac{e^{j\pi(n-m)} - e^{-j\pi(n-m)}}{j2\pi(n-m)} = \frac{\sin(\pi(n-m))}{\pi(n-m)} = 0$$

If  $m = n$ ,

$$\left( \frac{1}{\sqrt{2}} e^{j\pi nx}, \frac{1}{\sqrt{2}} e^{j\pi mx} \right) = \frac{1}{2} \int_{-1}^1 1 dx = 1$$

## (8) complete

若在 interval  $[a, b]$  之間，任何一個 function  $f(x)$  都可以表示成  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  的 linear combination

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \dots = \sum_{n=0}^{\infty} c_n\phi_n(x)$$

則  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  被稱作 complete

## (9) orthogonal series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x) \quad \text{where} \quad c_n = \frac{\int_a^b f(x)\phi_n^*(x)dx}{\int_a^b \phi_n(x)\phi_n^*(x)dx}$$

Specially, if  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  are orthonormal

$$c_n = \int_a^b f(x)\phi_n^*(x)dx$$

## (10) weight

inner product with weight function

$$(f_1(x), f_2(x)) = \int_a^b w(x) f_1(x) f_2^*(x) dx$$

$w(x)$  is called the weight function

With the weight function

(10-1) orthogonal 的定義改成

$$(f_m, f_n) = \int_a^b w(x) f_m(x) f_n^*(x) dx = 0 \quad \text{for } m \neq n$$

(10-2) square norm 的定義改成  $\|f(x)\|^2 = \int_a^b w(x) f(x) f^*(x) dx$

(10-3) norm 的定義改成  $\|f(x)\| = \sqrt{\int_a^b w(x) f(x) f^*(x) dx}$

(10-4) orthonormal 的定義改成

$$\int_a^b w(x) f_m(x) f_n^*(x) dx = 0 \quad \text{for } m \neq n$$

$$\int_a^b w(x) f_n(x) f_n^*(x) dx = 1$$

(10-5) normalize 的算法改成

$$v(x) = \frac{\psi(x)}{\|\psi(x)\|} = \frac{\psi(x)}{\sqrt{\int_a^b w(x) \psi(x) \psi^*(x) dx}}$$

(10-6) orthogonal series expansion of  $f(x)$  的算法改成

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \quad c_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x) dx}{\int_a^b w(x) \phi_n(x) \phi_n^*(x) dx}$$

$$\text{orthonormal case: } c_n = \int_a^b w(x) f(x) \phi_n^*(x) dx$$

trigonometric functions

$$\left\{ 1, \cos \frac{\pi}{p} x, \cos \frac{2\pi}{p} x, \cos \frac{3\pi}{p} x, \dots, \sin \frac{\pi}{p} x, \sin \frac{2\pi}{p} x, \sin \frac{3\pi}{p} x, \dots \right\}$$

form a complete and orthogonal set on the interval of  $[-p, p]$

$$\cos \frac{n\pi}{p} x, \quad \sin \frac{n\pi}{p} x \quad \text{週期} : \frac{2p}{n} \quad \text{頻率} : \frac{n}{2p}$$

$$\cos \frac{n\pi}{p} x = \cos \left( \frac{n\pi}{p} x + 2\pi \right) = \cos \left( \frac{n\pi}{p} \left( x + \frac{2p}{n} \right) \right)$$

Fourier Series (expanded by trigonometric functions)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$$

$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx \quad a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx$$

$$b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx$$

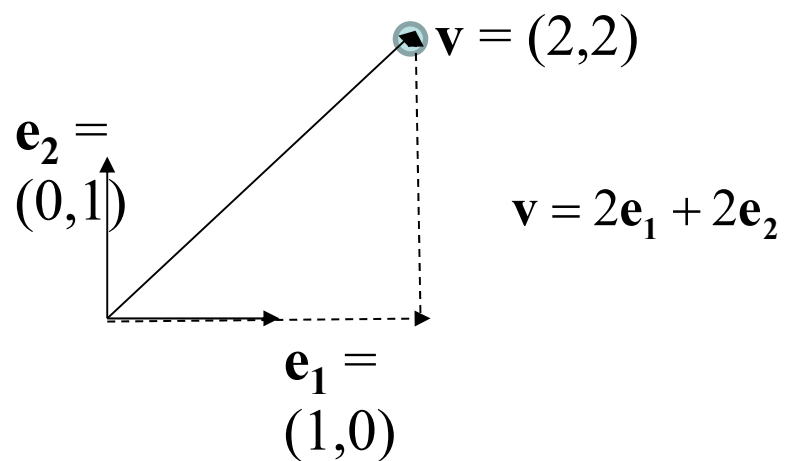
**Q: Why should we use the orthogonal basis?**

