

4. Fourier Analysis

Section 4.1 Definition of the Fourier Transform

Section 4.2 Dirac Delta Function

Section 4.3 Properties

Section 4.4 Uncertainty Principle

Section 4.5 Convolution and Correlation

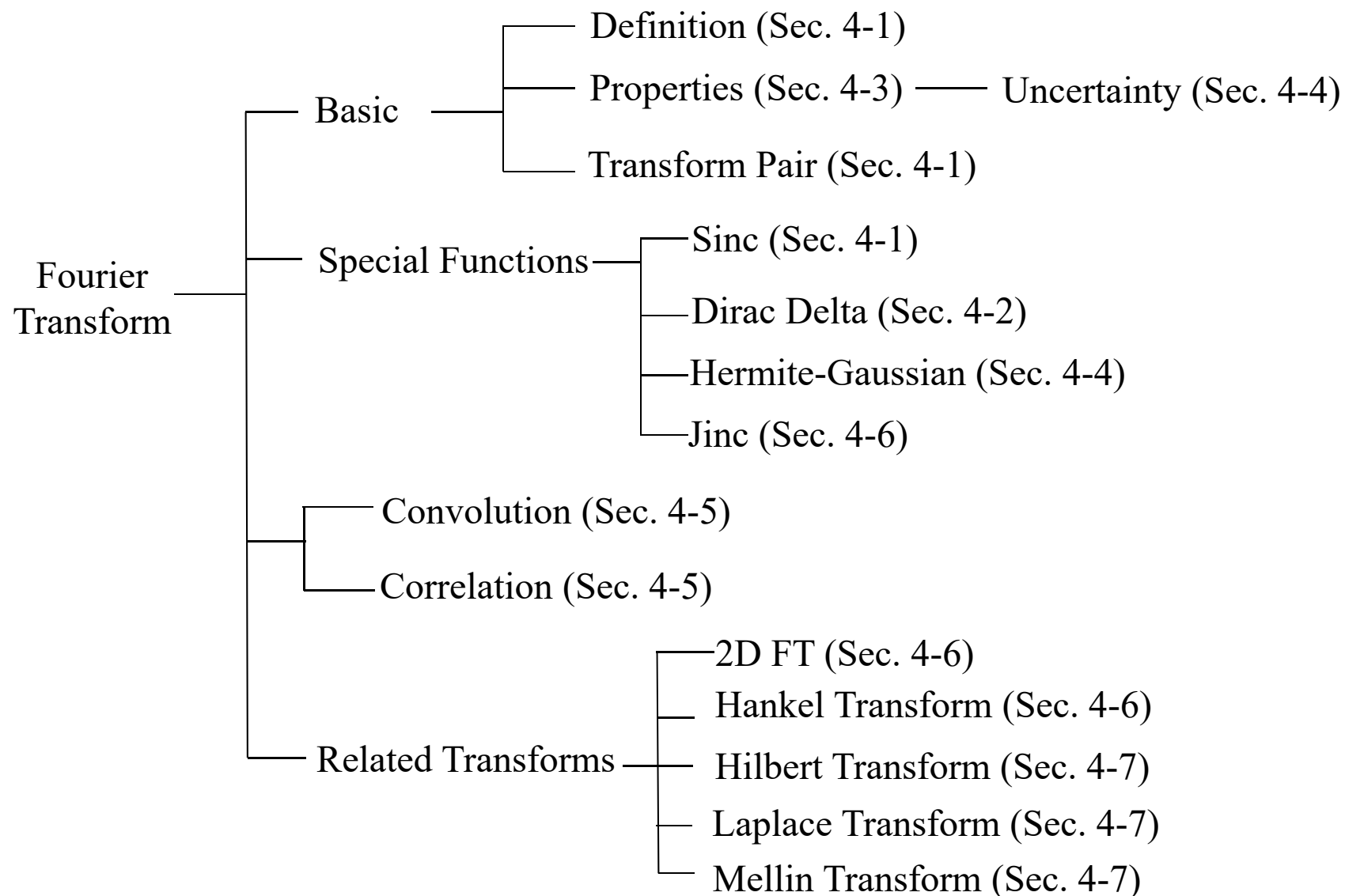
Section 4.6 2D Fourier Transforms

Section 4.7 The Operations Related to Fourier Transforms (只教不考)

[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

[2] D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017.

[3] D. G. Zill, W. S. Wright, and J. J. Ding, Engineering Mathematics, Metric Edition, Cengage Learning, Taipei, Taiwan, 2019, Chapter 15.

Fourier Transform

4.1 Definition of the Fourier Transform

Fourier transform

$$\mathfrak{F}[g(x)] = \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx = G(f)$$

\mathfrak{F} 代表 Fourier transform

inverse Fourier transform

$$\mathfrak{F}^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f) e^{j2\pi fx} df = g(x)$$

[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

[2] D. G. Zill, W. S. Wright, and J. J. Ding, Engineering Mathematics, Metric Edition, Cengage Learning, Taipei, Taiwan, 2019, Sections 15-2.

Review: Fourier Series of the Complex Form

$$\phi_n(x) = \exp\left(j \frac{2\pi}{T} nx\right) \quad n = \dots, -1, 0, 1, 2, 3, \dots$$

form a complete and **orthogonal** set within $x \in \left[-\frac{T}{2}, \frac{T}{2}\right]$

$$\int_{-T/2}^{T/2} \phi_n(x) \phi_m^*(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ T & \text{if } m = n \end{cases}$$

Review: Fourier Series of the Complex Form

(Compared to pages 277, 280)

If $g(x) = g(x+T)$, then

$$g(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j \frac{2\pi}{T} nx\right)$$

where

$$c_n = \frac{\int_{-T/2}^{T/2} g(x) \operatorname{conj}\left(\exp\left(j \frac{2\pi}{T} nx\right)\right) dx}{\int_{-T/2}^{T/2} \exp\left(j \frac{2\pi}{T} nx\right) \operatorname{conj}\left(\exp\left(j \frac{2\pi}{T} nx\right)\right) dx}$$

$$c_n = \frac{\int_{-T/2}^{T/2} g(x) \exp\left(-j \frac{2\pi}{T} nx\right) dx}{T}$$

4.1.1 Derivation and Physical Meaning

Fourier transform can be viewed as the Fourier series where

$$T \rightarrow \infty$$

Note that, if we set $g_n = c_n T$

$$g_n = \int_{-T/2}^{T/2} g(x) \exp\left(-j \frac{2\pi}{T} nx\right) dx \quad g(x) = \sum_{n=-\infty}^{\infty} \frac{g_n}{T} \exp\left(j \frac{2\pi}{T} nx\right)$$

Then we set $\Delta_f = 1/T$

$$g_n = \int_{-T/2}^{T/2} g(x) \exp(-j2\pi n \Delta_f x) dx \quad g(x) = \sum_{n=-\infty}^{\infty} g_n \Delta_f \exp(j2\pi n \Delta_f x)$$

$$G(f) = \int_{-T/2}^{T/2} g(x) \exp(-j2\pi fx) dx \quad g(x) = \sum_{n=-\infty}^{\infty} G(f) \exp(j2\pi fx) \Delta_f$$

where $f = n\Delta_f$, $G(f) = g_n$

If $T \rightarrow \infty$, $\Delta_f \rightarrow 0$

$$G(f) = \int_{-\infty}^{\infty} g(x) \exp(-j2\pi fx) dx \quad g(x) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi fx) df$$

Physical Meaning of the Fourier Transform:

expanding a signal as a combination of $\exp(j2\pi fx)$

$\exp(j2\pi fx)$ period: $1/f$, frequency: f

$G(f)$: the expansion coefficient for $\exp(j2\pi fx)$

$$g(x) = \int_{-\infty}^{\infty} G(f) e^{j2\pi fx} df$$

When will the Fourier transform exist?

Sufficient Conditions:

$$(1) \int_{-\infty}^{\infty} |g(x)| dx < \infty$$

(2) $g(x)$ is of bounded variations (It means that $g(x)$ can be represented by a curve of finite length in any finite interval of x).

4.1.2 Transform Pair

[Example 1] Find the Fourier transform of

$$g(x) = \exp(-3|x|)$$

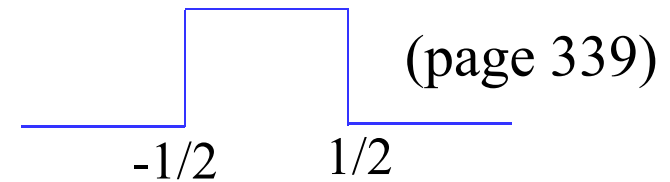
(Solution):

$$\begin{aligned}\mathcal{F}\{g(x)\} &= \int_{-\infty}^{\infty} e^{-3|x|} e^{-j2\pi fx} dx = \int_{-\infty}^0 e^{3x} e^{-j2\pi fx} dx + \int_0^{\infty} e^{-3x} e^{-j2\pi fx} dx \\ &= \left. \frac{e^{3x} e^{-j2\pi fx}}{3 - j2\pi f} \right|_{-\infty}^0 + \left. \frac{e^{-3x} e^{-j2\pi fx}}{-3 - j2\pi f} \right|_0^{\infty} = \frac{1}{3 - j2\pi f} - \frac{1}{-3 - j2\pi f} \\ &= \frac{6}{9 + (2\pi f)^2}\end{aligned}$$

[Example 2]

Find the Fourier transform of the **rectangular function** $\Pi(x)$ where

$$\Pi(x) = \begin{cases} 1 & \text{for } -1/2 < x < 1/2, \\ 0 & \text{otherwise.} \end{cases}$$



(Solution):

$$\mathcal{F}\{g(x)\} = \int_{-1/2}^{1/2} e^{-j2\pi fx} dx = \frac{e^{-j2\pi fx}}{-j2\pi f} \Big|_{-1/2}^{1/2} = \frac{\sin(\pi f)}{\pi f}$$

$$\mathcal{F}\{g(x)\} = \text{sinc } f \quad (\text{page 338})$$

[Example 3]

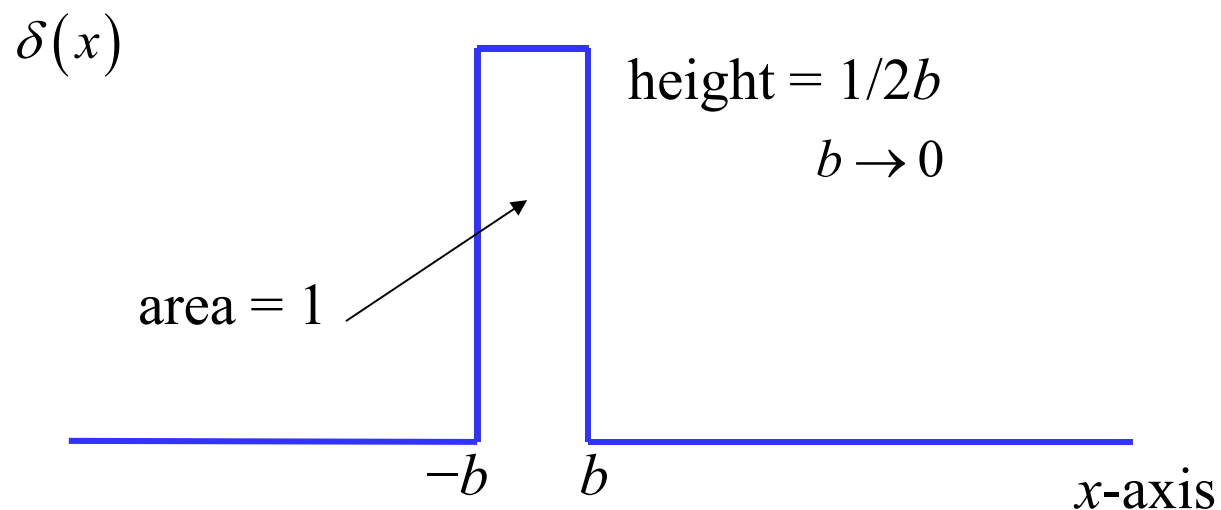
Find the Fourier transform of the **Dirac delta function** $\delta(x)$

(Solution): From the sifting property of $\delta(x)$:

$$\int_{-\infty}^{\infty} \delta(x - x_0) y(x) dx = y(x_0) \quad (\text{see page 344})$$

we have

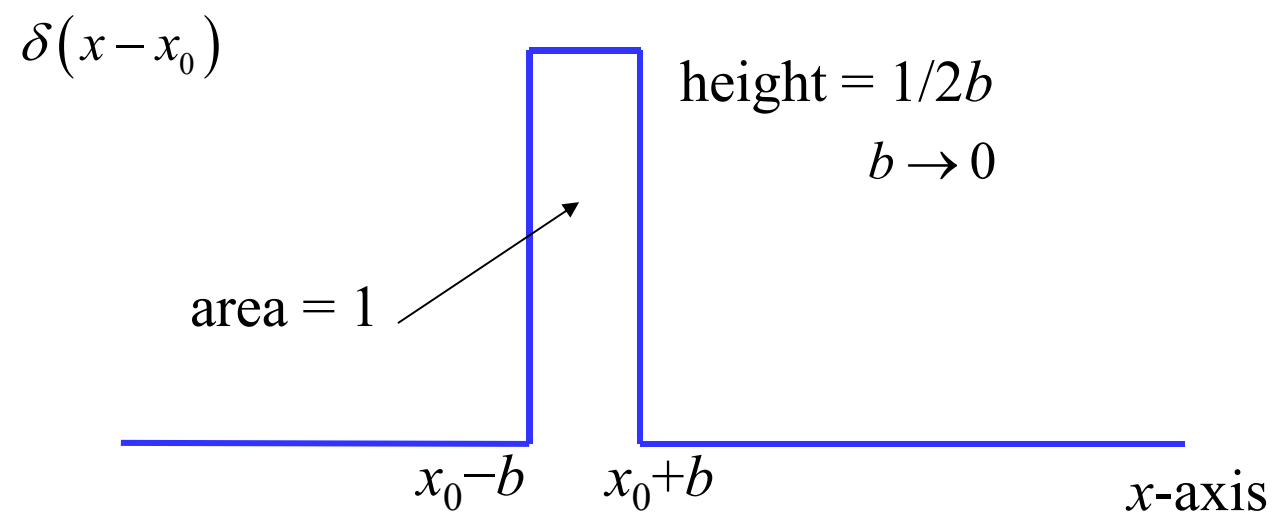
$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi fx} dx = e^{-j2\pi f \cdot 0} = 1$$



Note:

More generally,

$$\mathcal{F}\{\delta(x-x_0)\} = \int_{-\infty}^{\infty} \delta(x-x_0) e^{-j2\pi fx} dx = e^{-j2\pi x_0 f}$$



Linearity Property of the Fourier Transform

If

$$\mathfrak{F}[g_1(x)] = G_1(f) \qquad \mathfrak{F}[g_2(x)] = G_2(f)$$

then

$$\mathfrak{F}[\alpha g_1(x) + \beta g_2(x)] = \alpha G_1(f) + \beta G_2(f)$$

Duality Property of the Fourier Transform

If $\mathfrak{F}[g(x)] = G(f)$

then $\mathfrak{F}[G(x)] = g(-f)$

(Proof): Since

$$\mathfrak{F}^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f) e^{j2\pi fx} df = g(x)$$

$$\int_{-\infty}^{\infty} G(x) e^{j2\pi xf} dx = g(f)$$

$$\int_{-\infty}^{\infty} G(x) e^{-j2\pi fx} dx = g(-f) \implies \mathfrak{F}[G(x)] = g(-f)$$

Q: How do we compute the Fourier transforms of $\text{sinc}(x)$ and 1?

