7. Discrete Vector Set Approximation

$$\mathbf{Ax} \cong \mathbf{y}$$

A and **y** are known.

Problem: How do we find **x** such that

$$\left\| \mathbf{y} - \mathbf{Ax} \right\| \qquad (L_2 \text{ norm of } \mathbf{y} - \mathbf{Ax})$$

is minimized?

- Discrete Vector Set Expansion
  - Orthogonal
    - Complete (Case 1) (Sec. 7-1-2)
    - Incomplete (Case 2) (Sec. 7-1-3)
    - Discrete Orthogonal Transforms (Secs. 6-7, 7.1.1)
    - Discrete Orthogonal Polynomials (Sec. 7-4)
  - Non-Orthogonal
    - Independent Set (Sec. 7-2)
      - Complete (Case 3) (Secs. 7.2.1, 7.2.2)
      - Incomplete (Case 4) (Secs. 7.2.2, 7.2.3)
    - Dependent Set (Case 5) (Sec. 7-3)

7.1 Discrete Orthogonal Vector Set Expansion

7.1.1 Discrete Orthogonal Matrix

[Orthogonal (Column Form)]

$$\mathbf{A} = \begin{bmatrix} \phi_1[1] & \phi_2[1] & \phi_3[1] & \cdots & \phi_N[1] \\ \phi_1[2] & \phi_2[2] & \phi_3[2] & \cdots & \phi_N[2] \\ \phi_1[3] & \phi_2[3] & \phi_3[3] & \cdots & \phi_N[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_1[M] & \phi_2[M] & \phi_3[M] & \cdots & \phi_N[M] \end{bmatrix}$$

If

$$\sum_{m=1}^{M} \phi_n[m]\phi_k^*[m] = \begin{cases} 0 & for\ n \neq k \\ d_n & for\ n = k \end{cases}$$

then

$$\mathbf{A}^{\mathbf{H}}\mathbf{A} = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_N \end{bmatrix}$$

[Orthogonal (Column Form)]

Suppose that **A** is an *M*x*N* matrix. If all the columns of **A** are orthogonal, then

$$\mathbf{A}^{\mathbf{H}}\mathbf{A} = \mathbf{D}$$

where **D** is an *N*x*N* orthogonal matrix. Moreover, if all the columns of **A** are orthonormal, then

$(d_n = 1 \quad for\ all\ n)$

$$\mathbf{A}^{\mathbf{H}}\mathbf{A} = \mathbf{I}$$

where **I** is an *N*x*N* identity matrix.

(Note: An orthonormal matrix is also called a unitary matrix.)

[Orthogonal (Row Form)]

$$\mathbf{A} = \begin{bmatrix} \phi_1[1] & \phi_1[2] & \phi_1[3] & \cdots & \phi_1[N] \\ \hdashline \phi_2[1] & \phi_2[2] & \phi_2[3] & \cdots & \phi_2[N] \\ \hdashline \phi_3[1] & \phi_3[2] & \phi_3[3] & \cdots & \phi_3[N] \\ \hdashline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hdashline \phi_M[1] & \phi_M[2] & \phi_M[3] & \cdots & \phi_M[N] \end{bmatrix}$$

If $$\sum_{n=1}^{N} \phi_m[n]\phi_k^*[n] = \begin{cases} 0 & for\ m \neq k \\ d_m & for\ m = k \end{cases}$$

then $$\mathbf{AA^H} = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_M \end{bmatrix}$$

[Orthogonal (Row Form)]

Suppose that **A** is an MxN matrix. If all the rows of **A** are orthogonal, then

$$\mathbf{A}\mathbf{A}^{\mathbf{H}} = \mathbf{D}$$

where **D** is an MxM orthogonal matrix. Moreover, if all the rows of **A** are orthonormal, then

$$\mathbf{A}\mathbf{A}^{\mathbf{H}} = \mathbf{I}$$

where **I** is an MxM identity matrix.