(1) The expansion coefficients are easier to determined,

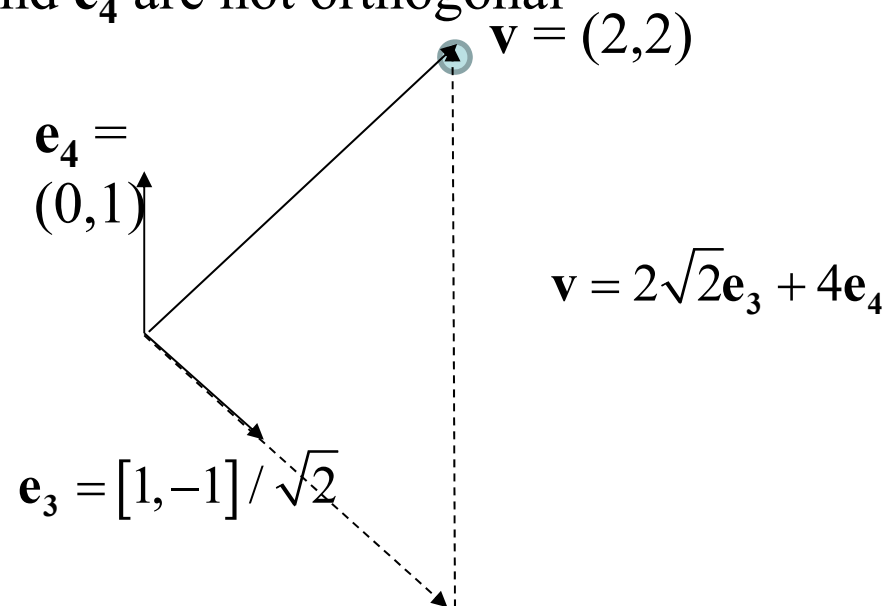
(2) No Interference

(3) More basis functions  $\rightarrow$  Less error

$\mathbf{e}_1$  and  $\mathbf{e}_2$  are orthogonal



$\mathbf{e}_3$  and  $\mathbf{e}_4$  are not orthogonal



## Section 3-2 Polynomial Approximation Using Legendre Polynomials

- Legendre's equation of order  $n$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

One of the solution: Legendre polynomials

$$n=0 \quad P_0(x) = 1$$

$$n=1 \quad P_1(x) = x$$

$$n=2 \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$n=3 \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$n=4 \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$n=5 \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9<sup>th</sup> edition, Cengage Learning, 2017, Sections 6-4, 11-5.

### 3.2.1 Legendre Polynomial

Legendre's Equation  $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

$$y(x) = \sum_{k=0}^{\infty} c_k x^k \quad \text{代入, 得出}$$

Two linearly independent solutions are

$$y_1(x) = c_0 \left[ 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} x^6 + \dots \right]$$

$$y_2(x) = c_1 \left[ x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} x^7 + \dots \right]$$

(a) When  $n$  is not an integer, both the two solutions have infinite number of terms.

(b) When  $n$  is an even integer,  $y_1(x)$  has finite number of terms.

In  $y_1(x)$ , the coefficient of  $x^k$  is zero when  $k > n$ .

(c) When  $n$  is an odd integer,  $y_2(x)$  has finite number of terms.

In  $y_2(x)$ , the coefficient of  $x^k$  is zero when  $k > n$ .

$y_1(x)$  when  $n$  is an even integer and  $y_2(x)$  when  $n$  is an odd integer are called the Legendre polynomials (denoted by  $P_n(x)$ ).

通常選

$$c_0 = (-1)^{n/2} \frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n}$$

(讓  $P_n(1)$  一律等於 1)

由  $y_1(x)$

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

In general,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$c_1 = (-1)^{(n-1)/2} \frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)}$$

由  $y_2(x)$

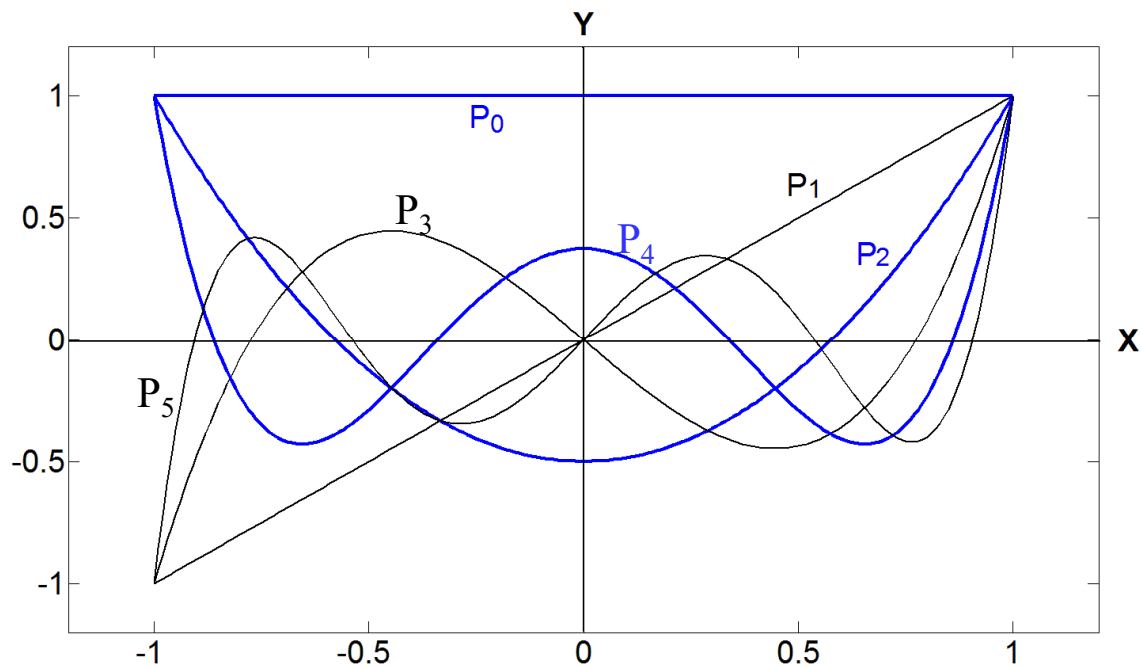
$$P_1(x) = x$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

**Rodrigues' formula**

## Legendre polynomials



Interval:

$$x \in [-1, 1]$$



$$(1) P_n(-x) = (-1)^n P_n(x) \quad \text{even / odd symmetry}$$

$$(2) P_n(1) = 1 \quad P_n(-1) = (-1)^n$$

$$(3) P_n(0) = 0 \quad \text{when } n \text{ is odd}$$

$$(4) P'_n(0) = 0 \quad \text{when } n \text{ is even}$$

$$(5) (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \quad \text{recursive relation}$$

$$(6) \int_{-1}^1 P_n(x)P_n(x)dx = \frac{2}{2n+1}$$

$$(7) \int_{-1}^1 P_m(x)P_n(x)dx = 0 \quad \text{If } m \neq n \quad \text{orthogonality property}$$