[Example 4] Find the Fourier transform of $\text{sinc}(x)$ where

(Solution): Since

$$\mathfrak{F}[\Pi(x)] = \text{sinc}(f)$$

from the duality property, we have

$$\mathfrak{F}[\text{sinc}(x)] = \Pi(-f) = \Pi(f)$$

[Example 5] Find the Fourier transform of $\exp(j2\pi kx)$

(Solution): Since

$$\mathcal{F}\{\delta(x-k)\} = e^{-j2\pi kf}, \quad \mathcal{F}\{\delta(x+k)\} = e^{j2\pi kf}$$

from the duality property, we have

$$\mathcal{F}\{e^{j2\pi kx}\} = \delta(-f+k) = \delta(f-k)$$

(Here we apply the fact that $\delta(x) = \delta(-x)$).

Specially,

$$\mathfrak{F}\{1\} = \delta(f)$$

(Note): Although 1 does not satisfy the sufficient condition on page 326, its Fourier transform exists.

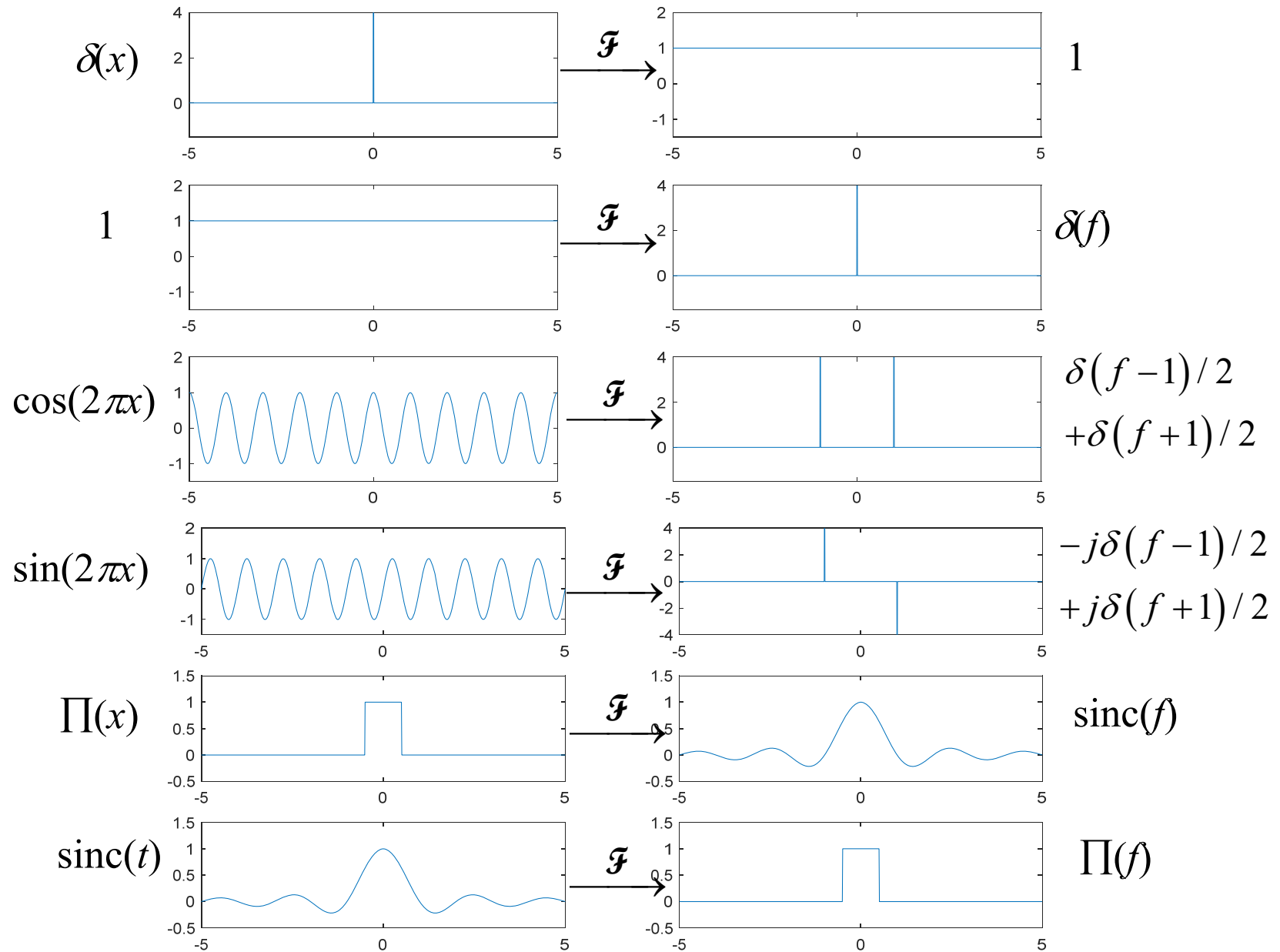
[Example 6] Find the Fourier transform of $\cos(2\pi kx)$

(Solution):

Some basic Fourier transform pairs

$g(x)$	$G(f) = \mathcal{F}\{g(x)\}$
(1) $\delta(x)$	1
(2) 1	$\delta(f)$
(3) $\delta(x-k)$	$\exp(-j2\pi kf)$
(4) $\exp(j2\pi kx)$	$\delta(f-k)$
(5) $\cos(2\pi kx)$	$\frac{1}{2}\delta(f-k) + \frac{1}{2}\delta(f+k)$
(6) $\sin(2\pi kx)$	$\frac{-j}{2}\delta(f-k) + \frac{j}{2}\delta(f+k)$
(7) $\Pi(x)$	$\text{sinc}(f)$
(8) $\text{sinc}(x)$	$\Pi(f)$
(9) $\exp(-kx)U(x)$ ($k > 0$)	$\frac{1}{k + j2\pi f}$
(10) $\exp(-k x)$ ($k > 0$)	$\frac{2k}{k^2 + 4\pi^2 f^2}$

Some basic Fourier transform pairs



附錄八 Summary of Popular Special Functions

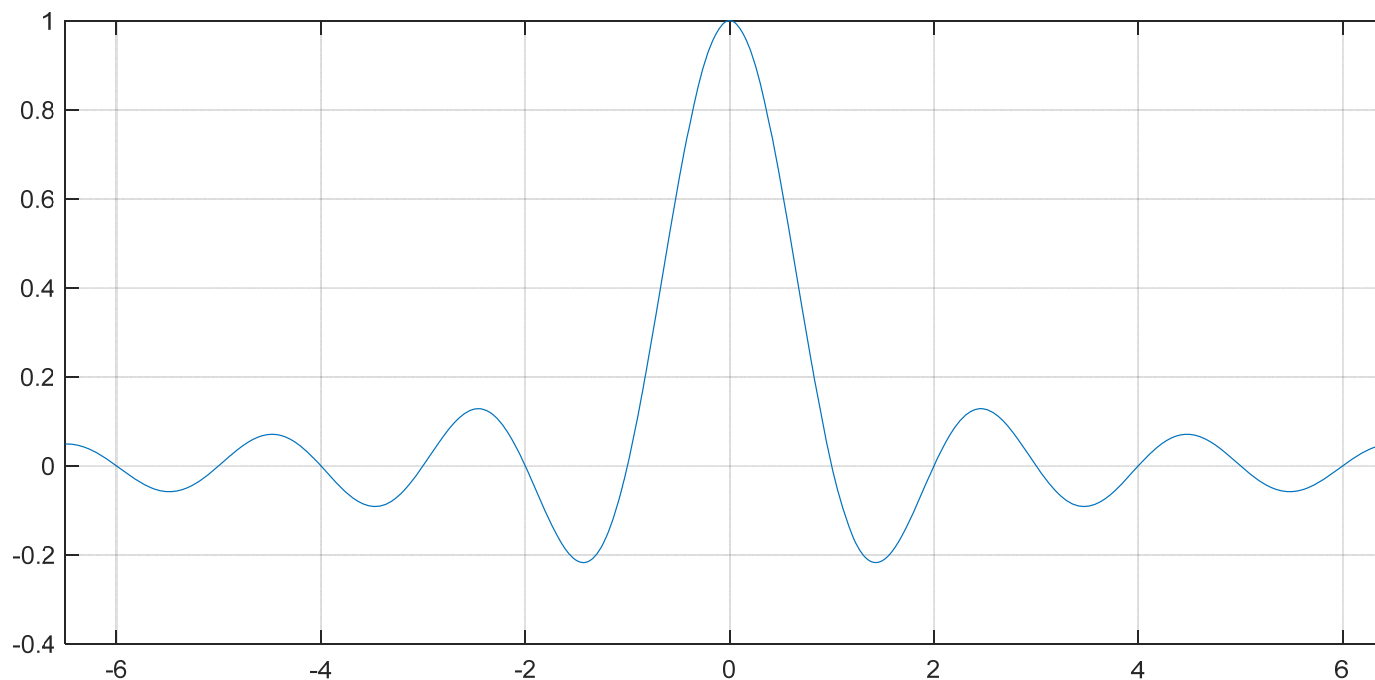
(1) Sinc Function $\text{sinc } x = \frac{\sin(\pi x)}{\pi x}$

$\text{sinc } 0 = 1$, (L'hospital's rule)

$\text{sinc } n = 0$ if n is a nonzero integer,

$\text{sinc } x = \text{sinc}(-x)$

Applications: sampling theorem; ideal filters

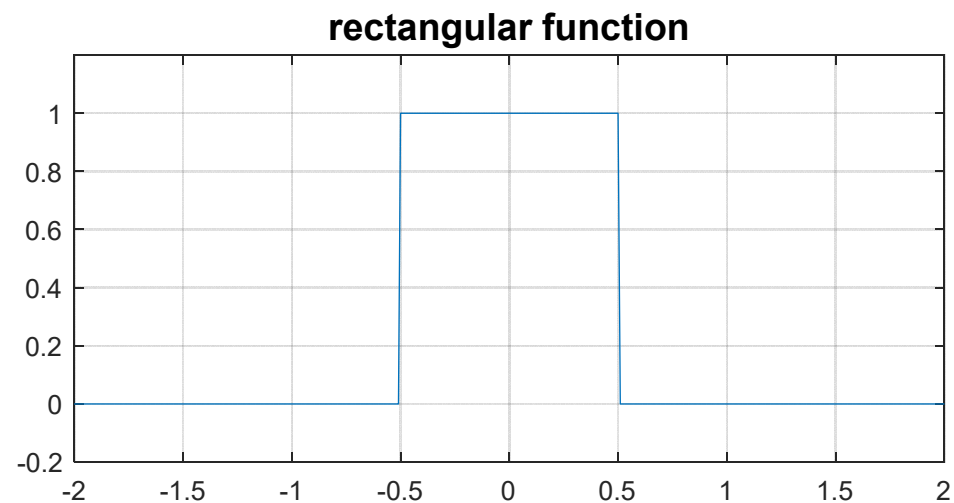


(2) Rectangular Function

$$\Pi(x) = \begin{cases} 1 & \text{for } |x| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

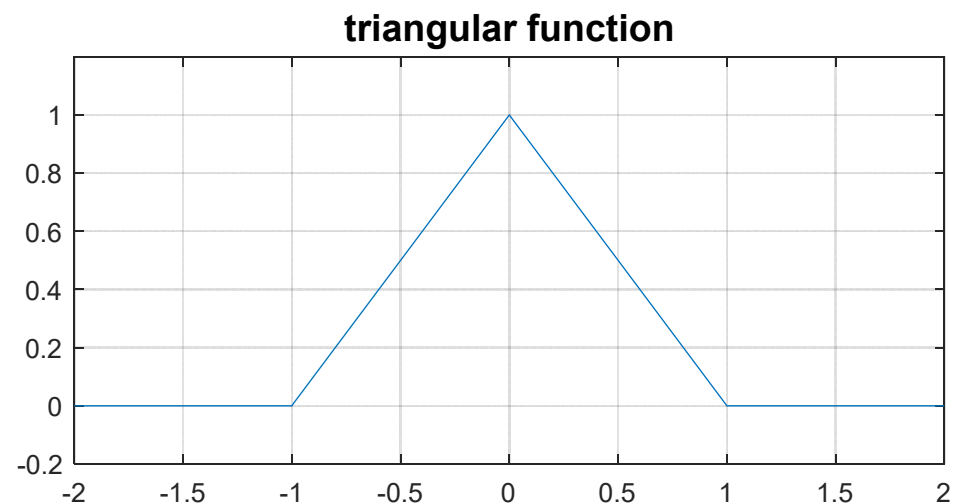
In general,

$$\Pi\left(\frac{x-a}{b}\right) = \begin{cases} 1 & \text{if } a - \frac{b}{2} < x < a + \frac{b}{2} \\ 0 & \text{otherwise} \end{cases}$$