(Note: If a set of vectors is orthogonal, then these vectors should be linearly independent. Therefore, if the rows of A are orthogonal, then $M \leq N$ should be satisfied.)

orthogonal (row form) $\neq$ orthogonal (column form)
orthonormal (row form) $=$ orthonormal (column form)

[Inverse of an Orthogonal Matrix]

If **A** is a square matrix (i.e., $M = N$)

(1) If all the columns of **A** are orthogonal, $\mathbf{A^H A} = \mathbf{D}$, then

$$\mathbf{A}^{-1} = \mathbf{D}^{-1}\mathbf{A}^{\mathbf{H}}$$

(2) If all the columns of **A** are orthonormal, $\mathbf{A^H A} = \mathbf{I}$, then

$$\mathbf{A}^{-1} = \mathbf{A}^{\mathbf{H}}$$

(3) If all the rows of **A** are orthogonal, $\mathbf{A A^H} = \mathbf{D}$, then

$$\mathbf{A}^{-1} = \mathbf{A}^{\mathbf{H}}\mathbf{D}^{-1}$$

(4) If all the rows of **A** are orthonormal, $\mathbf{A A^H} = \mathbf{I}$, then

$$\mathbf{A}^{-1} = \mathbf{A}^{\mathbf{H}}$$

[Example of Orthogonal Matrix]

- DFT
- Discrete Cosine Transform
- Walsh (Hadamard Transform)

both row-form and column-form orthogonal

- Haar Transform (row-form orthogonal)
- Discrete Orthogonal Polynomial Matrices (row-form orthogonal)

[**Example 1**]

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad \mathbf{W}_4^{-1} = \frac{1}{4}\mathbf{I}\mathbf{W}_4^{H} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & -1/4 \\ 1/4 & -1/4 & -1/4 & 1/4 \\ 1/4 & -1/4 & 1/4 & -1/4 \end{bmatrix}$$

$$\mathbf{W}_4^{\mathbf{H}}\mathbf{W}_4 = 4\mathbf{I}$$

[Duality Property of Orthogonal Matrices]

If all the columns of a square matrix **A** are orthonormal, then all the rows of **A** are orthonormal, too.

(Proof): If

$$\mathbf{A}^{\mathbf{H}}\mathbf{A} = \mathbf{I}$$

then since $\mathbf{A}^{\mathbf{H}} = \mathbf{A}^{-1}$, we have

$$\mathbf{A}\mathbf{A}^{\mathbf{H}} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

Therefore, all the rows of **A** are orthonormal, too.

[**Example 2**] Note that, if

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

then the columns of **A** are orthogonal. However, the rows of **A** are not orthogonal.

If we perform normalization for the columns **A** and obtain **B**:

$$\mathbf{B} = \left[\begin{array}{c:c:c} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{array}\right]$$

then both the columns and the rows of **B** are orthonormal:

$$\mathbf{B}^H\mathbf{B} = \mathbf{I}, \qquad \mathbf{B}\mathbf{B}^H = \mathbf{I}$$

7.1.2 Discrete Orthogonal Vector Set Expansion of the Complete Case (Case 1)

Suppose that $b_1[n]$, $b_2[n]$, …. $b_N[n]$ forms a complete and orthogonal set in C^N:

$$\sum_{n=1}^{N} b_m[n] b_k^*[n] = \begin{cases} 0 & for\ m \neq k \\ d_m & for\ m = k \end{cases}$$

If we want to expand $y[n]$ by a linear combination of $b_m[n]$ (m = 1, 2, …, N):

$$y[n] = \sum_{m=1}^{N} x_m b_m[n]$$

then, analogous to page 277, $$x_m = \frac{\sum_{n=1}^{N} y[n] b_m^*[n]}{\sum_{n=1}^{N} b_m[n] b_m^*[n]}$$