Orthogonality property 才是 Legendre polynomials 最重要的性質

### 3.2.2 Expansion by Legendre Polynomials

若任何在  $x \in [-1, 1]$  區間為 continuous 的函式  $f(x)$

皆可表示為

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

$$\text{由於 } \int_{-1}^1 f(x) P_m(x) dx = \sum_{n=0}^{\infty} a_n \int_{-1}^1 P_m(x) P_n(x) dx = a_m \int_{-1}^1 P_m(x) P_m(x) dx$$

根據 orthogonality property


所以

$$a_n = \frac{\int_{-1}^1 f(x) P_n(x) dx}{\int_{-1}^1 P_n(x) P_n(x) dx} = \left(n + \frac{1}{2}\right) \int_{-1}^1 f(x) P_n(x) dx$$

**Q: How do we use the  $N^{\text{th}}$  order polynomial to approximate a function  $f(x)$  for  $x \in [-1, 1]$ ?**

Answer:

$$f(x) = \sum_{n=0}^N a_n P_n(x)$$

*note* 

where  $a_n = \left(n + \frac{1}{2}\right) \int_{-1}^1 f(x) P_n(x) dx$

The detail of proof can be seen from Section 3.3.

**[Example 2]** Suppose that

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \quad \text{when } -1 \leq x \leq 1.$$

Try to approximate  $f(x)$  by a 2<sup>nd</sup> order polynomial

$$f(x) \cong f_2(x) = a_0 + a_1x + a_2x^2$$

to minimize  $\|f(x) - f_2(x)\|^2 = \int_{-1}^1 (f(x) - f_2(x))^2 dx$

**(Solution):**

Note that  $1, x, x^2$  are not an orthonormal set with  $x \in [-1, 1]$ .

Therefore, instead, we may adopt the Legendre polynomials.

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

Then  $f(x) \cong f_2(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)$

where  $c_0 = \frac{1}{2} \int_{-1}^1 f(x) P_0(x) dx = \frac{1}{2} \int_{-1}^1 \cos\left(\frac{\pi}{2} x\right) dx = \frac{2}{\pi}$

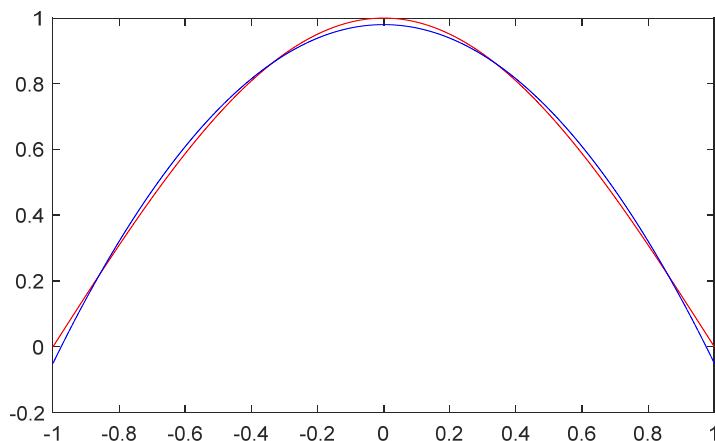
$$c_1 = \frac{3}{2} \int_{-1}^1 f(x) P_1(x) dx = \frac{3}{2} \int_{-1}^1 x \cos\left(\frac{\pi}{2} x\right) dx = 0$$

$$\begin{aligned} c_2 &= \frac{5}{2} \int_{-1}^1 f(x) P_2(x) dx = \frac{5}{2} \int_{-1}^1 \frac{1}{2} (3x^2 - 1) \cos\left(\frac{\pi}{2} x\right) dx \\ &= \frac{1}{2} \frac{5}{2} \left( \frac{2}{\pi} \sin\left(\frac{\pi}{2} x\right) (3x^2 - 1) \Big|_{-1}^1 - \int_{-1}^1 \frac{12}{\pi} x \sin\left(\frac{\pi}{2} x\right) dx \right) \\ &= \frac{5}{4} \left( \frac{8}{\pi} - \frac{96}{\pi^3} \right) \end{aligned}$$

$$f_2(x) = \frac{2}{\pi} (1) + 0 \cdot x + \frac{5}{4} \left( \frac{8}{\pi} - \frac{96}{\pi^3} \right) \frac{1}{2} (3x^2 - 1)$$

$$f_2(x) = \left( \frac{15}{\pi} - \frac{180}{\pi^3} \right) x^2 + \left( -\frac{3}{\pi} + \frac{60}{\pi^3} \right)$$

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \quad f_2(x) = \left(\frac{15}{\pi} - \frac{180}{\pi^3}\right)x^2 + \left(-\frac{3}{\pi} + \frac{60}{\pi^3}\right)$$



red line:  $f(x)$

blue line:  $f_2(x)$

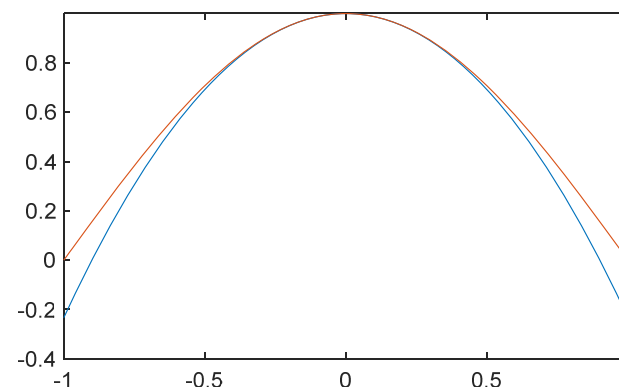
Approximation error (expressed by  $\|f(x) - f_2(x)\|^2$ ) is