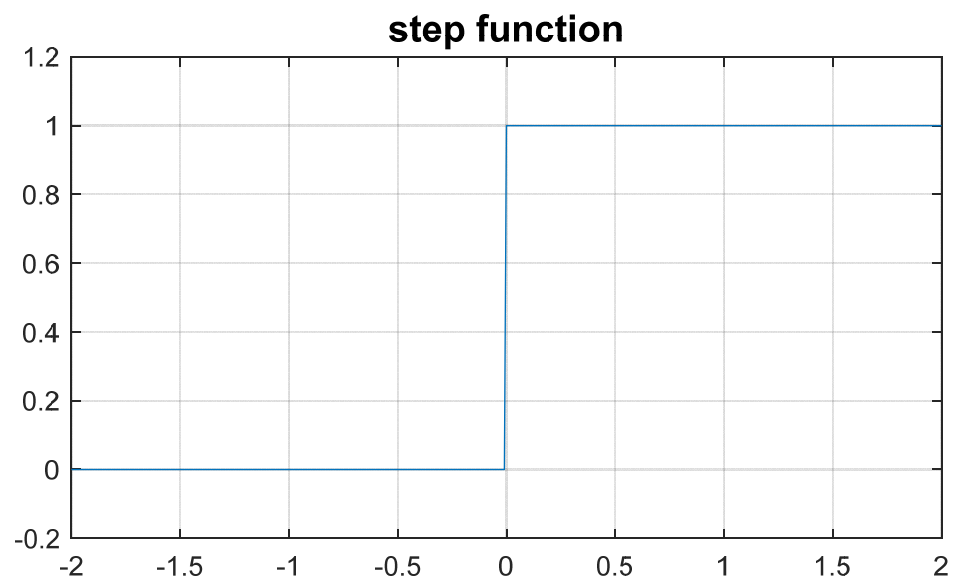
(3) Triangular Function

$$\text{tri}(x) = \begin{cases} 1 - |x| & \text{for } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$



(4) Step Function

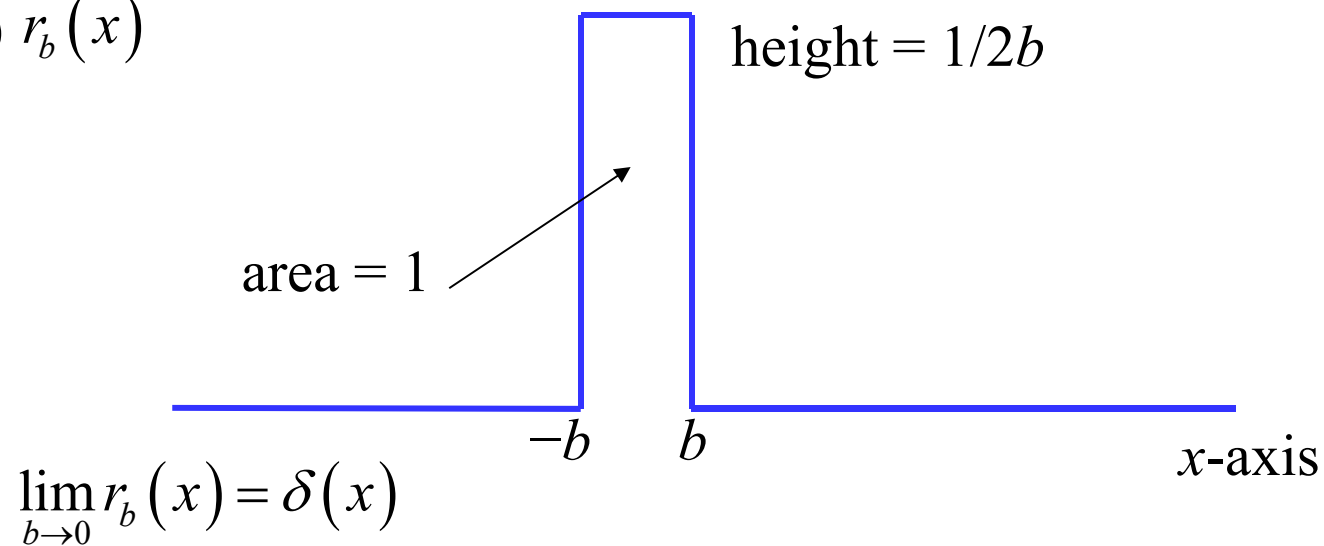
$$U(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

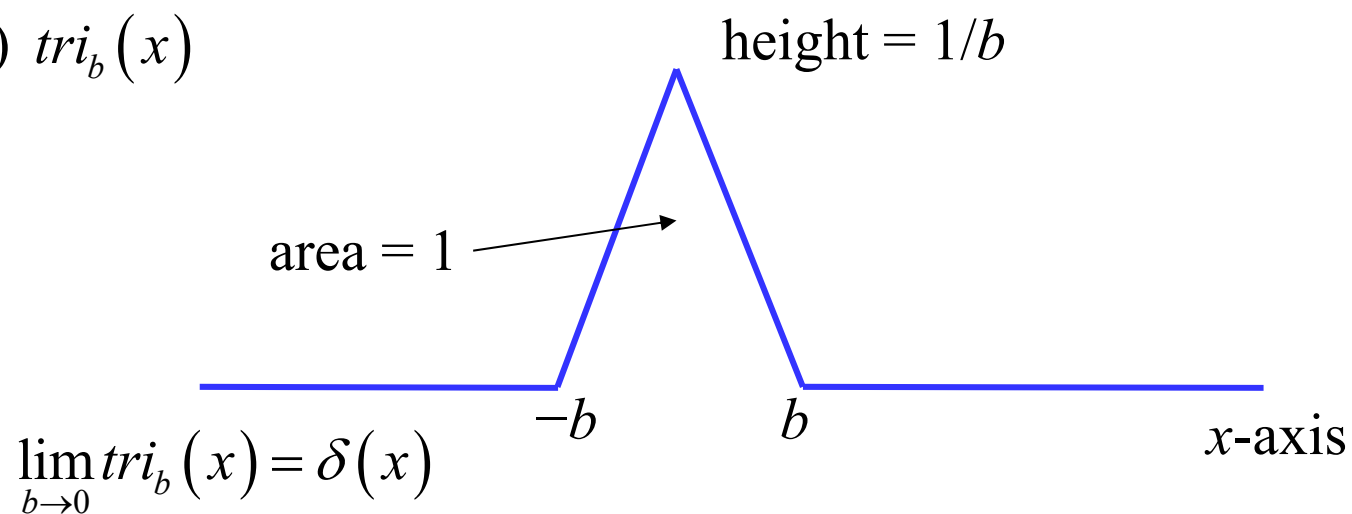
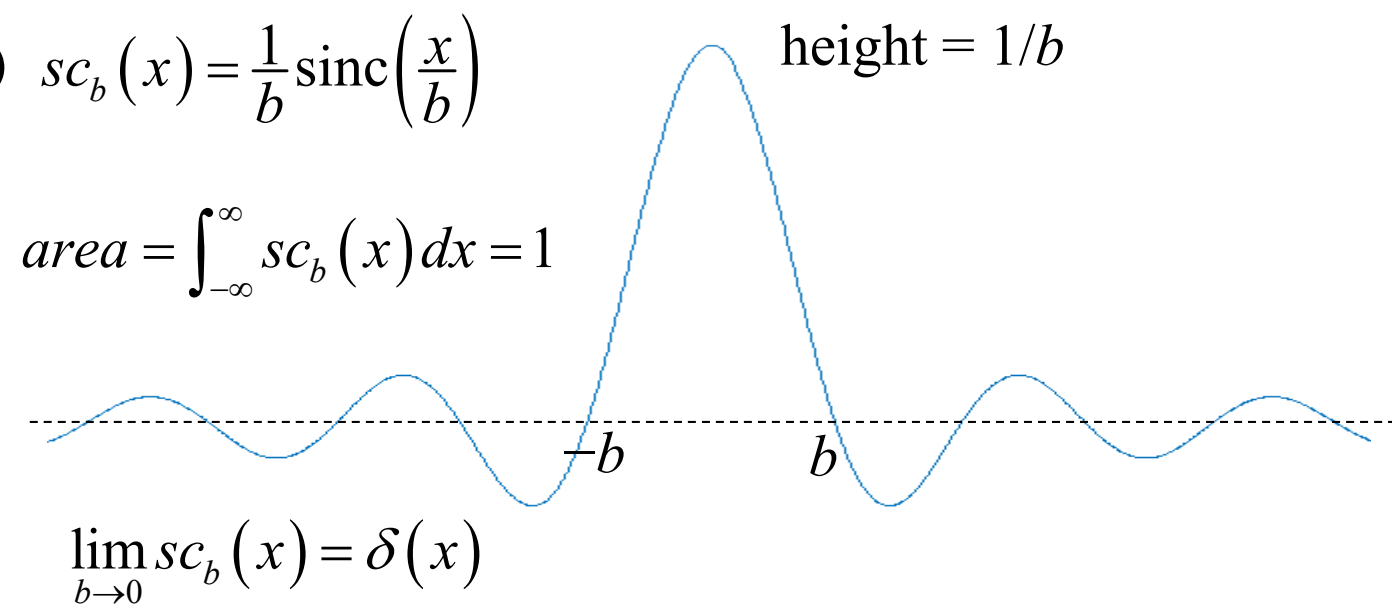


4.2 Dirac Delta Functions

The Dirac delta function does not have a fixed definition. It is in fact the limitation of a distribution.

(1) $r_b(x)$



(2) $tri_b(x)$ (3) $sc_b(x) = \frac{1}{b} \text{sinc}\left(\frac{x}{b}\right)$ 

Definition of the Dirac delta function:

$$(1) \quad \int_{-\infty}^{\infty} \delta(x) dx = 1$$

$$(2) \quad \delta(x) = 0 \quad \text{if } x \neq 0$$

$$(3) \quad \delta(x) = \delta(-x)$$

Properties of the Dirac delta function:

$$(1) \delta(x) = \frac{d}{dx}U(x) \quad U(x): \text{unit step function}$$

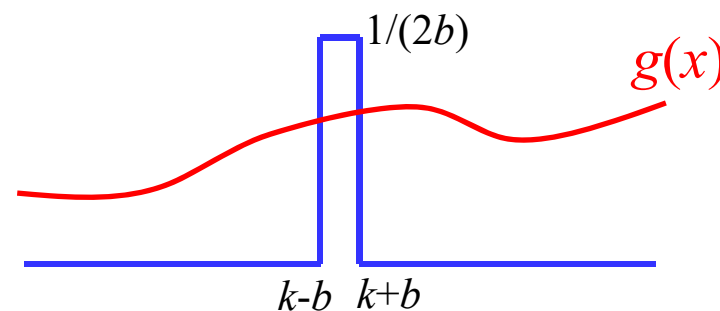
$$\delta(x-k) = \frac{d}{dx}U(x-k)$$

(2) Sifting property

$$\int_{-\infty}^{\infty} \delta(x-k)g(x)dx = g(k)$$

(Proof):

$$\begin{aligned} & \int_{-\infty}^{\infty} \delta(x-k)g(x)dx \\ &= \lim_{b \rightarrow 0} \frac{1}{2b} \int_{k-b}^{k+b} g(x)dx \\ &= \lim_{b \rightarrow 0} \frac{1}{2b} g(k) \int_{k-b}^{k+b} dx = g(k) \end{aligned}$$



(3) Sifting property (without integral)

$$\delta(x-k)g(x) = \delta(x-k)g(k)$$

(4) Scaling property

$$\delta(ax) = \frac{1}{|a|} \delta(x)$$

(to balance the integral)

$$\text{for } a > 0 \quad \int_{-\infty}^{\infty} \delta(ax) dx = \int_{-\infty}^{\infty} \delta(x') \frac{dx'}{a} = \frac{1}{a} \int_{-\infty}^{\infty} \delta(x) dx$$

$$\text{for } a < 0 \quad \int_{-\infty}^{\infty} \delta(ax) dx = \int_{\infty}^{-\infty} \delta(x') \frac{dx'}{a} = \frac{-1}{a} \int_{-\infty}^{\infty} \delta(x) dx$$

(5) Convolution property

$$g(x) * \delta(x) = g(x)$$

$$\text{Specially, } \delta(x) * \delta(x) = \delta(x)$$

(6) Integral for exponential functions

$$\int_{-\infty}^{\infty} e^{j2\pi fx} df = \delta(x)$$

It is directly from the fact that

$$\mathfrak{F}\{\delta(x)\} = 1, \quad \mathfrak{F}^{-1}\{1\} = \delta(x)$$

(7) Generalization of the integral for exponential functions

$$\int_{-\infty}^{\infty} e^{j2\pi f g(x)} df = \delta(g(x))$$

How do we define it?