From the view point of the matrix

If

$$\mathbf{A} = \left[\begin{array}{c:c:c:c:c} b_1[1] & b_2[1] & b_3[1] & \cdots & b_N[1] \\ b_1[2] & b_2[2] & b_3[2] & \cdots & b_N[2] \\ b_1[3] & b_2[3] & b_3[3] & \cdots & b_N[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1[N] & b_2[N] & b_3[N] & \cdots & b_N[N] \end{array}\right]$$

$$\mathbf{x} = [x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_N]^T \qquad \mathbf{y} = [y[1] \quad y[2] \quad y[3] \quad \cdots \quad y[N]]^T$$

then the problem can be re-expressed as

$$\mathbf{Ax} = \mathbf{y}$$

Since

$$\mathbf{A}^H\mathbf{A} = \mathbf{D} \qquad \text{where} \quad D[m,n] = \begin{cases} 0 & \text{if } m \neq n \\ \sum_{k=1}^{N} b_m[k] b_m^*[k] & \text{if } m = n \end{cases}$$

we have

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} = \mathbf{D}^{-1}\mathbf{A}^H\mathbf{y}, \qquad x_m = \frac{\sum_{n=1}^{N} y[n] b_m^*[n]}{\sum_{n=1}^{N} b_m[n] b_m^*[n]}$$

[Parseval's Theorem for Discrete Orthogonal Matrix]

If

$$\mathbf{Ax} = \mathbf{y}$$

and the columns of **A** are orthogonal, then

$$\sum_{n=1}^{N} \left| y[n] \right|^2 = \sum_{n=1}^{N} d_n \left| x[n] \right|^2 \quad \text{where} \quad d_n = \sum_{k=1}^{N} \left| A[k,n] \right|^2$$

(Proof):

$$\mathbf{y}^H \mathbf{y} = \mathbf{x}^H \mathbf{A}^H \mathbf{A}\mathbf{x} = \mathbf{x}^H \mathbf{D}\mathbf{x}$$

[Example 3]

Parseval's theorem for the DFT and the Walsh transform:

$$\sum_{n=1}^{N} \left| y[n] \right|^2 = N \sum_{n=1}^{N} \left| x[n] \right|^2$$

Parseval's theorem for the DCT

$$\sum_{n=1}^{N} \left| y[n] \right|^2 = \sum_{n=1}^{N} \left| x[n] \right|^2$$

7.1.3 Discrete Orthogonal Basis Expansion of the Incomplete Case (Case 2)

Suppose that $b_1[n]$, $b_2[n]$, …. $b_M[n]$ forms an incomplete and orthogonal set in C^N but $M < N$:

$$\sum_{n=1}^{N} b_m[n] b_k^*[n] = \begin{cases} 0 & for\ m \neq k \\ d_m & for\ m = k \end{cases}$$

If we want to expand $y[n]$ by a linear combination of $b_m[n]$ (m = 1, 2, …, M):

$$y[n] \cong \sum_{m=1}^{M} x_m b_m[n]$$

then

$$x_m = \frac{\sum_{n=1}^{N} y[n] b_m^*[n]}{\sum_{n=1}^{N} b_m[n] b_m^*[n]}$$

The formulas are similar to those of Case 1, except for that $y[n] =$ is replaced by $y[n] \cong$

Note:

(1) Since $b_1[n]$, $b_2[n]$, …. $b_M[n]$ can be viewed as a subset of a complete and orthogonal set $\{b_1[n], b_2[n], …., b_M[n], b_{M+1}[n], …, b_N[n]\}$, the method to determine the linear combination coefficients x_m is all the same as that of the complete case.

In fact, this is related QR decomposition (Gram--Schmidt):
For any invertible matrix A, c_1 the first column of A is always an orthogonal set.

The first two columns, c_1 and c_2, is not necessarily orthogonal. So we compute a number r such that $c_2 - rc_1$ is orthogonal to c1.
Now that's an orthogonal set of size 2.

For the third column, c3, we compute s and t such that
$c_3 - sc_1 - tc_2$ is orthogonal to the first two vectors.
Repeating this we get A = QR, where Q is unitary and R is upper-triangular

Note:

(2) Determine x_m by $x_m = \sum_{n=1}^{N} y[n] b_m^*[n] / \sum_{n=1}^{N} b_m[n] b_m^*[n]$ can minimize

$$\left\| y[n] - \sum_{m=1}^{M} x_m b_m[n] \right\| = \sqrt{\sum_{n=1}^{N} \left(y[n] - \sum_{m=1}^{M} x_m b_m[n] \right)^2}$$

$$\left\| y[n] - \sum_{m=1}^{M} x_m b_m[n] \right\|^2 = \sum_{n=1}^{N} \left(\sum_{m=M+1}^{N} x_m b_m[n] \right)^2 = \sum_{m=M+1}^{N} d_m \left| x_m \right|^2$$

$$= \sum_{m=1}^{N} d_m \left| x_m \right|^2 - \sum_{m=1}^{M} d_m \left| x_m \right|^2$$