$$\int_{-1}^1 (f(x) - f_2(x))^2 dx = 0.00059606$$

**Comparison:** When using the Taylor series,

$$f(x) \cong T(x) = 1 - \frac{\pi^2}{8}x^2$$

$$\int_{-1}^1 (f(x) - T(x))^2 dx = 0.0125$$



**[Example 3]** (Zill page 456)

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 \leq x < 1. \end{cases}$$

**SOLUTION**

Using 
$$c_n = \frac{\int_a^b f(x) P_n^*(x) dx}{\int_a^b P_n(x) P_n^*(x) dx}$$

$$c_0 = \frac{1}{2} \int_{-1}^1 f(x) P_0(x) dx = \frac{1}{2} \int_0^1 1 \cdot 1 dx = \frac{1}{2}$$

$$c_1 = \frac{3}{2} \int_{-1}^1 f(x) P_1(x) dx = \frac{3}{2} \int_0^1 1 \cdot x dx = \frac{3}{4}$$

$$c_2 = \frac{5}{2} \int_{-1}^1 f(x) P_2(x) dx = \frac{5}{2} \int_0^1 1 \cdot \frac{1}{2} (3x^2 - 1) dx = 0$$

$$c_3 = -\frac{7}{16}, \quad c_4 = 0, \quad c_5 = \frac{11}{32}, \quad \dots\dots\dots$$

$$f(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x) + \frac{11}{32} P_5(x) + \dots$$

If we want to approximate  $f(x)$  by the 1<sup>st</sup> order polynomial,

$$f(x) \cong c_0 P_0(x) + c_1 P_1(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) = \frac{1}{2} + \frac{3}{4} x$$

If we want to approximate  $f(x)$  by the 2<sup>nd</sup> order polynomial,

$$f(x) \cong c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + 0 P_2(x) = \frac{1}{2} + \frac{3}{4} x$$

If we want to approximate  $f(x)$  by the 3<sup>rd</sup> order polynomial,

$$\begin{aligned} f(x) &\cong c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x) \\ &= \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + 0 P_2(x) - \frac{7}{16} P_3(x) = \frac{1}{2} + \frac{45}{32} x - \frac{35}{32} x^3 \end{aligned}$$

### 3.2.3 Generalization for the Interval

Problem :

How do we expand  $f(x)$  where the range of  $x$  is  $[a, b]$  and  $a \neq -1, b \neq 1$ ?

(1)  $g(x) = f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right)$       Note: The range of  $x$  is changed into  $[-1, 1]$

(2) Expand  $g(x)$  by Legendre polynomials

$$g(x) \cong \sum_{n=0}^N c_n P_n(x) \quad \text{where} \quad c_n = \left(n + \frac{1}{2}\right) \int_{-1}^1 g(x) P_n(x) dx$$

(3)  $f(x) = g\left(\frac{2}{b-a}\left(x - \frac{a+b}{2}\right)\right) \cong \sum_{n=0}^N c_n P_n\left(\frac{2}{b-a}\left(x - \frac{a+b}{2}\right)\right)$

## Section 3.3 Generalization for Function Set Approximation

Suppose that  $\phi_0(x), \phi_1(x), \phi_2(x), \dots, \phi_N(x)$  are a set of **independent functions** where  $x \in [a, b]$  and the weight function is  $w(x)$ .

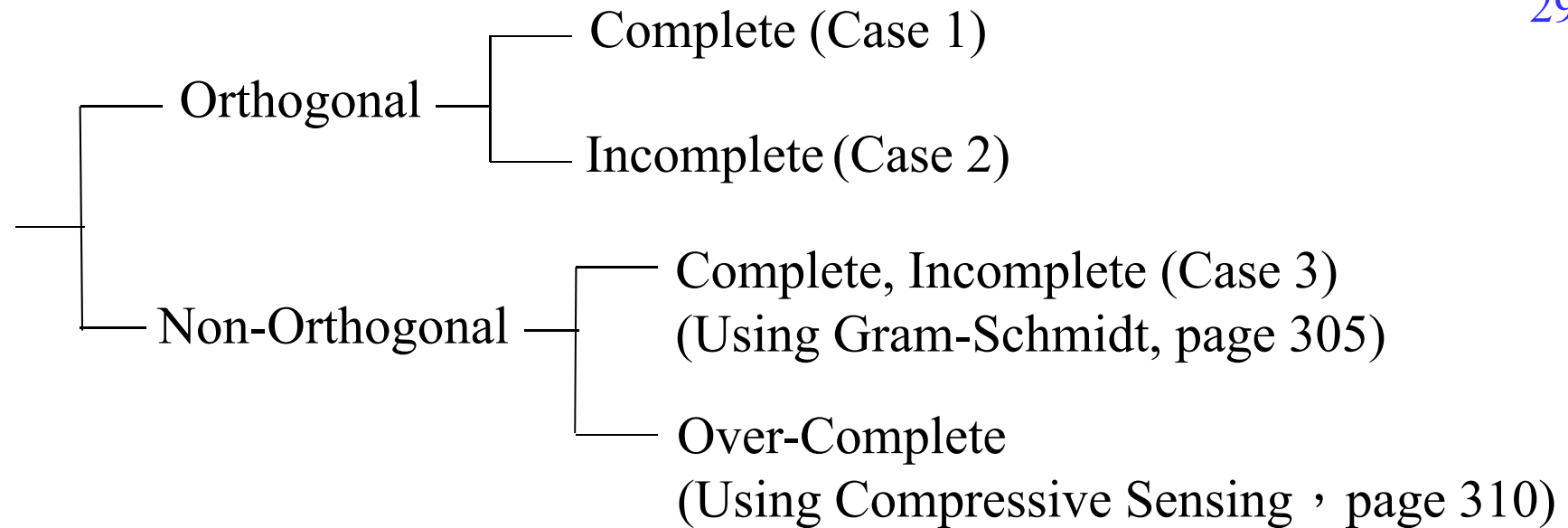
**Problem :**

How do we expand  $f(x)$  by  $\phi_0(x), \phi_1(x), \phi_2(x) \dots \phi_N(x)$

$$f_N(x) = \sum_{n=0}^N c_n \phi_n(x)$$

where  $error = \|f(x) - f_N(x)\| = \sqrt{\int_a^b w(x)(f(x) - f_N(x))^2 dx}$

is minimized.