(8) $\delta(g(x))$

If $g(x) = 0$ only at $x = x_0$, then

$$\delta(g(x)) = \frac{\delta(x - x_0)}{|g'(x_0)|}$$

(Proof):

$$g(x) \cong g'(x_0)(x - x_0) \quad \text{when } x \cong x_0$$

$$\delta(g(x)) = \delta(g'(x_0)(x - x_0)) = \frac{\delta(x - x_0)}{|g'(x_0)|}$$

In general, if $g(x) = 0$ only at $x = x_1, x_2, \dots, x_N$, then

$$\delta(g(x)) = \sum_{n=1}^N \frac{\delta(x - x_n)}{|g'(x_n)|}$$

(9) Derivative of $\delta(x)$ $\delta'(x) = \frac{d}{dx} \delta(x)$

$$\delta'(x) * g(x) = g'(x)$$

(Proof):

$$\begin{aligned} \delta'(x) * g(x) &= \int_{-\infty}^{\infty} \delta'(\tau) g(x - \tau) d\tau \\ &= \delta(\tau) g(x - \tau) \Big|_{\tau \rightarrow -\infty}^{\tau \rightarrow \infty} - \int_{-\infty}^{\infty} \delta(\tau) \frac{d}{d\tau} g(x - \tau) d\tau \\ &= \int_{-\infty}^{\infty} \delta(\tau) g'(x - \tau) d\tau \quad (\text{from the sifting property}) \\ &= g'(x) \end{aligned}$$

(10) Properties related to derivative of $\delta(x)$

$$(i) \quad \delta'(x) = -\delta'(-x)$$

$$(ii) \quad \int_{-\infty}^{\infty} \delta'(x - x_0) g(x) dx = -g'(x_0)$$

$$(iii) \quad \delta'(x - x_0) g(x) = \delta'(x - x_0) g(x_0) - \delta(x - x_0) g'(x_0)$$

(Proof): Since

$$\delta(x - x_0) g(x) = \delta(x - x_0) g(x_0)$$

$$\frac{d}{dx} \delta(x - x_0) g(x) = \frac{d}{dx} \delta(x - x_0) g(x_0)$$

$$\delta'(x - x_0) g(x) + \delta(x - x_0) g'(x) = \delta'(x - x_0) g(x_0)$$

$$\delta'(x - x_0) g(x) = \delta'(x - x_0) g(x_0) - \delta(x - x_0) g'(x_0)$$

(11) Higher order derivative of $\delta(x)$

$$\delta^{(n)}(x) * g(x) = g^{(n)}(x)$$

$$\int_{-\infty}^{\infty} \delta^{(n)}(x - x_0) g(x) dx = (-1)^n g^{(n)}(x_0)$$

$$\int_{-\infty}^{\infty} \delta^{(n)}(x - x_0) dx = 0 \quad \text{when } n > 0$$

$$\delta^{(n)}(x) = 0 \quad \text{when } x \neq 0$$

$$\delta^{(n)}(x) = (-1)^n \delta^{(n)}(-x)$$

4.3 Properties of the Fourier Transform

4.3.1 List of Properties

$$G(f) = \mathcal{F}[g(x)] = \int_{-\infty}^{\infty} g(x) \exp(-j2\pi f x) dx$$

(1) Recovery (inverse Fourier transform)	$g(x) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi f x) df$
(2) Integration (DC property)	$G(0) = \int_{-\infty}^{\infty} g(x) dx \quad g(0) = \int_{-\infty}^{\infty} G(f) df$
(3) Modulation	$\mathcal{F}[g(x) e^{j2\pi f_0 x}] = G(f - f_0)$
(4) Time Shifting	$\mathcal{F}[g(x - x_0)] = G(f) e^{-j2\pi f x_0}$
(5) Scaling	$\mathcal{F}[g(ax)] = \frac{1}{ a } G\left(\frac{f}{a}\right)$
(6) Time Reverse	$\mathcal{F}[g(-x)] = G(-f)$

(7) Real / Imaginary Input	<p>If $g(x)$ is real, then $G(f) = G^*(-f)$; If $g(x)$ is pure imaginary, then $G(f) = -G^*(-f)$</p>
(8) Even / Odd Input	<p>If $g(x) = g(-x)$, then $G(f) = G(-f)$; If $g(x) = -g(-x)$, then $G(f) = -G(-f)$;</p>
(9) Conjugation	$\mathcal{F}[g^*(x)] = G^*(-f) \quad \mathcal{F}[g^*(-x)] = G^*(f)$
(10) Differentiation	$\mathcal{F}[g'(x)] = j2\pi f G(f)$
(11) Multiplication by x	$\mathcal{F}[xg(x)] = \frac{j}{2\pi} G'(f)$
(12) Division by x	$\mathcal{F}\left[\frac{g(x)}{x}\right] = -j2\pi \int_{-\infty}^f G(\mu) d\mu$
(13) Parseval's Theorem (Energy Preservation)	$\int_{-\infty}^{\infty} g(x) ^2 dx = \int_{-\infty}^{\infty} G(f) ^2 df$
(14) Generalized Parseval's Theorem	$\int_{-\infty}^{\infty} g(x)h^*(x) dx = \int_{-\infty}^{\infty} G(f)H^*(f) df$

(15) Linearity	$\mathcal{F}[ag(x) + bh(x)] = aG(f) + bH(f)$
(16) Convolution	If $z(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(\tau)h(x - \tau)d\tau$, then $Z(f) = G(f)H(f)$
(17) Multiplication	If $z(x) = g(x)h(x)$, then $Z(f) = G(f) * H(f) = \int_{-\infty}^{\infty} G(\mu)H(f - \mu)d\mu$
(18) Correlation	If $z(x) = \int_{-\infty}^{\infty} g(\tau)h^*(\tau - x)d\tau$, then $Z(f) = G(f)H^*(f)$
(19) Two Times of Fourier Transforms	$\mathcal{F}\{\mathcal{F}[g(x)]\} = g(-x)$
(20) Four Times of Fourier Transforms	$\mathcal{F}\left[\mathcal{F}\left(\mathcal{F}\left\{\mathcal{F}[g(x)]\right\}\right)\right] = g(x)$

(Proof of (2) Integration Property)

$$G(f) = \int_{-\infty}^{\infty} \exp(-j2\pi f x) g(x) dx$$

$$G(0) = \int_{-\infty}^{\infty} \exp(-j2\pi 0 x) g(x) dx = \int_{-\infty}^{\infty} g(x) dx$$

(Proof of (5) Scaling Property)

$$\mathcal{F}[g(ax)] = \int_{-\infty}^{\infty} \exp(-j2\pi f x) g(ax) dx$$

$$= \int_{-\infty}^{\infty} \exp\left(-j2\pi f \frac{x'}{a}\right) g(x') \frac{dx'}{|a|}$$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} \exp\left(-j2\pi \frac{f}{a} x'\right) g(x') dx' = \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

Property (6) is a special case of Property (5) where $a = -1$.

(Proof of (7) and (9))

$$\begin{aligned} G^*(-f) &= \int_{-\infty}^{\infty} \overline{\exp(j2\pi f x) g(x)} dx \\ &= \int_{-\infty}^{\infty} \exp(-j2\pi f x) g^*(x) dx = \mathcal{F}[g^*(x)] \quad ((9) \text{ is proven}) \end{aligned}$$

If $g(x)$ is real, then

$$G^*(-f) = \mathcal{F}[g^*(x)] = \mathcal{F}[g(x)] = G(f)$$

(Proof of (10) Differentiation Property)

$$\begin{aligned} g(x) &= \mathcal{F}^{-1}[G(f)] = \int_{-\infty}^{\infty} \exp(j2\pi f x) G(f) df \\ \frac{d}{dx} g(x) &= \int_{-\infty}^{\infty} j2\pi f \exp(j2\pi f x) G(f) df = \mathcal{F}^{-1}[j2\pi f G(f)] \end{aligned}$$

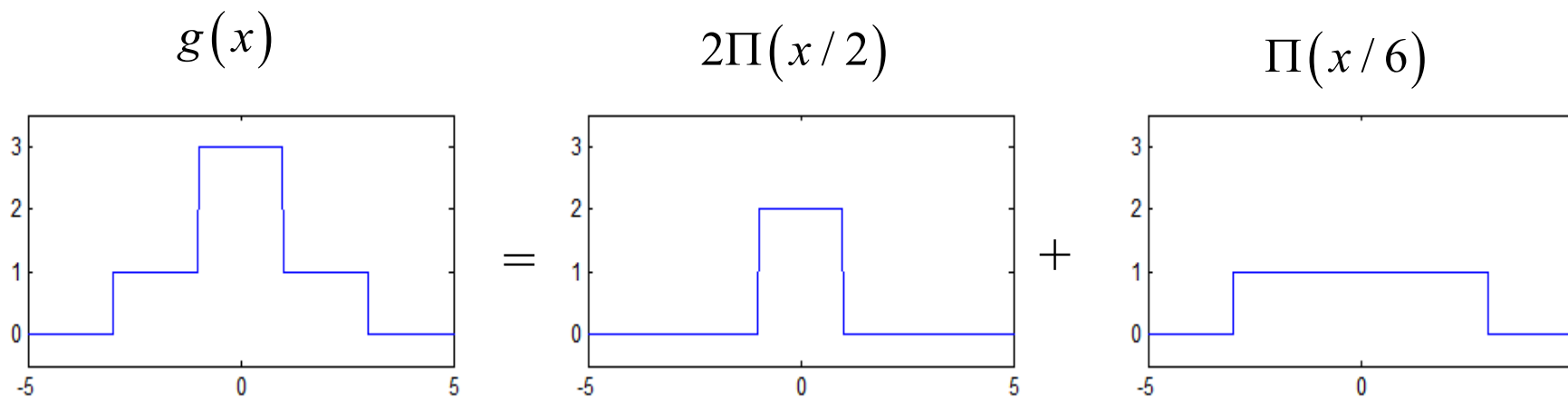
[Example 1] Determine the Fourier transform of the following signal.

$$g(x) = 3 \quad \text{for } |x| < 1,$$

$$g(x) = 1 \quad \text{for } 1 < |x| < 3, \quad g(x) = 0 \quad \text{for } |x| > 3$$

(Solution): Note that

$$g(x) = 2\Pi\left(\frac{x}{2}\right) + \Pi\left(\frac{x}{6}\right)$$



Therefore,
$$G(f) = 2 \cdot 2 \operatorname{sinc}(2f) + 6 \operatorname{sinc}(6f)$$

$$= 4 \operatorname{sinc}(2f) + 6 \operatorname{sinc}(6f)$$

[Example 2] Determine the Fourier transform of the following signal.

$$g(x) = x \exp(-|x|)$$

(Solution): From page 336, we have

$$\mathcal{F}[\exp(-|x|)] = \frac{2}{1 + 4\pi^2 f^2}$$

Then, from the differentiation property $\mathcal{F}[xg(x)] = \frac{j}{2\pi} G'(f)$

$$\begin{aligned} \mathcal{F}[x \exp(-|x|)] &= \frac{j}{2\pi} \frac{d}{df} \frac{2}{1 + 4\pi^2 f^2} \\ &= -j \frac{8\pi f}{(1 + 4\pi^2 f^2)^2} \end{aligned}$$

[Example 3] Determine the Fourier transform of the following signal.

$$g(x) = \exp(-3|x-1| + j6\pi x)$$

(Solution): Since

$$\mathcal{F}[\exp(-3|x|)] = \frac{6}{9 + 4\pi^2 f^2}$$

$$\mathcal{F}[\exp(-3|x-1|)] = \frac{6}{9 + 4\pi^2 f^2} e^{-j2\pi f} \quad \text{time shifting property}$$

$$\mathcal{F}[\exp(-3|x-1| + j6\pi x)] = \frac{6}{9 + 4\pi^2 (f-3)^2} e^{-j2\pi(f-3)}$$

modulation property

[Example 4] Determine the Fourier transform of the following signal.

$$g(x) = \cos(6\pi x) \quad \text{for } 0 < x < 8,$$

$$g(x) = 0 \quad \text{otherwise}$$

(Solution): Note that

$$\begin{aligned} g(x) &= \cos(6\pi x) \Pi\left(\frac{x-4}{8}\right) \\ &= \frac{1}{2} \exp(j6\pi x) \Pi\left(\frac{x-4}{8}\right) + \frac{1}{2} \exp(-j6\pi x) \Pi\left(\frac{x-4}{8}\right) \end{aligned}$$

Since

$$\mathcal{F}[\Pi(x)] = \text{sinc}(f)$$

$$\mathcal{F}\left[\Pi\left(\frac{x}{8}\right)\right] = 8 \text{sinc}(8f) \quad (\text{scaling})$$

$$\mathcal{F}\left[\Pi\left(\frac{x-4}{8}\right)\right] = 8e^{-j8\pi f} \text{sinc}(8f) \quad (\text{time shifting})$$

$$\mathcal{F}\left[\Pi\left(\frac{x-4}{8}\right)\right] = 8e^{-j8\pi f} \operatorname{sinc}(8f)$$

$$\mathcal{F}\left[\exp(j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right] = 8e^{-j8\pi(f-3)} \operatorname{sinc}(8(f-3)) \quad (\text{modulation})$$

$$\mathcal{F}\left[\exp(-j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right] = 8e^{-j8\pi(f+3)} \operatorname{sinc}(8(f+3))$$

Therefore,

$$\begin{aligned} & \mathcal{F}[g(x)] \\ &= \frac{1}{2} \mathcal{F}\left[\exp(j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right] + \frac{1}{2} \mathcal{F}\left[\exp(-j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right] \\ &= 4e^{-j8\pi(f-3)} \operatorname{sinc}(8(f-3)) + 4e^{-j8\pi(f+3)} \operatorname{sinc}(8(f+3)) \end{aligned}$$

4.3.2 Real, Imaginary, Even, and Odd Parts

Moreover, from Properties (7), (8), (9), we can conclude that

$$(i) \quad \mathfrak{F}[\mathcal{R}e\{g(x)\}] = \frac{1}{2}(G(f) + G^*(-f))$$

$$(ii) \quad \mathfrak{F}[j\mathcal{I}m\{g(x)\}] = \frac{1}{2}(G(f) - G^*(-f))$$

(Practice to prove them)

Also, any function can be decomposed into

$$(1) \quad g(x) = g_e(x) + g_o(x)$$

where $g_e(x) = \frac{1}{2}(g(x) + g(-x))$

$$g_o(x) = \frac{1}{2}(g(x) - g(-x))$$

$$(2) \quad g(x) = g_{e,r}(x) + g_{e,i}(x) + g_{o,r}(x) + g_{o,i}(x)$$

where

$$g_{e,r}(x) = \mathcal{Re} \left\{ \frac{1}{2}(g(x) + g(-x)) \right\} = \frac{g(x) + g^*(x) + g(-x) + g^*(-x)}{4}$$

$$g_{e,i}(x) = j \mathcal{Im} \left\{ \frac{1}{2}(g(x) + g(-x)) \right\}$$

$$g_{o,r}(x) = \mathcal{Re} \left\{ \frac{1}{2}(g(x) - g(-x)) \right\}$$

$$g_{o,i}(x) = j \mathcal{Im} \left\{ \frac{1}{2}(g(x) - g(-x)) \right\}$$

One can prove that

$$\mathfrak{I}[g_e(x)] = G_e(f)$$

$$\mathfrak{I}[g_o(x)] = G_o(f)$$

$$\mathfrak{I}[g_{e,r}(x)] = G_{e,r}(f)$$

$$\mathfrak{I}[g_{e,i}(x)] = G_{e,i}(f)$$

$$\mathfrak{I}[g_{o,r}(x)] = G_{o,r}(f)$$

$$\mathfrak{I}[g_{o,i}(x)] = G_{o,i}(f)$$

where

note

$$G_e(f) = \frac{1}{2}(G(f) + G(-f)) \quad G_o(f) = \frac{1}{2}(G(f) - G(-f))$$

$$G_{e,r}(f) = \mathcal{Re} \left\{ \frac{1}{2}(G(f) + G(-f)) \right\}$$

$$G_{e,i}(f) = j \mathcal{Im} \left\{ \frac{1}{2}(G(f) + G(-f)) \right\}$$

$$G_{o,r}(f) = \mathcal{Re} \left\{ \frac{1}{2}(G(f) - G(-f)) \right\}$$

$$G_{o,i}(f) = j \mathcal{Im} \left\{ \frac{1}{2}(G(f) - G(-f)) \right\}$$

$$g(x) = g_{e,r}(x) + g_{e,i}(x) + g_{o,r}(x) + g_{o,i}(x)$$

$G(f) = G_{e,r}(f) + G_{e,i}(f) + G_{o,r}(f) + G_{o,i}(f)$

4.3.3 Parseval's Theorem

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |G(f)|^2 df$$

Parseval's theorem is also called the **energy preservation property**, **Rayleigh's Theorem**, or **Plancherel's Theorem**.