(from Parseval's theorem on page 618) where $d_m = \sum_{n=1}^{N} \left| b_m[n] \right|^2$

$$\left\| y[n] - \sum_{m=1}^{M} x_m b_m[n] \right\|^2 = \sum_{n=1}^{N} \left| y[n] \right|^2 - \sum_{m=1}^{M} d_m \left| x_m \right|^2 = \left\| y[n] \right\|_2^2 - \sum_{m=1}^{M} \left| x_m \right|^2 \left\| b_m[n] \right\|_2^2$$

[**Example 4**] Suppose that

$$\mathbf{y} = [1 \quad 1 \quad 5 \quad 5 \quad 6 \quad 6 \quad 5 \quad 4 \quad 4 \quad 3 \quad 3]^T$$

Try to expand **y** as a linear combination of

$$\mathbf{b_1} = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]^T$$

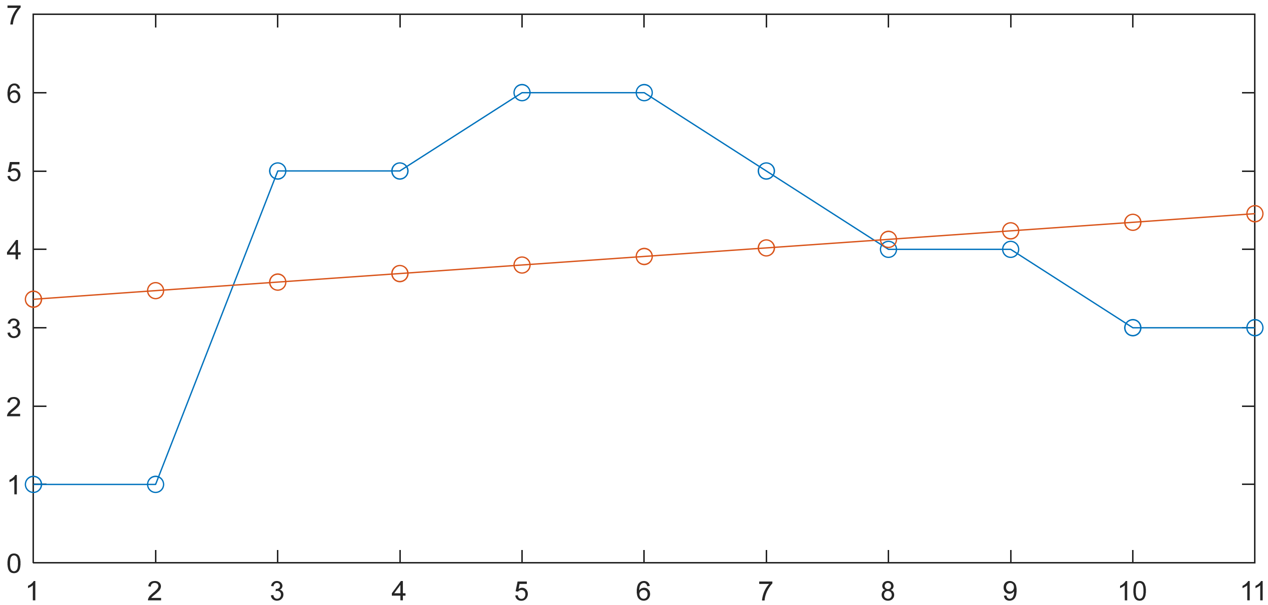
and $\mathbf{b_2} = [-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5]^T$

such that $\left\| \mathbf{y} - x_1 \mathbf{b_1} - x_2 \mathbf{b_2} \right\|$ is minimized.