(Case 1): The function set is **complete and orthogonal / orthonormal**

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

$$c_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x) dx}{\int_a^b w(x) |\phi_n(x)|^2 dx}$$

(orthogonal case)

$$error = 0$$

$$c_n = \int_a^b w(x) f(x) \phi_n^*(x) dx$$

(orthonormal case)

(Case 2): The function set is **incomplete and orthogonal**

$$f(x) \cong f_N(x) = \sum_{n=0}^N d_n \phi_n(x)$$

$$d_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x) dx}{\int_a^b w(x) |\phi_n(x)|^2 dx}$$

$$\begin{aligned} \text{error} &= \|f(x) - f_N(x)\| = \sqrt{\sum_{n=N+1}^{\infty} |d_n|^2 \int_a^b w(x) |\phi_n(x)|^2 dx} \\ &= \sqrt{\int_a^b w(x) f^2(x) dx - \sum_{n=0}^N |d_n|^2 \int_a^b w(x) |\phi_n(x)|^2 dx} \end{aligned}$$

Specially, if the function set is **orthonormal**

$$f(x) \cong f_N(x) = \sum_{n=0}^N d_n \phi_n(x)$$

$$d_n = \int_a^b w(x) f(x) \phi_n^*(x) dx$$

$$\text{error} = \sqrt{\sum_{n=N+1}^{\infty} |d_n|^2} = \sqrt{\int_a^b w(x) f^2(x) dx - \sum_{n=0}^N |d_n|^2}$$

(Proof): The formula of  $d_n$  is directly from page 279.

$$\text{If } \psi_n(x) = \frac{\phi_n(x)}{\|\phi_n(x)\|}$$

then  $\psi_0(x), \psi_1(x), \psi_2(x), \dots, \psi_N(x)$  form an **orthonormal** set and

$$f_N(x) = \sum_{n=0}^N d_n \|\phi_n(x)\| \psi_n(x)$$

Suppose that  $\psi_0(x), \psi_1(x), \psi_2(x), \dots, \psi_N(x)$  is a subset of the complete and orthonormal function set  $\psi_0(x), \psi_1(x), \psi_2(x), \dots, \psi_N(x), \psi_{N+1}(x), \dots$ . Then,  $f(x)$  can be expressed as

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x) \quad \text{where } c_n = \int_a^b w(x) f(x) \psi_n^*(x) dx$$

Therefore,

$$f(x) - f_N(x) = \sum_{n=0}^N (c_n - d_n \|\phi_n(x)\|) \psi_n(x) + \sum_{n=N+1}^{\infty} c_n \psi_n(x)$$

$$f(x) - f_N(x) = \sum_{n=0}^N (c_n - d_n \|\phi_n(x)\|) \psi_n(x) + \sum_{n=N+1}^{\infty} c_n \psi_n(x)$$

Then, from Parseval's theorem on page 302,

$$\|f(x) - f_N(x)\|^2 = \sum_{n=0}^N |c_n - d_n \|\phi_n(x)\||^2 + \sum_{n=N+1}^{\infty} |c_n|^2$$

To minimize  $\|f(x) - f_N(x)\|^2$ , we should set

$$d_n = \frac{c_n}{\|\phi_n(x)\|} = \frac{\int_a^b w(x) f(x) \phi_n^*(x) dx}{\int_a^b w(x) |\phi_n(x)|^2 dx} \quad \text{for } n = 0, 1, \dots, N$$

In this case,

$$\begin{aligned} \text{error} = \|f(x) - f_N(x)\| &= \sqrt{\sum_{n=N+1}^{\infty} |c_n|^2} = \sqrt{\sum_{n=0}^{\infty} |c_n|^2 - \sum_{n=0}^N |c_n|^2} \\ &= \sqrt{\sum_{n=0}^{\infty} |c_n|^2 - \sum_{n=0}^N |d_n|^2 \int_a^b w(x) |\phi_n(x)|^2 dx} \\ &= \sqrt{\int_a^b w(x) f^2(x) dx - \sum_{n=0}^N |d_n|^2 \int_a^b w(x) |\phi_n(x)|^2 dx} \end{aligned}$$

Parseval's  
theorem on  
page 302

### 3.3.1 Parseval's Theorem

#### [Theorem 3.3.1] Parseval's Theorem (Energy Preservation Theorem)

Suppose that  $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots$  are a **complete** and **orthonormal** function set for  $x \in [a, b]$  and

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \quad \text{where} \quad c_n = (f(x), \phi_n(x)) \\ = \int_a^b w(x) f(x) \phi_n^*(x) dx$$

Then

$$\|f(x)\|^2 = (f(x), f(x)) = \sum_{n=0}^{\infty} |c_n|^2 \quad \int_a^b w(x) |f(x)|^2 dx = \sum_{n=0}^{\infty} |c_n|^2$$