(Proof):

$$\begin{aligned} \int_{-\infty}^{\infty} G(f)G^*(f)df &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)e^{-j2\pi fx} dx \overline{\int_{-\infty}^{\infty} g(\tau)e^{-j2\pi f\tau} d\tau} df \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)g^*(\tau) \left[\int_{-\infty}^{\infty} e^{j2\pi f(\tau-x)} df \right] dx d\tau \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x)g^*(\tau)\delta(\tau-x)d\tau dx \quad (\text{from page 346}) \\ &= \int_{-\infty}^{\infty} g(x)g^*(x)dx \quad (\text{from the sifting property}) \\ &= \int_{-\infty}^{\infty} |g(x)|^2 dx \end{aligned}$$

Generalized Parseval's Theorem

$$\int_{-\infty}^{\infty} g(x)h^*(x)dx = \int_{-\infty}^{\infty} G(f)H^*(f)df$$

It also called the **power theorem**.

[Example 5] Determine the following integral:

$$\int_{-\infty}^{\infty} \text{sinc}^2(x) dx$$

[Example 6] Determine the following integral:

$$\int_{-\infty}^{\infty} \cos(8\pi x) \text{sinc}(3x) dx$$

(Solution): Since

$$\mathcal{F}[\cos(8\pi x)] = \frac{1}{2}(\delta(f-4) + \delta(f+4))$$

$$\mathcal{F}[\text{sinc}(3x)] = \frac{1}{3}\Pi\left(\frac{f}{3}\right)$$

$$\begin{aligned} \int_{-\infty}^{\infty} \cos(8\pi x) \text{sinc}(3x) dx &= \int_{-\infty}^{\infty} \frac{1}{2}(\delta(f-4) + \delta(f+4)) \frac{1}{3}\Pi\left(\frac{f}{3}\right) df \\ &= \frac{1}{6} \int_{-3/2}^{3/2} (\delta(f-4) + \delta(f+4)) df = 0 \end{aligned}$$

4.4 Uncertainty Principles

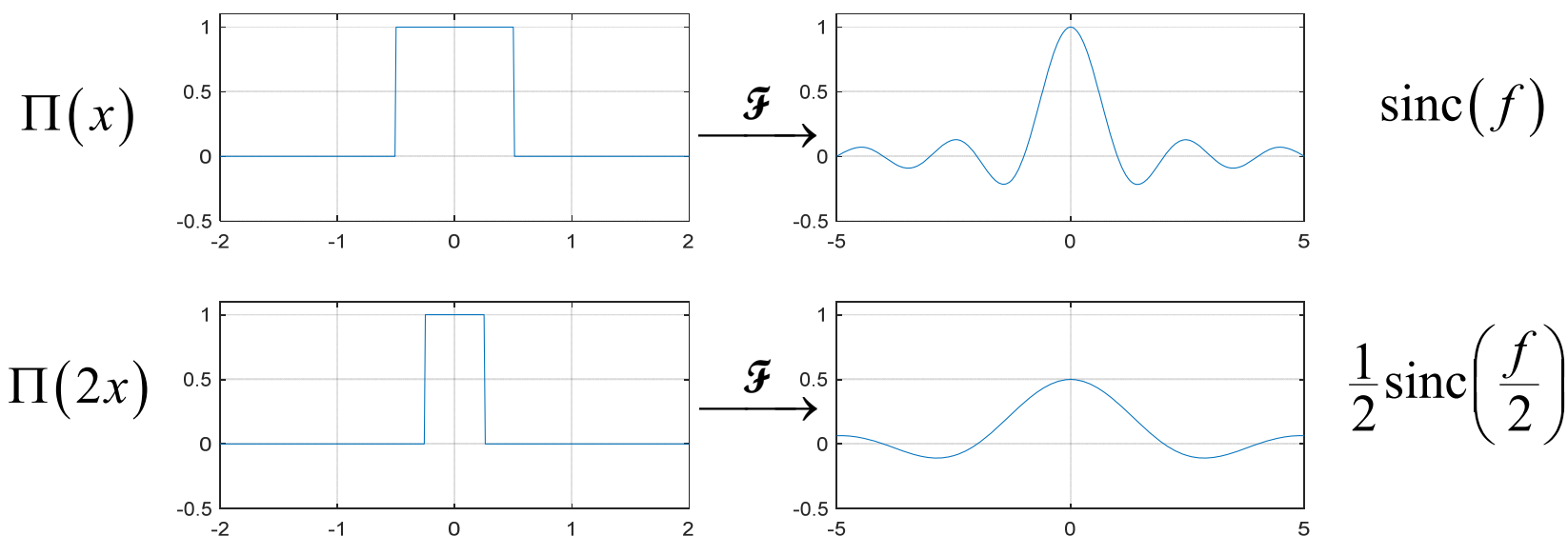
4.4.1 Uncertainty Principles from Different Views

(1) From the Point of View of the Scaling Property

$$\mathcal{F}[g(ax)] = \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

wide in the time domain \rightarrow narrow in the frequency domain

narrow in the time domain \rightarrow wide in the frequency domain



(2) From the Point of View of Equivalent Width

Equivalent Width in the Time Domain:

$$W_g = \frac{\int_{-\infty}^{\infty} g(x) dx}{g(0)}$$

from the integration property $W_g = \frac{G(0)}{g(0)}$

Equivalent Width in the Frequency Domain:

$$W_G = \frac{\int_{-\infty}^{\infty} G(f) df}{G(0)} = \frac{g(0)}{G(0)}$$

Product of the Two Equivalent Widths:

$$W_g W_G = \frac{G(0)}{g(0)} \frac{g(0)}{G(0)} = 1$$

(3) Heisenberg's Uncertainty Principle

For a signal $g(x)$, if $\sqrt{|x|} g(x) = 0$ when $|x| \longrightarrow \infty$, then

$$\sigma_x \sigma_f \geq 1/4\pi$$

where $\sigma_x^2 = \int (x - \mu_x)^2 P_g(x) dx$ $\sigma_f^2 = \int (f - \mu_f)^2 P_G(f) df$,

$$\mu_x = \int x P_g(x) dx, \quad \mu_f = \int f P_G(f) df$$

$$P_g(x) = \frac{|g(x)|^2}{\int |g(x)|^2 dx}, \quad P_G(f) = \frac{|G(f)|^2}{\int |G(f)|^2 df},$$

(Proof of Henseinberg's uncertainty principle):

For simplification, we consider the case where $\mu_x = \mu_f = 0$

Then, use Parseval's theorem

$$\begin{aligned}\sigma_x^2 \sigma_f^2 &= \frac{\int x^2 |g(x)|^2 dx}{\int |g(x)|^2 dx} \frac{\int f^2 |G(f)|^2 df}{\int |G(f)|^2 df} \\ &= \frac{1}{4\pi^2} \frac{\int x^2 |g(x)|^2 dx}{\int |g(x)|^2 dx} \frac{\int |g'(x)|^2 dx}{\int |g(x)|^2 dx}\end{aligned}$$

Here, we apply the fact that

$$\begin{aligned}\int |g(x)|^2 dx &= \int |G(f)|^2 df \\ \mathcal{F}[g'(x)] &= j2\pi f G(f) \qquad \int |g'(x)|^2 dx = 4\pi^2 \int f^2 |G(f)|^2 df\end{aligned}$$

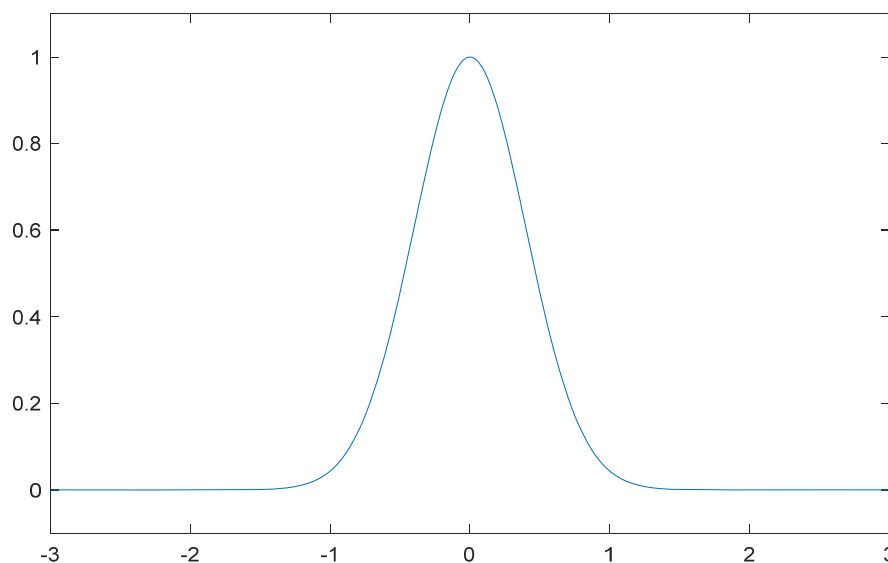
From Schwarz's inequality $\langle g(x), g(x) \rangle \langle h(x), h(x) \rangle \geq |\langle g(x), h(x) \rangle|^2$

$$\begin{aligned} \int x^2 |g(x)|^2 dx \int |g'(x)|^2 dx &\geq \left(\left| \int x g^*(x) \frac{d}{dx} g(x) dx \right|^2 + \left| \int x g(x) \frac{d}{dx} g^*(x) dx \right|^2 \right) / 2 \\ &\geq \left| \int \left(x g^*(x) \frac{d}{dx} g(x) + x g(x) \frac{d}{dx} g^*(x) \right) dx \right|^2 / 4 \quad (\text{using } |a+b|^2 + |a-b|^2 \geq 2|a|^2) \\ &= \left| \int x \frac{d}{dx} [g(x)g^*(x)] dx \right|^2 / 4 = \left| xg(x)g^*(x) \Big|_{-\infty}^{\infty} - \int g^*(x)g(x) dx \right|^2 / 4 \\ &= \left[xg(x)g^*(x) \Big|_{x \rightarrow \infty} - xg(x)g^*(x) \Big|_{x \rightarrow -\infty} \right] - \int g^*(x)g(x) dx \Big|^2 / 4 \\ &= \left| \int |g(x)|^2 dx \right|^2 / 4 \end{aligned}$$

$$\sigma_t^2 \sigma_f^2 \geq \frac{1}{16\pi^2} \implies \sigma_t \sigma_f \geq \frac{1}{4\pi}$$

4.4.2 Gaussian and Hermite-Gaussian Functions

Gaussian function: $\exp(-\pi x^2)$



The Gaussian function is an eigenfunction of the Fourier transform with eigenvalue = 1:

$$\mathfrak{F}[\exp(-\pi x^2)] = \exp(-\pi f^2)$$

$$\mathfrak{F}\left[\exp(-\pi x^2)\right] = \exp(-\pi f^2)$$

(Proof): From the fact that

$$\int_{-\infty}^{\infty} e^{-(at^2+bt)} dt = \sqrt{\pi/a} \cdot e^{b^2/4a}$$

M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 3rd Ed., 2009.

we have

$$\mathfrak{F}\left\{e^{-\pi x^2}\right\} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-j2\pi fx} dx = \sqrt{\frac{\pi}{\pi}} e^{\frac{(j2\pi f)^2}{4\pi}} = e^{-\pi f^2}$$

The Gaussian function is not the only eigenfunction of the Fourier transform.

The Gaussian function satisfies the lower bound of Heisenberg's uncertainty principle.

$$\mathcal{F}\left[e^{-\pi x^2}\right] = e^{-\pi f^2}$$

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} e^{-2\pi x^2} dx = \sqrt{1/2}$$

$$\text{use } \int_{-\infty}^{\infty} e^{-(at^2+bt)} dt = \sqrt{\pi/a} \cdot e^{b^2/4a}$$

$$\int_{-\infty}^{\infty} x^2 |g(x)|^2 dx = \int_{-\infty}^{\infty} x^2 e^{-2\pi x^2} dx = 2 \int_0^{\infty} x^2 e^{-2\pi x^2} dx =$$

$$= \frac{2\Gamma[3/2]}{2(2\pi)^{3/2}} = \frac{\sqrt{\pi}/2}{(2\pi)^{3/2}} = \frac{1}{4\sqrt{2}\pi}$$

$$\text{use } \int_0^{\infty} x^m e^{-ax^2} dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$$

$$\Gamma(1/2) = \sqrt{\pi} \quad \Gamma(n+1) = n\Gamma(n)$$

$$\mu_x = \frac{\int_{-\infty}^{\infty} x |g(x)|^2 dx}{\int_{-\infty}^{\infty} |g(x)|^2 dx} = 0$$

$$\sigma_x^2 = \frac{\int_{-\infty}^{\infty} x^2 |g(x)|^2 dx}{\int_{-\infty}^{\infty} |g(x)|^2 dx} = \frac{1}{4\pi}$$

$$\sigma_x^2 = \frac{1}{4\pi} \quad \sigma_x = \sqrt{\frac{1}{4\pi}}$$

Since $G(f) = g(f)$,

$$\sigma_f = \sqrt{\frac{1}{4\pi}}$$

Therefore,

$$\sigma_x \sigma_f = \frac{1}{4\pi} \quad \text{if} \quad g(x) = e^{-\pi x^2}$$

Note: Other Hermite Gaussian functions do not satisfy the lower bound of Heisenberg's uncertainty principle.

$$\mathfrak{F}\left[H_n(\sqrt{2\pi}x)\exp(-\pi x^2)\right] = (-j)^n \exp(-\pi f^2) H_n(\sqrt{2\pi}f)$$

$H_n(x)$: The Hermite polynomial of order n (see page 318).