(Solution): It is obvious that $\mathbf{b_1}$ and $\mathbf{b_2}$ are orthogonal. Therefore,

$$x_1 = \frac{\sum_{n=1}^{11} y[n] b_1^*[n]}{\sum_{n=1}^{11} b_1[n] b_1^*[n]} = \frac{43}{11} \qquad x_2 = \frac{\sum_{n=1}^{11} y[n] b_2^*[n]}{\sum_{n=1}^{11} b_2[n] b_2^*[n]} = \frac{12}{110}$$

$$\mathbf{y} \cong \frac{43}{11} \mathbf{b_1} + \frac{6}{55} \mathbf{b_2}$$



Blue: $\mathbf{y}$ Red: $\frac{43}{11}\mathbf{b_1} + \frac{6}{55}\mathbf{b_2}$

$$\left\|\mathbf{y} - x_1\mathbf{b_1} - x_2\mathbf{b_2}\right\|^2 = \left\|\mathbf{y}\right\|^2 - \left|x_1\right|^2 \left\|\mathbf{b_1}\right\|^2 - \left|x_2\right|^2 \left\|\mathbf{b_2}\right\|^2 = 29.6$$

$$\left\|\mathbf{y} - x_1\mathbf{b_1} - x_2\mathbf{b_2}\right\| = 5.4406$$

7.2 Non-Orthogonal Discrete Basis Expansion

7.2.1 Method 1: Matrix Inverse

Suppose that $\{b_1[n], b_2[n], b_3[n], \ldots\ldots\ldots\ldots, b_N[n]\}$ are linearly independent and complete vector set in C^N but are not orthogonal. (Case 3)

To express $y[n] \in C^N$ by a linear combination of $b_1[n], b_2[n], b_3[n], \ldots\ldots\ldots\ldots, b_N[n]\}$

$$y[n] = \sum_{m=1}^{N} x_m b_m[n]$$

we first construct a matrix **A**:

$$\mathbf{A} = \left[\begin{array}{c:c:c:c:c} b_1[1] & b_2[1] & b_3[1] & \cdots & b_N[1] \\ b_1[2] & b_2[2] & b_3[2] & \cdots & b_N[2] \\ b_1[3] & b_2[3] & b_3[3] & \cdots & b_N[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1[N] & b_2[N] & b_3[N] & \cdots & b_N[N] \end{array}\right]$$

Then,

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_N \end{bmatrix}^T \qquad \mathbf{y} = \begin{bmatrix} y[1] & y[2] & y[3] & \cdots & y[N] \end{bmatrix}^T$$

[Dual Orthogonal]

$\{b_1[n], b_2[n], b_3[n], \ldots\ldots\ldots., b_N[n]\}$ are dual orthogonal to $\{\phi_1[n], \phi_2[n], \phi_3[n], \ldots\ldots\ldots., \phi_N[n]\}$ if:

$$\sum_{m=1}^{N} b_m[n]\phi_k^*[n] = \begin{cases} 0 & \text{if } m \neq k \\ u_m & \text{if } m = k \end{cases}$$

In fact, they are also dual orthonormal if $u_m = 1$.

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

If

$$\mathbf{A} = \begin{bmatrix} b_1[1] & b_2[1] & b_3[1] & \cdots & b_N[1] \\ b_1[2] & b_2[2] & b_3[2] & \cdots & b_N[2] \\ b_1[3] & b_2[3] & b_3[3] & \cdots & b_N[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1[N] & b_2[N] & b_3[N] & \cdots & b_N[N] \end{bmatrix}$$

conjugation

$$\overline{\mathbf{A}}^{-1} = \begin{bmatrix} \phi_1[1] & \phi_1[2] & \phi_1[3] & \cdots & \phi_1[N] \\ \phi_2[1] & \phi_2[2] & \phi_2[3] & \cdots & \phi_2[N] \\ \phi_3[1] & \phi_3[2] & \phi_3[3] & \cdots & \phi_3[N] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_N[1] & \phi_N[2] & \phi_N[3] & \cdots & \phi_N[N] \end{bmatrix}$$

then $\{b_1[n], b_2[n], b_3[n], \ldots\ldots\ldots, b_N[n]\}$ are dual orthonormal to $\{\phi_1[n], \phi_2[n], \phi_3[n], \ldots\ldots\ldots, \phi_N[n]\}$:

$$\sum_{m=1}^{N} b_m[n]\phi_k^*[n] = \begin{cases} 0 & if\ m \neq k \\ 1 & if\ m = k \end{cases}$$

7.2.2 Method 2: Gram-Schmidt (Cases 3, 4)

Suppose that $\{b_1[n], b_2[n], \ldots\ldots, b_M[n]\}$ are linearly independent but not orthogonal. Then we can follow the Gram-Schmidt process to convert it into an orthogonal set $\{a_1[n], a_2[n], \ldots\ldots, a_M[n]\}$ and perform expansion.