(Proof):

$$\begin{aligned}
 \int_a^b w(x) |f(x)|^2 dx &= \int_a^b w(x) f(x) f^*(x) dx \\
 &= \int_a^b w(x) \sum_{n=0}^{\infty} c_n \phi_n(x) \sum_{m=0}^{\infty} c_m^* \phi_m^*(x) dx \\
 &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n c_m^* \int_a^b w(x) \phi_n(x) \phi_m^*(x) dx
 \end{aligned}$$

Since  $\int_a^b w(x) \phi_n(x) \phi_m^*(x) dx = 0$  if  $n \neq m$

$$\begin{aligned}
 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n c_m^* \int_a^b w(x) \phi_n(x) \phi_m^*(x) dx &= \sum_{n=0}^{\infty} c_n c_n^* \int_a^b w(x) \phi_n(x) \phi_n^*(x) dx \\
 &= \sum_{n=0}^{\infty} |c_n|^2 \int_a^b w(x) |\phi_n(x)|^2 dx
 \end{aligned}$$

$$\int_a^b w(x) |f(x)|^2 dx = \sum_{n=0}^{\infty} |c_n|^2 \int_a^b w(x) \phi_n(x) \phi_n^*(x) dx$$

The result on the previous page leads to Parseval's theorem:

$$\int_a^b w(x) |f(x)|^2 dx = \sum_{n=0}^{\infty} |c_n|^2 \int_a^b w(x) |\phi_n(x)|^2 dx$$

Specially, if the function set  $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$  is **orthonormal**

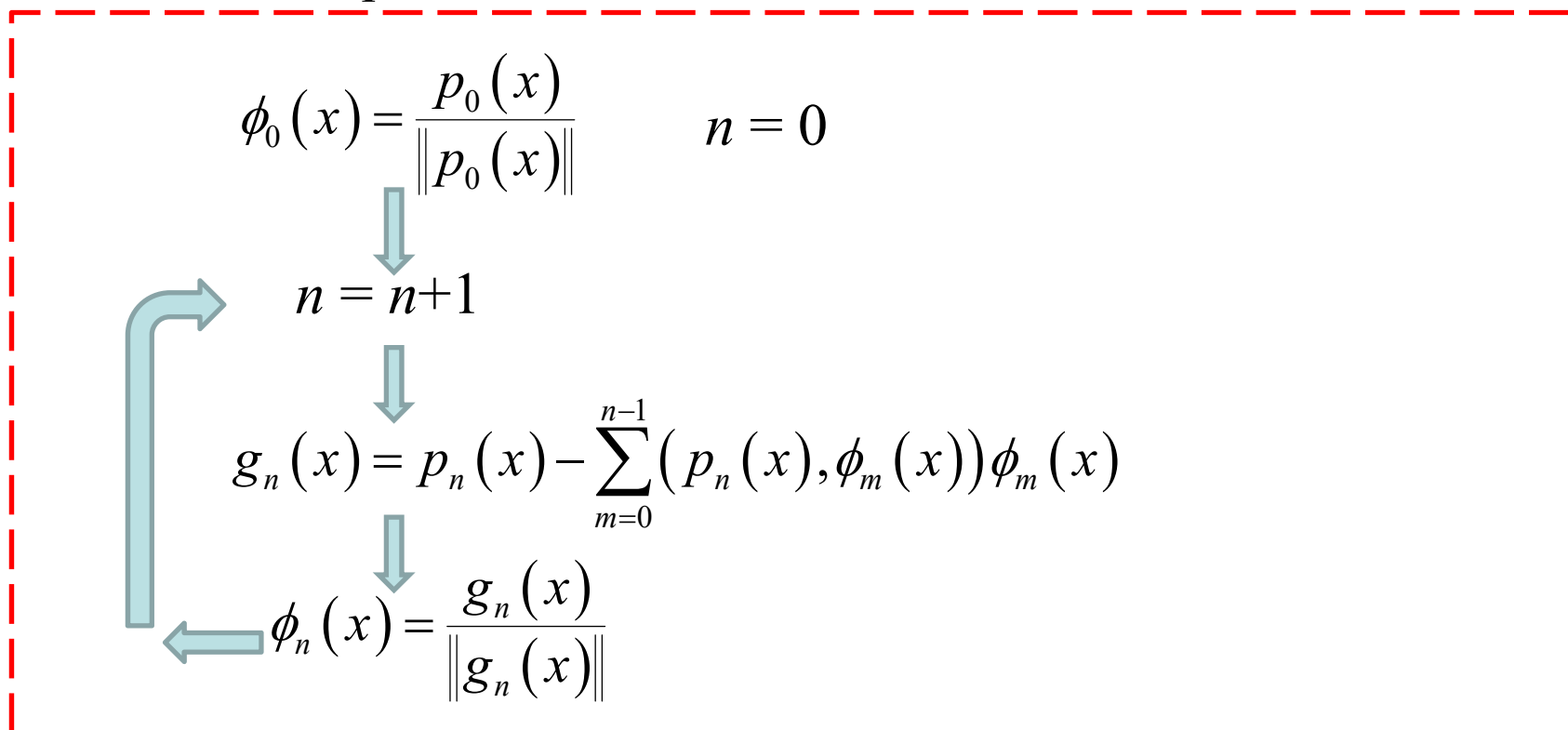
$$\int_a^b w(x) |f(x)|^2 dx = \sum_{n=0}^{\infty} |c_n|^2$$

### 3.3.2 Non-Orthogonal Function Set Expansion

(Case 3)

How do we generate an orthonormal function set  $\{\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \dots\}$  if the input  $\{p_0(x), p_1(x), p_2(x), p_3(x), \dots\}$  is independent but **not an orthogonal function set**?

Gram-Schmidt process



## Proof for orthogonality of the function set generated from the Gram-Schmidt process

Suppose that we have known that  $\phi_0(x), \phi_1(x), \phi_2(x), \dots, \phi_{n-1}(x)$  are orthonormal.