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

附錄八 Convolution

Convolution: $g(x) * h(x) = \int_{-\infty}^{\infty} g(x-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} g(\tau)h(x-\tau)d\tau$

Specially, if $g(x) = 0$ for $x < 0$ and $h(x) = 0$ for $x < 0$, then

Convolution (causal form):

$$g(x) * h(x) = \int_0^{\infty} g(x-\tau)h(\tau)d\tau = \int_0^{\infty} g(\tau)h(x-\tau)d\tau$$

Physical meaning: The effect of the input on the output is determined by their time difference.

$$y(x) = \int_0^{\infty} g(\tau)h(x-\tau)d\tau$$

$$y(x) = \int_0^{\infty} g(\tau) h(x - \tau) d\tau$$

output input effect of $g(\tau)$ on $y(x)$

$$y(x) = g(0)h(x)\Delta + g(\Delta)h(x-\Delta)\Delta + g(2\Delta)h(x-2\Delta)\Delta \\ + \cdots g(x)h(0)\Delta$$

Any linear time-invariant system can be expressed as the convolution form.

$$z(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(\tau)h(x-\tau)d\tau$$

[Support Theorem]:

If the support of $g(x)$ is $x \in [x_1, x_2]$

(i.e., $g(x) = 0$ for $x < x_1$ and $x > x_2$)

and the support of $h(x)$ is $x \in [x_3, x_4]$,

then the support of $z(x)$ is

$$x \in [x_1 + x_3, x_2 + x_4]$$

(Proof):

$$z(x) = \int_{-\infty}^{\infty} g(\tau)h(x-\tau)d\tau = \int_{x_1}^{x_2} g(\tau)h(x-\tau)d\tau$$

$$h(x-\tau) \neq 0 \quad \text{when } x-\tau \in [x_3, x_4],$$

$$x \in [x_3 + \min(\tau), x_4 + \max(\tau)] = [x_1 + x_3, x_2 + x_4]$$

附錄九 Change of Independent Variables for Integrals

$$\iint \dots \dots \dots dx dy = \iint \dots \dots \dots C^{-1} dw dv$$

$$\text{where } C = \det \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det \begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial v} \end{bmatrix}^{-1}, \quad C^{-1} = \det \begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial v} \end{bmatrix}$$

For the indefinite integral case

$$C = \left| \det \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \right| = \left| \det \begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial v} \end{bmatrix} \right|^{-1}$$

4.5 Convolution and Correlation

4.5.1 Convolution Property

$$\text{If } z(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(\tau)h(x - \tau) d\tau,$$

$$\text{then } Z(f) = G(f)H(f)$$

convolution \implies multiplication

(Proof): If $z(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(\tau)h(x-\tau)d\tau$,

$$\begin{aligned}
 \mathcal{F}^{-1}[G(f)H(f)] &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-j2\pi f\tau} g(\tau) d\tau \int_{-\infty}^{\infty} e^{-j2\pi ft} h(t) dt \right] e^{j2\pi fx} df \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi f\tau} g(\tau) e^{-j2\pi ft} h(t) e^{j2\pi fx} d\tau dt df \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-j2\pi f\tau} e^{-j2\pi ft} e^{j2\pi fx} df \right] g(\tau) h(t) d\tau dt \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x-\tau-t) g(\tau) h(t) d\tau dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \delta(t+\tau-x) h(t) dt \right] g(\tau) d\tau \\
 &= \int_{-\infty}^{\infty} h(x-\tau) g(\tau) d\tau = z(x)
 \end{aligned}$$

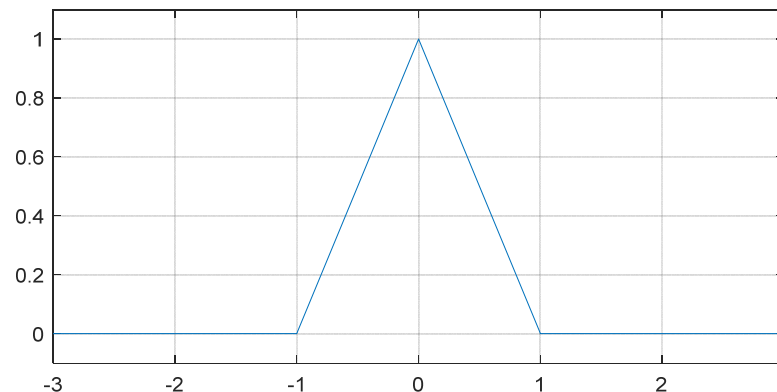
Therefore,

$$\mathcal{F}[z(x)] = G(f)H(f) \qquad Z(f) = G(f)H(f)$$

[Example 1] Determine the Fourier transform of

$$\Lambda(x) = \begin{cases} x+1 & \text{for } -1 \leq x \leq 0 \\ 1-x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

We call it the **triangular function**.



(Solution):

Note that $\Lambda(x) = \Pi(x) * \Pi(x)$

$$\Pi(x) * \Pi(x) = \int_{-\infty}^{\infty} \Pi(\tau) \Pi(x - \tau) d\tau = \int_{-1/2}^{1/2} \Pi(x - \tau) d\tau$$

$$\Pi(x - \tau) = 1 \quad \text{for } -\frac{1}{2} < x - \tau < \frac{1}{2}, \text{ i.e., } x - \frac{1}{2} < \tau < x + \frac{1}{2}$$

When $x - 1/2 > 1/2$ or $x + 1/2 < -1/2$, i.e., $x > 1$ or $x < -1$,

$$\int_{-1/2}^{1/2} \Pi(x - \tau) d\tau = 0$$

$$\text{When } -1 < x < 0 \quad \int_{-1/2}^{1/2} \Pi(x - \tau) d\tau = \int_{-1/2}^{x+1/2} 1 \cdot d\tau = x + 1$$

$$\text{When } 0 < x < 1 \quad \int_{-1/2}^{1/2} \Pi(x - \tau) d\tau = \int_{x-1/2}^{1/2} 1 \cdot d\tau = 1 - x$$

Therefore,

$$\begin{aligned}\mathcal{F}[\Lambda(x)] &= \mathcal{F}[\Pi(x) * \Pi(x)] \\ &= \mathcal{F}[\Pi(x)]\mathcal{F}[\Pi(x)] = \text{sinc}^2 x\end{aligned}$$

[Example 2] Determine the inverse Fourier transform of

$$G(f) = \frac{1}{1 + \pi^2 f^2} \frac{1}{j\pi f + 1}$$

(Solution): Note that

$$G(f) = 2 \frac{2 \cdot 2}{4 + 4\pi^2 f^2} \frac{1}{j2\pi f + 2}$$

From page 336

$$\mathcal{F}^{-1} \left[\frac{2 \cdot 2}{4 + 4\pi^2 f^2} \right] = \exp(-2|x|) \quad \mathcal{F}^{-1} \left[\frac{1}{j2\pi f + 2} \right] = \exp(-2x)U(x)$$

$$\begin{aligned} g(x) &= 2 \exp(-2|x|) * \exp(-2x)U(x) \\ &= 2 \int_{-\infty}^{\infty} \exp(-2|\tau|) \exp(-2x + 2\tau) U(x - \tau) d\tau \\ &= 2 \int_{-\infty}^x \exp(-2|\tau|) \exp(-2x + 2\tau) d\tau \end{aligned}$$

$$g(x) = 2 \int_{-\infty}^x \exp(-2|\tau|) \exp(-2x + 2\tau) d\tau$$

When $x \leq 0$, since $\tau \in (-\infty, x]$, $\tau \leq 0$ is always satisfied,

$$\exp(-2|\tau|) = \exp(2\tau)$$

$$g(x) = 2 \exp(-2x) \int_{-\infty}^x \exp(4\tau) d\tau = \frac{\exp(2x)}{2}$$

When $x > 0$,

$$\begin{aligned} g(x) &= 2 \exp(-2x) \left[\int_{-\infty}^0 \exp(4\tau) d\tau + \int_0^x d\tau \right] \\ &= \exp(-2x) \left[\frac{1}{2} + 2x \right] \end{aligned}$$

[Example 3] Determine

$$\text{sinc}(t) * \text{sinc}(t)$$

[Example 4] Determine

$$\text{sinc}(t) * \text{sinc}(2t) * \text{sinc}(3t)$$

[Example 5] Determine

$$\delta(t) * \delta(t)$$

[Example 6] Determine

$$\text{sinc}(4t) * \sin(2\pi t)$$

[Theorem 4.5.1]

$$g(x) * \delta(x - x_0) = g(x - x_0)$$

Specially,

$$g(x) * \delta(x) = g(x)$$

(Proof):

$$\begin{aligned} \mathcal{F}[g(x) * \delta(x - x_0)] &= \mathcal{F}[g(x)] \mathcal{F}[\delta(x - x_0)] \\ &= G(f) \exp(-j2\pi x_0 f) \end{aligned}$$

$$\mathcal{F}^{-1}\{\mathcal{F}[g(x) * \delta(x - x_0)]\} = \mathcal{F}^{-1}[G(f) \exp(-j2\pi x_0 f)]$$

$$g(x) * \delta(x - x_0) = g(x - x_0)$$

(from the time-shifting property)

4.5.2 Multiplication Property

$$\text{If } z(x) = g(x)h(x)$$

$$\text{then } Z(f) = G(f) * H(f) = \int_{-\infty}^{\infty} G(s)H(f-s)ds$$

multiplication \implies convolution

$$\begin{aligned}
(\text{Proof}): \quad \mathcal{F}[z(x)] &= \mathcal{F}[g(x)h(x)] \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{j2\pi s x} G(s) ds \int_{-\infty}^{\infty} e^{j2\pi u x} H(u) du \right] e^{-j2\pi f x} dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j2\pi s x} G(s) e^{j2\pi u x} H(u) e^{-j2\pi f x} ds du dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{j2\pi s x} e^{j2\pi u x} e^{-j2\pi f x} dx \right] G(s) H(u) ds du \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(s+u-f) G(s) H(u) ds du \\
&= \int_{-\infty}^{\infty} G(s) H(f-s) ds
\end{aligned}$$

Therefore,

$$Z(f) = G(f) * H(f) = \int_{-\infty}^{\infty} G(s) H(f-s) ds$$

[Example 7] Determine the Fourier transform of

$$g(x) = \cos(4\pi x) \operatorname{rect}\left(\frac{x}{6}\right)$$

(Solution):

$$\mathcal{F}\left\{\operatorname{rect}\left(\frac{x}{6}\right)\right\} = 6 \operatorname{sinc}(6f)$$

$$\mathcal{F}\{\cos(4\pi x)\} = \frac{1}{2}[\delta(f-2) + \delta(f+2)]$$

Therefore,

$$G(f) = 6 \operatorname{sinc}(6f) * \frac{1}{2}[\delta(f-2) + \delta(f+2)]$$

$$G(f) = 3 \operatorname{sinc}(6(f-2)) + 3 \operatorname{sinc}(6(f+2))$$

4.5.3 Correlation

Correlation

$$z(x) = \text{corr}(g(x), h(x)) = \int_{-\infty}^{\infty} g(\tau + x) h^*(\tau) d\tau$$

Auto-Correlation

$$a_g(x) = \text{corr}(g(x), g(x)) = \int_{-\infty}^{\infty} g(\tau + x) g^*(\tau) d\tau$$

Applications: Matched filter, communication, pattern recognition, signal detection

[Theorem 4.5.2] In fact, correlation is equivalent to convolution with the conjugate + time reverse of a signal.

$$\text{corr}(g(x), h(x)) = g(x) * h^*(-x)$$

(Proof):

$$\begin{aligned} g(x) * h^*(-x) &= \int_{-\infty}^{\infty} g(x - \tau) h_1(\tau) d\tau \quad \text{where } h_1(x) = h^*(-x) \\ &= \int_{-\infty}^{\infty} g(x - \tau) h^*(-\tau) d\tau \\ &= -\int_{\infty}^{-\infty} g(x + \tau) h^*(\tau) d\tau \quad \tau_{\text{new}} = -\tau_{\text{old}} \\ &= \int_{-\infty}^{\infty} g(x + \tau) h^*(\tau) d\tau \\ &= \text{corr}(g(x), h(x)) \end{aligned}$$

Since

$$\text{corr}(g(x), h(x)) = g(x) * h^*(-x)$$

we have

$$\mathcal{F}[\text{corr}(g(x), h(x))] = G(f)H^*(f)$$

Specially, if $a_g(x)$ is the auto-correlation of $g(x)$:

$$a_g(x) = \text{corr}(g(x), g(x))$$

then

$$\mathcal{F}[a_g(x)] = |G(f)|^2$$

4.6 Two-Dimensional Fourier Transform

4.6.1 Rectangular Coordinate

Two Dimensional Fourier Transform

$$\mathfrak{F}_{2D}[g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi fx} e^{-j2\pi hy} dx dy = G(f, h)$$

Two Dimensional Inverse Fourier Transform

$$\mathfrak{F}_{2D}^{-1}[G(f, h)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(f, h) e^{j2\pi fx} e^{j2\pi hy} df dh = g(x, y)$$

Possible Applications: Image processing, optics, electromagnet wave propagation analysis,

[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

Physical meaning: Express a signal by a linear combination of

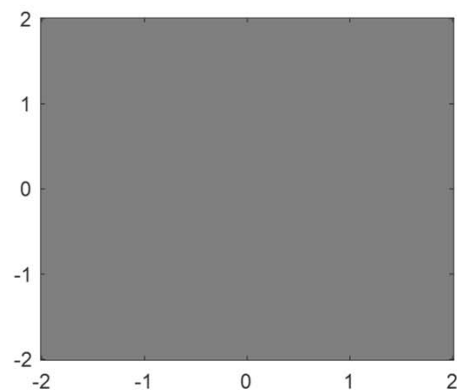
$$e^{j2\pi fx} e^{j2\pi hy}$$

f is the number of periods per unit of x

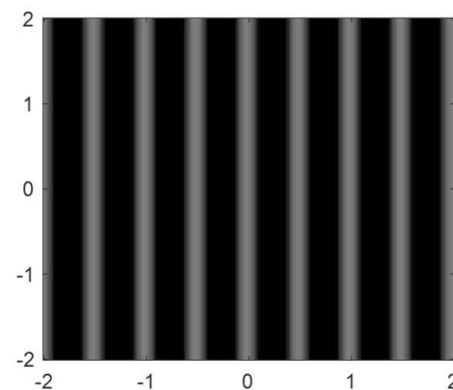
h is the number of periods per unit of y

real part of $e^{j2\pi fx} e^{j2\pi hy}$

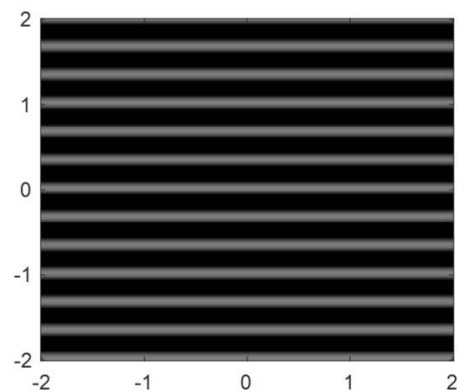
Bright colors mean higher values and dark colors mean lower values.