(applicable for both complete and incomplete case)

$$a_1[n] = \frac{b_1[n]}{\|b_1[n]\|} \qquad m = 1$$

$$\Downarrow$$

$$g_m[n] = b_m[n] - \sum_{k=1}^{m-1}\left(\sum_{n=1}^{N} b_m[n]\, a_k^*[n]\right) a_k[n]$$

$$\Downarrow$$

$$a_m[n] = \frac{g_m[n]}{\|g_m[n]\|}$$

$$\Downarrow$$

$$m = m+1$$

Find $x_1, x_2, \ldots, x_M$ to minimize $\left\| \mathbf{y} - x_1\mathbf{b_1} - x_2\mathbf{b_2} - \cdots - x_M\mathbf{b_M} \right\|$

by the Gram-Schmidt method.

Step 1: Convert $\{b_1[n], b_2[n], \ldots\ldots, b_M[n]\}$ into an orthogonal set $\{a_1[n], a_2[n], \ldots\ldots, a_M[n]\}$ by the Gram-Schmidt method.

Step 2: Expand $y[n]$ by $\{a_1[n], a_2[n], \ldots\ldots, a_M[n]\}$

$$y[n] \cong \sum_{m=1}^{M} z_m a_m[n] \qquad z_m = \sum_{n=1}^{N} y[n] a_m^*[n] \qquad \text{(from page 619)}$$

Step 3: If

$$a_k[n] \cong \sum_{m=1}^{k} c_{k,m} b_m[n]$$

then

$$y[n] \cong \sum_{k=1}^{M} z_k \sum_{m=1}^{k} c_{k,m} b_m[n] = \sum_{m=1}^{M} \sum_{k=m}^{M} z_k c_{k,m} b_m[n] = \sum_{m=1}^{M} x_m b_m[n]$$

$$x_m = \sum_{k=m}^{M} z_k c_{k,m}$$

[Example 1] Suppose that

$$\mathbf{y} = \begin{bmatrix} 2 & 3 & 3 & 4 & 5 & 4 & 5 \end{bmatrix}^T$$

Try to express $\mathbf{y}$ as $x_1\mathbf{b_1} + x_2\mathbf{b_2} + x_3\mathbf{b_3}$ where

$$\mathbf{b_1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

$$\mathbf{b_2} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}^T$$

$$\mathbf{b_3} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}^T$$

such that

$$\left\| \mathbf{y} - x_1\mathbf{b_1} - x_2\mathbf{b_2} - x_3\mathbf{b_3} \right\| \text{ is minimized}$$

using the Gram-Schmidt method.

(Solution):

$$\mathbf{a_1} = \frac{\mathbf{b_1}}{\|\mathbf{b_1}\|} = \frac{1}{\sqrt{7}}\mathbf{b_1} = \frac{1}{\sqrt{7}}[1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$$

$$\mathbf{g_2} = \mathbf{b_2} - \mathbf{a_1}\sum_{n=1}^{7} b_2[n]a_1[n] = \mathbf{b_2} - 4\sqrt{7}\mathbf{a_1} = \mathbf{b_2} - 4\mathbf{b_1}$$

$$= [-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3]$$

$$\mathbf{a_2} = \frac{\mathbf{g_2}}{\|\mathbf{g_2}\|} = \frac{\mathbf{g_2}}{2\sqrt{7}} = -\frac{2}{\sqrt{7}}\mathbf{b_1} + \frac{1}{2\sqrt{7}}\mathbf{b_2} = \frac{1}{2\sqrt{7}}[-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3]$$