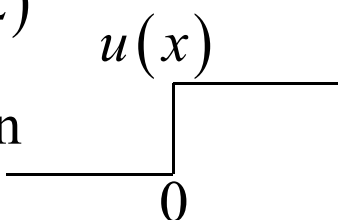
If  $k < n$ ,

$$\begin{aligned}
 (\phi_n(x), \phi_k(x)) &= \left( \frac{g_n(x)}{\|g_n(x)\|}, \phi_k(x) \right) \\
 &= \frac{1}{\|g_n(x)\|} \left( p_n(x) - \sum_{m=0}^{n-1} (p_n(x), \phi_m(x)) \phi_m(x), \phi_k(x) \right) \\
 &= \frac{1}{\|g_n(x)\|} \left[ (p_n(x), \phi_k(x)) - \sum_{m=0}^{n-1} (p_n(x), \phi_m(x)) (\phi_m(x), \phi_k(x)) \right] \\
 &= \frac{1}{\|g_n(x)\|} \left[ (p_n(x), \phi_k(x)) - (p_n(x), \phi_k(x)) \right] = 0
 \end{aligned}$$

**[Example 4]**

Derive  $\phi_0(x)$ ,  $\phi_1(x)$ ,  $\phi_2(x)$ , ..... such that

$$\phi_n(x) = \sum_{k=0}^n \tau_{n,k} p_k(x), \quad p_k(x) = u(x-k) - u(x-k-2)$$

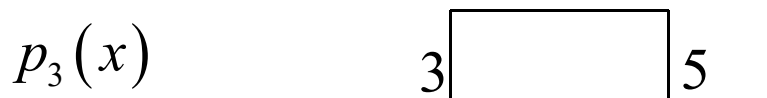
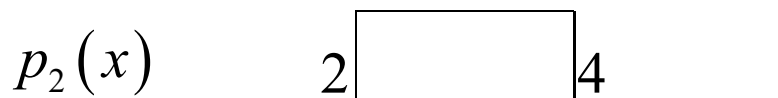
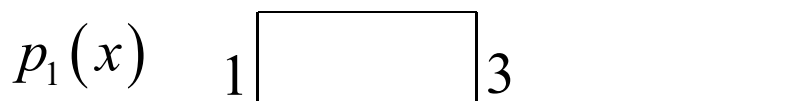
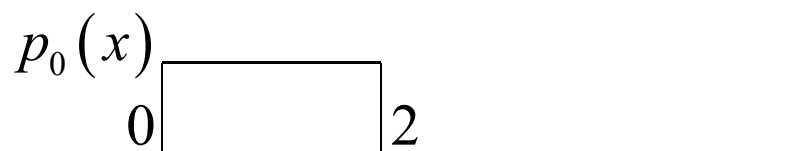
$u(x)$ : unit step function 

and

$$\int_0^{\infty} \phi_n(x) \phi_m(x) dx = 0$$

when  $m \neq n$

$$\int_0^{\infty} \phi_n(x) \phi_n(x) dx = 1$$



→ orthonormalization

Answer:

$$\text{Set } p_k(x) = u(x-k) - u(x-k-2)$$

$$\phi_0(x) = \frac{p_0(x)}{\|p_0(x)\|} = \frac{u(x) - u(x-2)}{\sqrt{2}}$$

$$\begin{aligned} g_1(x) &= p_1(x) - (p_1(x), \phi_0(x))\phi_0(x) \\ &= u(x-1) - u(x-3) - \frac{1}{\sqrt{2}} \left( \frac{u(x) - u(x-2)}{\sqrt{2}} \right) \end{aligned}$$

$$\phi_1(x) = \frac{g_1(x)}{\|g_1(x)\|} = \frac{u(x-1) - u(x-3) - \frac{1}{2}(u(x) - u(x-2))}{\sqrt{6}/2}$$

$$\begin{aligned}
g_2(x) &= p_2(x) - (p_2(x), \phi_0(x))\phi_0(x) - (p_2(x), \phi_1(x))\phi_1(x) \\
&= u(x-2) - u(x-4) - \frac{2}{\sqrt{6}} \frac{u(x-1) - u(x-3) - \frac{1}{2}(u(x) - u(x-2))}{\sqrt{6}/2} \\
&= u(x-2) - u(x-4) - \frac{2}{3}u(x-1) + \frac{2}{3}u(x-3) + \frac{1}{3}(u(x) - u(x-2)) \\
\phi_2(x) &= \frac{g_2(x)}{\|g_2(x)\|} \\
&= \frac{u(x-2) - u(x-4) - \frac{2}{3}(u(x-1) - u(x-3)) + \frac{1}{3}(u(x) - u(x-2))}{2/\sqrt{3}}
\end{aligned}$$

.....

.....

## 附錄七 Compressive Sensing

The problems that compressive sensing deals with:

Suppose that  $b_0(t), b_1(t), b_2(t), b_3(t) \dots$  form an **over-complete** and **non-orthogonal** spanning set.

(Problem 1) We want to minimize  $\|c\|_0$  ( $\|\cdot\|_0$  是 zero-order norm,  $\|c\|_0$  意指  $c_m$  的值不為 0 的個數) such that

$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize  $\|c\|_0$  such that

$$\int \left( x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When  $\|c\|_0$  is limited to  $M$ , we want to minimize

$$\int \left( x(t) - \sum_m c_m b_m(t) \right)^2 dt$$

Examples for the spanning set used for compressive sensing  
(Since they are not linearly independent, they are not basis).