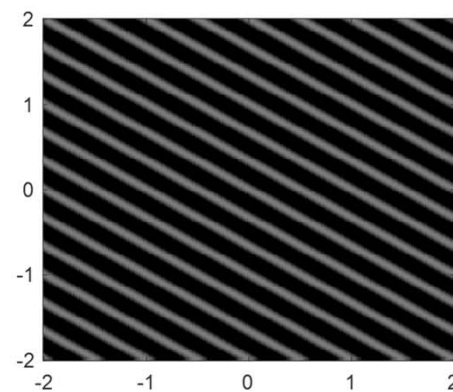
(a) $f = h = 0$



(b) $f = 2, h = 0$



(c) $f = 0, h = 3$



(d) $f = 2, h = 3$

[Example 1] Find the 2D Fourier transform of

$$g(x, y) = \text{sinc}(x)\text{sinc}(y)$$

(Solution):

$$\begin{aligned} G(f, h) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi fx} e^{-j2\pi hy} \text{sinc}(x)\text{sinc}(y) dx dy \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-j2\pi fx} \text{sinc}(x) dx \right] e^{-j2\pi hy} \text{sinc}(y) dy \\ &= \int_{-\infty}^{\infty} \Pi(f) e^{-j2\pi hy} \text{sinc}(y) dy \\ &= \Pi(f) \int_{-\infty}^{\infty} e^{-j2\pi hy} \text{sinc}(y) dy = \Pi(f)\Pi(h) \end{aligned}$$

[Example 2] Find the 2D Fourier transform of

$$g(x, y) = \Pi(x)$$

(Solution):

$$\begin{aligned} G(f, h) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi fx} e^{-j2\pi hy} \Pi(x) dx dy \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-j2\pi fx} \Pi(x) dx \right] e^{-j2\pi hy} dy = \text{sinc}(f) \int_{-\infty}^{\infty} e^{-j2\pi hy} dy \\ &= \text{sinc}(f) \delta(h) \end{aligned}$$

[Example 3] Find the 2D Fourier transform of

$$g(x, y) = \sin(2\pi(x + 2y))$$

4.6.2 Circular Coordinate Conversion

$$g(x, y) \longrightarrow g(r, \theta) \quad x = r \cos \theta, \quad y = r \sin \theta$$

$$G(f, h) \longrightarrow G(s, \phi) \quad f = s \cos \phi, \quad h = s \sin \phi$$

$$G(f, h) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi fx} e^{-j2\pi hy} g(x, y) dx dy$$

$$G(s, \phi) = \int_0^{\infty} \int_0^{2\pi} e^{-j2\pi sr \cos \phi \cos \theta} e^{-j2\pi sr \sin \phi \sin \theta} g(r, \theta) C^{-1} d\theta dr$$

where $C^{-1} = \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} = r$
 (page 381)

$$G(s, \phi) = \int_0^{\infty} \int_0^{2\pi} e^{-j2\pi sr \cos(\phi - \theta)} g(r, \theta) r d\theta dr$$

$$G(s, \phi) = \int_0^{\infty} \int_0^{2\pi} e^{-j2\pi sr \cos(\phi - \theta)} g(r, \theta) r d\theta dr$$

Specially, if $g(r, \theta)$ is independent of θ

$$g(r, \theta) = g(r)$$

$$G(s, \phi) = \int_0^{\infty} \left[\int_0^{2\pi} e^{-j2\pi sr \cos(\phi - \theta)} d\theta \right] r g(r) dr$$

From the fact that

$$(\varphi = \phi - \theta)$$

$$\int_0^{2\pi} e^{-jx \cos(\phi - \theta)} d\theta = - \int_{\phi}^{\phi - 2\pi} e^{-jx \cos \varphi} d\varphi = \int_{\phi - 2\pi}^{\phi} e^{-jx \cos \varphi} d\varphi$$

$$= \int_0^{2\pi} e^{-jx \cos(\varphi)} d\varphi = 2\pi J_0(x)$$

($\cos \varphi$ has a period of 2π)

Bessel function of the 1st kind of zero order, see pages 201-203

We have

$$G(s, \phi) = 2\pi \int_0^{\infty} J_0(2\pi sr) r g(r) dr \quad (\text{Note that it is independent of } \phi)$$

$$G(s) = 2\pi \int_0^{\infty} J_0(2\pi sr) r g(r) dr$$

Hankel Transform

$$G(s) = 2\pi \int_0^{\infty} J_0(2\pi sr) r g(r) dr$$

It is in fact the 2D Fourier transform for a rotationally symmetric signal.

$$g(r, \theta) = g(r)$$

Inverse Hankel Transform

$$g(r) = 2\pi \int_0^{\infty} J_0(2\pi sr) s G(s) ds$$

It has the same form as the forward transform.

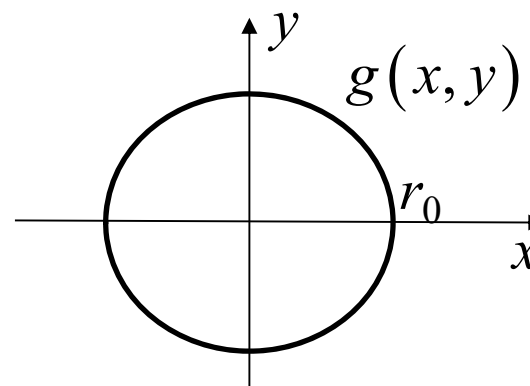
Several Hankel Transform Pairs

(i) If $g(r) = \delta(r - r_0)$

$$\text{then } G(s) = 2\pi r_0 J_0(2\pi r_0 s)$$

Note that

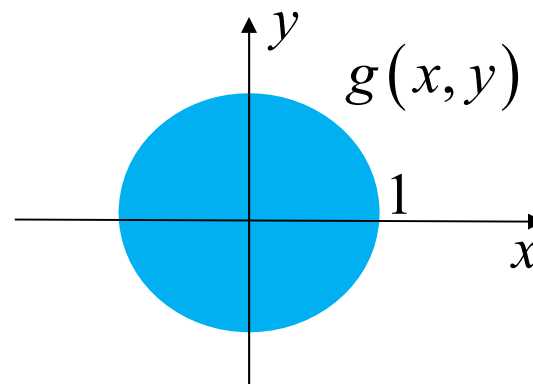
$$\begin{aligned} & 2\pi \int_0^{\infty} J_0(2\pi sr) r \delta(r - r_0) dr \\ &= 2\pi J_0(2\pi sr_0) r_0 \end{aligned}$$



(from page 344(2))

(ii) If $g(r) = \text{circ}(r)$

$$\text{circ}(r) = \begin{cases} 1 & \text{for } r < 1 \\ 0 & \text{for } r > 1 \end{cases}$$



then $G(s) = \frac{J_1(2\pi s)}{s}$ ←

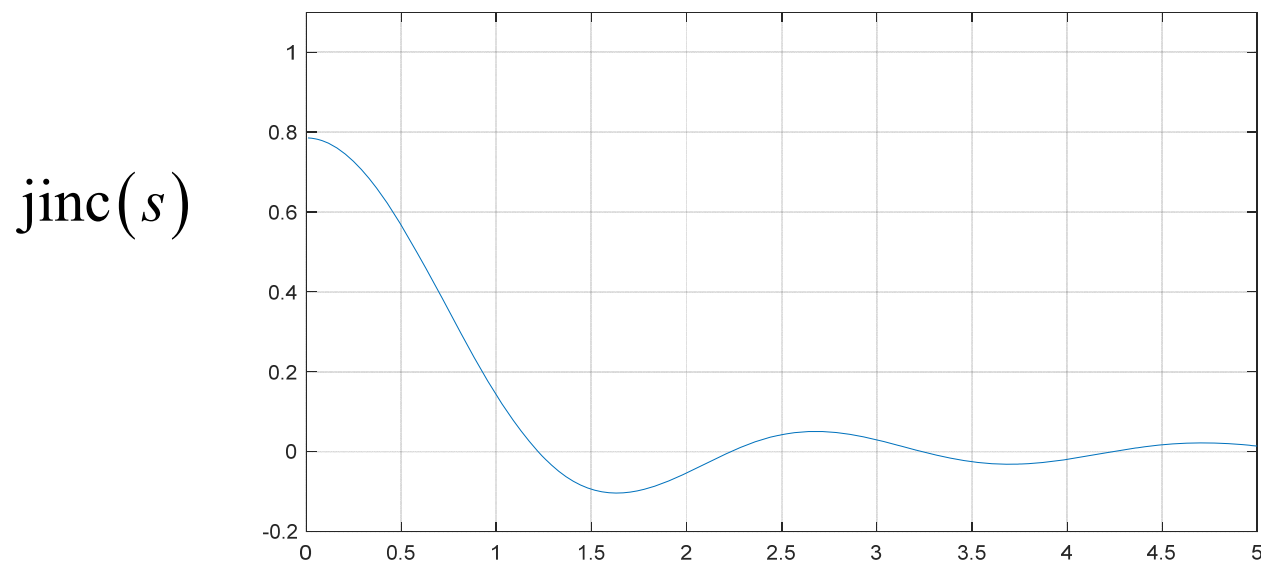
Bessel function of the 1st kind of 1st order,
See pages 201-203.

Jinc function (also called the Besinc function).

$$\text{jinc}(s) = \frac{J_1(\pi s)}{2s}$$

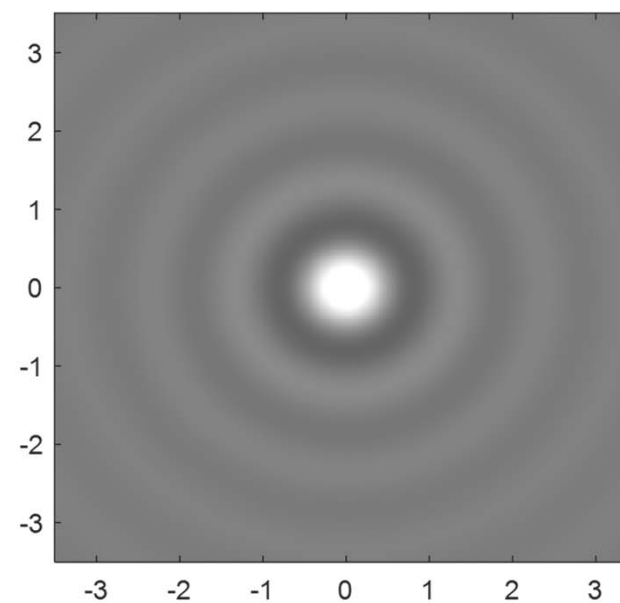
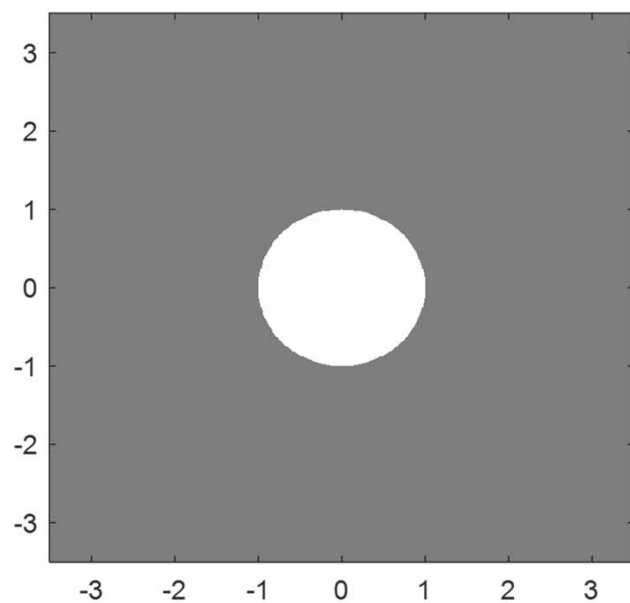
$$\text{If } g(r) = \text{circ}(r) \quad G(s) = \frac{J_1(2\pi s)}{s} = 4 \text{jinc}(2s)$$

It plays a similar role as the sinc function.



Hankel
transform

$$g(r) = \text{circ}(r) \longrightarrow G(s) = 4 \text{jinc}(2s)$$



4.7 The Operations Closely Related to the Fourier Transform (只教不考)

(1) Two-Sided Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t) dt$$

Note that when

$$s = j2\pi f$$

it is reduced to the Fourier transform. When

$$s = \sigma + j2\pi f$$

it is equivalent to the Fourier transform of $\exp(-\sigma t)f(t)$.

$$\mathcal{L}\{f(t)\}_{s=\sigma+j2\pi f} = \int_{-\infty}^{\infty} e^{-j2\pi ft} e^{-\sigma t} f(t) dt$$

$$\mathcal{L}\{f(t)\}_{s=\sigma+j2\pi f} = \mathfrak{F}[e^{-\sigma t} f(t)]$$

(2) One-Sided Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

When

$$s = \sigma + j2\pi f$$

it is equivalent to the Fourier transform of $\exp(-\sigma t)f(t)U(t)$.

$$\mathcal{L}\{f(t)\}_{s=\sigma+j2\pi f} = \mathfrak{F}\left[e^{-\sigma t} f(t)U(t)\right]$$

$U(t)$: unit step function

It has less physical meaning, but the probability that the transform exists is higher. It is suitable for solving the **initial value problem**.

$$\text{FT} \quad \mathfrak{F}[g(x)] = \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx = G(f)$$

$$\text{inverse FT} \quad \mathfrak{F}^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f) e^{j2\pi fx} df = g(x)$$

(3) Fourier Cosine Transform

When $g(x)$ is even, the FT is reduced to the Fourier cosine transform.

$$\mathfrak{F}_c[g(x)] = \int_0^{\infty} g(x) \cos(2\pi fx) dx = G_c(f)$$

$$\mathfrak{F}_c^{-1}[G_c(f)] = 4 \int_0^{\infty} G_c(f) \cos(2\pi fx) df = g(x)$$

(4) Fourier Sine Transform

When $g(x)$ is odd, the FT is reduced to the Fourier sine transform.

$$\mathfrak{F}_s[g(x)] = \int_0^{\infty} g(x) \sin(2\pi fx) dx = G_s(f)$$

$$\mathfrak{F}_s^{-1}[G_s(f)] = 4 \int_0^{\infty} G_s(f) \sin(2\pi fx) df = g(x)$$

(5) Hartley Transform

$$\mathfrak{S}_{ha} [g(x)] = \int_{-\infty}^{\infty} g(x) \text{cas}(2\pi fx) dx = G_{ha}(f)$$

where $\text{cas}(x) = \cos(x) + \sin(x)$

$$\mathfrak{S}_{ha}^{-1} [G_{ha}(f)] = \int_{-\infty}^{\infty} G_{ha}(f) \text{cas}(2\pi fx) dx = g(x)$$

real input \rightarrow real output

(6) Mellin Transform

$$\mathfrak{S}_M [g(x)] = \int_0^{\infty} g(x) x^{s-1} dx = G_M(s)$$

If we set $x = \exp(-t)$, $\frac{dx}{dt} = -x$, $dt = -x^{-1} dx$, then

$$G_M(s) = \int_{-\infty}^{\infty} g(e^{-t}) e^{-st} dt$$

It is the two-sided Laplace transform of $g(e^{-t})$

It is suitable to deal with the fractal (碎形), since if

$$g(cx) = \lambda g(x)$$

then

$$g(e^{-t+\ln c}) = \lambda g(e^{-t})$$

(7) Hilbert Transform

$$g_H(x) = \mathfrak{F}^{-1} \{ \mathfrak{F}[g(x)] H(f) \}$$

where

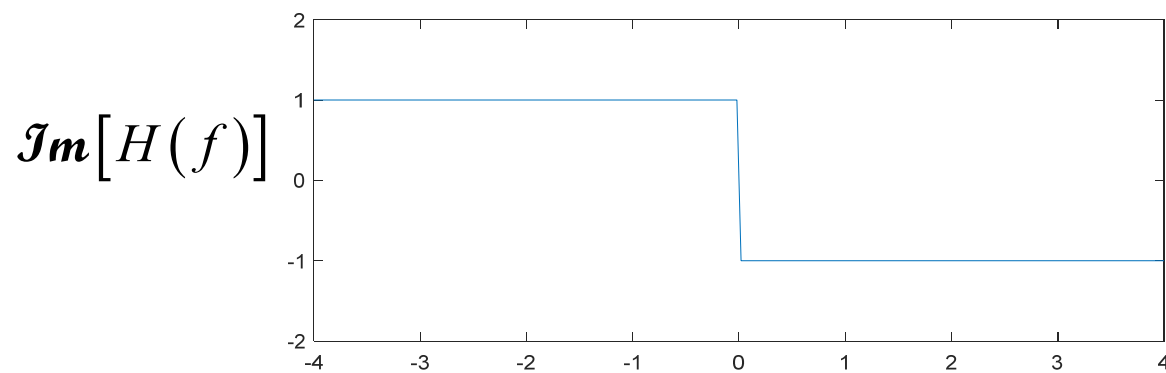
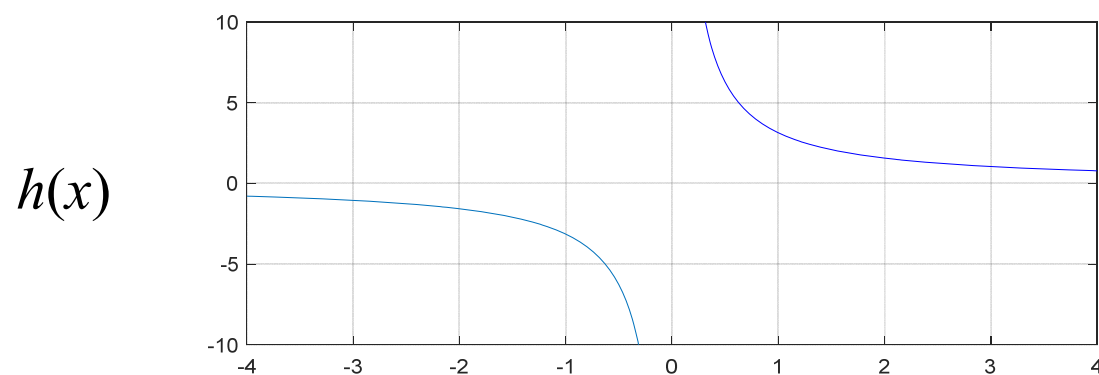
$$H(f) = \begin{cases} -j & \text{if } f > 0 \\ 0 & \text{if } f = 0 \\ j & \text{if } f < 0 \end{cases}$$

Note that

$$\begin{aligned} \mathfrak{F}^{-1}[H(f)] &= j \int_{-\infty}^0 e^{j2\pi fx} df - j \int_0^{\infty} e^{j2\pi fx} df \\ &= \lim_{\sigma \rightarrow 0} \left[j \int_{-\infty}^0 e^{\sigma f} e^{j2\pi fx} df - j \int_0^{\infty} e^{-\sigma f} e^{j2\pi fx} df \right] \\ &= \lim_{\sigma \rightarrow 0} \left[j \frac{e^{\sigma f} e^{j2\pi xf}}{\sigma + j2\pi x} \Big|_{-\infty}^0 - j \frac{e^{-\sigma f} e^{j2\pi xf}}{-\sigma + j2\pi x} \Big|_0^{\infty} \right] \\ &= \lim_{\sigma \rightarrow 0} \left[j \frac{1}{\sigma + j2\pi x} + j \frac{1}{-\sigma + j2\pi x} \right] = \lim_{\sigma \rightarrow 0} \frac{4\pi x}{\sigma^2 + 4\pi^2 x^2} = \frac{1}{\pi x} \end{aligned}$$

Therefore,

$$g_H(x) = g(x) * \overset{h(x)}{\frac{1}{\pi x}} = \int_{-\infty}^{\infty} \frac{g(\tau)}{\pi(x-\tau)} d\tau$$



$$g(x) = \cos(2\pi kx) \xrightarrow{\text{Hilbert}} g_H(x) = \sin(2\pi kx)$$

$k \neq 0$

$$g(x) = \sin(2\pi kx) \xrightarrow{\text{Hilbert}} g_H(x) = -\cos(2\pi kx)$$

$k \neq 0$

(Proof): If $g(x) = \cos(2\pi kx)$

$$\text{then } G(f) = \frac{1}{2}\delta(f-k) + \frac{1}{2}\delta(f+k)$$

$$H(f)G(f) = \frac{-j}{2}\delta(f-k) + \frac{j}{2}\delta(f+k)$$

$$g_H(x) = \mathfrak{F}^{-1}\{G(f)H(f)\} = \sin(2\pi kx)$$

(8) Analytic Signal

$$g_a(x) = g(x) + jg_H(x)$$

reconstruction:

$$g(x) = \mathcal{Re}\{g_a(x)\}$$

if $g(x)$ is real

Since

$$\mathcal{F}[g_a(x)] = \mathcal{F}[g(x)] + j\mathcal{F}[g_H(x)]$$

$$G_a(f) = G(f) + jH(f)G(f) = (1 + jH(f))G(f)$$

$$1 + jH(f) = \begin{cases} 2 & \text{if } f > 0 \\ 1 & \text{if } f = 0 \\ 0 & \text{if } f < 0 \end{cases}$$

we have

$$G_a(f) = \begin{cases} 2G(f) & \text{if } f > 0 \\ G(f) & \text{if } f = 0 \\ 0 & \text{if } f < 0 \end{cases}$$

(It is called the 'single sided band signal')

(halve the bandwidth)

(9) Fractional Fourier Transform

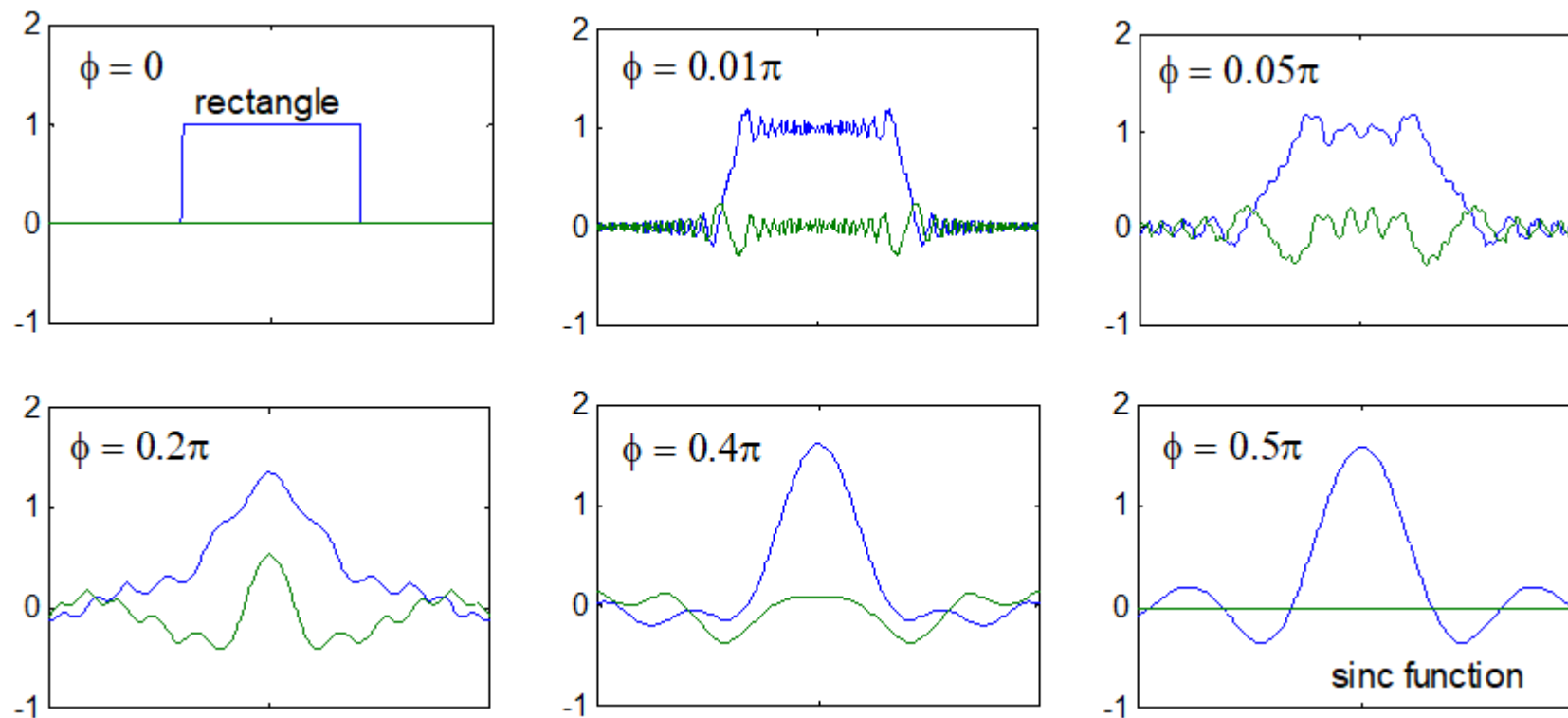
$$X_{\phi}(u) = \sqrt{1 - j \cot \phi} e^{j\pi \cot \phi \cdot u^2} \int_{-\infty}^{\infty} e^{-j2\pi \cdot \csc \phi \cdot ut} e^{j\pi \cdot \cot \phi \cdot t^2} x(t) dt$$

When $\phi = 0.5\pi$, the FRFT becomes the FT.

Physical meaning: Performing the FT a times, $\phi = 0.5a\pi$

- [Ref] L. B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE Trans. Signal Processing*, vol. 42, no. 11, pp. 3084-3091, Nov. 1994.
- [Ref] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, New York, John Wiley & Sons, 2000.
- [Ref] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *J. Inst. Maths. Applics.*, vol. 25, pp. 241-265, 1980.

Fractional Fourier transforms for a rectangular function



blue lines: real parts; green lines: imaginary part

(10) Linear Canonical Transform

$$X_{(a,b,c,d)}(u) = \sqrt{\frac{1}{jb}} e^{j\pi \frac{d}{b} u^2} \int_{-\infty}^{\infty} e^{-j2\pi \frac{1}{b} ut} e^{j\pi \frac{a}{b} t^2} x(t) dt$$

where $ad - bc = 1$

when $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \implies$ Fourier transform

when $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \implies$ fractional Fourier transform

[Ref] K. B. Wolf, “*Integral Transforms in Science and Engineering*,” Ch. 9: Canonical transforms, New York, Plenum Press, 1979.

Summary of Transforms

Fourier Transform	$G(f) = \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx$
2D Fourier Transform	$G(f, h) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi fx} e^{-j2\pi hy} dx dy$
Hankel Transform	$G(s) = 2\pi \int_0^{\infty} J_0(2\pi sr) r g(r) dr$
Two-Sided Laplace Transform	$G(s) = \int_{-\infty}^{\infty} e^{-st} g(t) dt$
One-Sided Laplace Transform	$G(s) = \int_0^{\infty} e^{-st} g(t) dt$
Fourier Cosine Transform	$G_c(f) = \int_0^{\infty} g(x) \cos(2\pi fx) dx$
Fourier Sine Transform	$G_s(f) = \int_0^{\infty} g(x) \sin(2\pi fx) dx$

Hartley Transform	$G_{ha}(f) = \int_{-\infty}^{\infty} g(x) \text{cas}(2\pi fx) dx$
Mellin Transform	$G_M(s) = \int_0^{\infty} g(x) x^{s-1} dx$
Hilbert Transform	$g_H(x) = \mathfrak{I}^{-1} \{ \mathfrak{I}[g(x)] H(f) \}$ $H(f) = -j \text{ for } f > 0,$ $H(f) = j \text{ for } f < 0, \quad H(0) = 0$
Analytic Signal	$g_a(x) = g(x) + jg_H(x)$
Fractional Fourier Transform	$X_{\phi}(u) = \sqrt{1 - j \cot \phi} e^{j\pi \cot \phi \cdot u^2}$ $\int_{-\infty}^{\infty} e^{-j2\pi \cdot \csc \phi \cdot ut} e^{j\pi \cdot \cot \phi \cdot t^2} x(t) dt$
Linear Canonical Transform	$X_{(a,b,c,d)}(u) = \sqrt{1/jb} e^{j\pi du^2/b}$ $\int_{-\infty}^{\infty} e^{-j2\pi ut/b} e^{j\pi at^2/b} x(t) dt$