### 3-atom form

$$b_m(t) = \exp(j2\pi f_m t) \exp\left(-\frac{\pi(t-t_m)^2}{\sigma_m^2}\right)$$

### 4-atom form

$$b_m(t) = \exp(j2\pi(f_m t + \eta_m t^2)) \exp\left(-\frac{\pi(t-t_m)^2}{\sigma_m^2}\right)$$

Problems: over-complete, non-orthogonal, too many functions in the sets

## Section 3.4 Other Orthogonal Polynomials

(只教不考)

In addition to Legendre Polynomials, there are many other orthogonal polynomials. However, their weight functions and intervals are different.

- [1] R. Beals, Special Functions and Orthogonal Polynomials, Cambridge Studies in Advanced Mathematics, vol. 153, Cambridge University Press, 2016.
- [2] M. R. Spiegel, Mathematical Handbook, Schaum, 1990.

# Summary of Weights and Supports of Orthogonal Polynomials

Name	Weight Function $w(x)$	Support
Legendre	1	$x \in [-1, 1]$
Jacobi	$(1+x)^n(1-x)^m$	$x \in [-1, 1]$
Associated Legendre (Ultraspherical, Gegenbauer)	$(1-x^2)^m$	$x \in [-1, 1]$
Chebyshev (1 <sup>st</sup> Kind)	$(1-x^2)^{-1/2}$	$x \in [-1, 1]$
Chebyshev (2 <sup>st</sup> Kind)	$(1-x^2)^{1/2}$	$x \in [-1, 1]$
Laguerre	$e^{-x}$	$x \in [0, \infty)$
Associated Laguerre	$x^m e^{-x}$	$x \in [0, \infty)$
Hermite	$e^{-x^2}$	$x \in (-\infty, \infty)$

- Associated Legendre Functions

$$P_{n,m}(x) = \frac{d^m}{dx^m} P_n(x)$$

$P_n(x)$ : the Legendre polynomial of order  $n$ , in fact,  $P_n(x) = P_{n,0}(x)$

They are orthogonal for  $x \in [-1, 1]$ ,  $w(x) = (1-x^2)^m$

$$\int_{-1}^1 (1-x^2)^m P_{n,m}(x) P_{k,m}(x) dx = 0 \quad \text{if } n \neq k$$

$$\int_{-1}^1 (1-x^2)^m (P_{n,m}(x))^2 dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$$

Ex: When  $m = 1$ ,

$$P_{1,1}(x) = x$$

$$P_{2,1}(x) = 3x$$

$$P_{3,1}(x) = \frac{5}{2}x^2 - \frac{3}{2}x$$

$$P_{4,1}(x) = \frac{7}{2}x^3 - \frac{7}{2}x$$

- Chebychev polynomials 電子學和 filter design 常用

$$P_n(\cos \theta) = \cos n\theta$$

$$P_n(x) = \sum_{k=0}^{n/2} \binom{n}{2k} x^{n-2k} (1-x^2)^k$$

They are orthogonal for  $x \in [-1, 1]$   $w(x) = \frac{1}{\sqrt{1-x^2}}$

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} P_n(x) P_k(x) dx = 0 \quad \text{if } n \neq k$$

$$\int_{-1}^1 \frac{P_n^2(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi & \text{if } n = 0, \\ \pi/2 & \text{otherwise} \end{cases}$$

They are the solutions of  $(1-x^2)P_n''(x) - xP_n'(x) + n^2P_n(x) = 0$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = 2x^2 - 1$$

$$P_3(x) = 4x^3 - 3x$$

- Laguerre polynomials

$$P_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x})$$

They are orthogonal for  $x \in [0, \infty)$   $w(x) = e^{-x}$

$$\int_0^\infty e^{-x} P_n(x) P_k(x) dx = 0 \quad \int_0^\infty e^{-x} (P_n(x))^2 dx = (n!)^2$$

if  $n \neq k$

They are the solutions of  $xP_n''(x) + (1-x)P_n'(x) + nP_n(x) = 0$

$$P_0(x) = 1$$

$$P_1(x) = -x + 1$$

$$P_2(x) = x^2 - 4x + 2$$

$$P_3(x) = -x^3 + 9x^2 - 18x + 6$$

- Associated Laguerre polynomials

$$P_{n,m}(x) = \frac{d^m}{dx^m} (P_n(x))$$

where  $P_n(x)$  is the  $n^{\text{th}}$  order Laguerre polynomial

They are orthogonal for  $x \in [0, \infty)$   $w(x) = x^m e^{-x}$

$$\int_0^\infty x^m e^{-x} P_{n,m}(x) P_{k,m}(x) dx = 0 \quad \int_0^\infty x^m e^{-x} (P_{n,m}(x))^2 dx = \frac{(n!)^2}{(n-m)!}$$

if  $n \neq k$

They are the solutions of  $xP_n''(x) + (m+1-x)P_n'(x) + (n-m)P_n(x) = 0$

$$P_{1,1}(x) = -1$$

$$P_{2,1}(x) = 2x - 4$$

$$P_{3,1}(x) = -3x^2 + 18x - 18$$

$$P_{4,1}(x) = 4x^3 - 48x^2 + 144x - 96$$

- Hermite polynomials 電磁波、光學、頻譜分析常用

$$P_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$$

They are orthogonal for  $x \in (-\infty, \infty)$   $w(x) = e^{-x^2}$

$$\int_{-\infty}^{\infty} e^{-x^2} P_n(x) P_k(x) dx = 0 \quad \int_{-\infty}^{\infty} e^{-x^2} (P_n(x))^2 dx = 2^n n! \sqrt{\pi}$$

if  $n \neq k$

They are the solutions of  $P_n''(x) - xP_n'(x) + nP_n(x) = 0$

$$P_0(x) = 1$$

$$P_1(x) = 2x$$

$$P_2(x) = 4x^2 - 2$$

$$P_3(x) = 8x^3 - 12x$$

$$P_4(x) = 16x^4 - 48x^2 + 12$$

$$P_5(x) = 32x^5 - 160x^3 + 120x$$