$$\mathbf{g_3} = \mathbf{b_3} - \mathbf{a_1}\sum_{n=1}^{7} b_3[n]a_1[n] - \mathbf{a_2}\sum_{n=1}^{7} b_3[n]a_2[n] = \mathbf{b_3} - \frac{1}{\sqrt{7}}\mathbf{a_1} - 0\mathbf{a_2} = \mathbf{b_3} - \frac{1}{7}\mathbf{b_1}$$

$$= \frac{2}{7}[3 \quad -4 \quad 3 \quad -4 \quad 3 \quad -4 \quad 3]$$

$$\mathbf{a_3} = \frac{\mathbf{g_3}}{\|\mathbf{g_3}\|} = \frac{7\mathbf{g_3}}{4\sqrt{21}} = \frac{-1}{4\sqrt{21}}\mathbf{b_1} + \frac{7}{4\sqrt{21}}\mathbf{b_3} = \frac{1}{2\sqrt{21}}[3 \quad -4 \quad 3 \quad -4 \quad 3 \quad -4 \quad 3]$$

Since

$$\sum_{n=1}^{7} y[n]a_1[n] = \frac{26}{\sqrt{7}} \qquad \sum_{n=1}^{7} y[n]a_2[n] = \frac{13}{2\sqrt{7}}$$

$$\sum_{n=1}^{7} y[n]a_3[n] = \frac{1}{2\sqrt{21}}$$

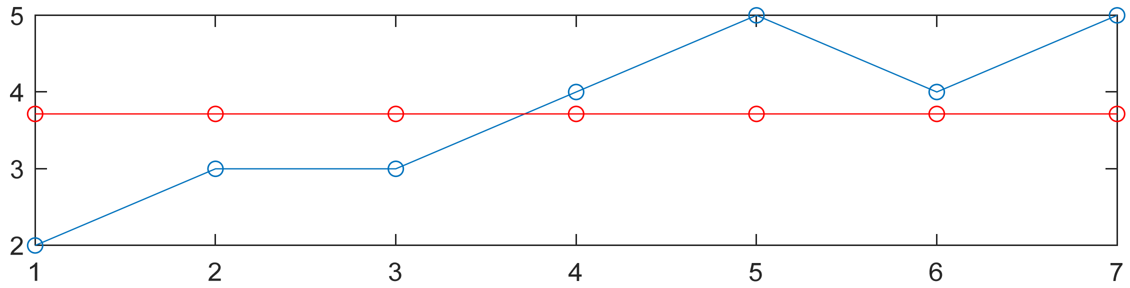
Therefore

$$y[n] \cong \frac{26}{\sqrt{7}} a_1[n] + \frac{13}{2\sqrt{7}} a_2[n] + \frac{1}{2\sqrt{21}} a_3[n]$$

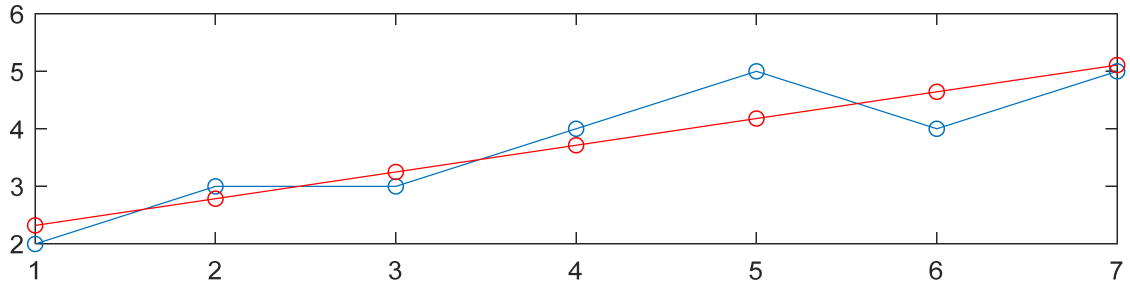
$$y[n] \cong \frac{311}{168} b_1[n] + \frac{13}{28} b_2[n] + \frac{1}{24} b_3[n]$$

$$= \begin{bmatrix} \frac{99}{42} & \frac{115}{42} & \frac{138}{42} & \frac{154}{42} & \frac{177}{42} & \frac{193}{42} & \frac{216}{42} \end{bmatrix}$$

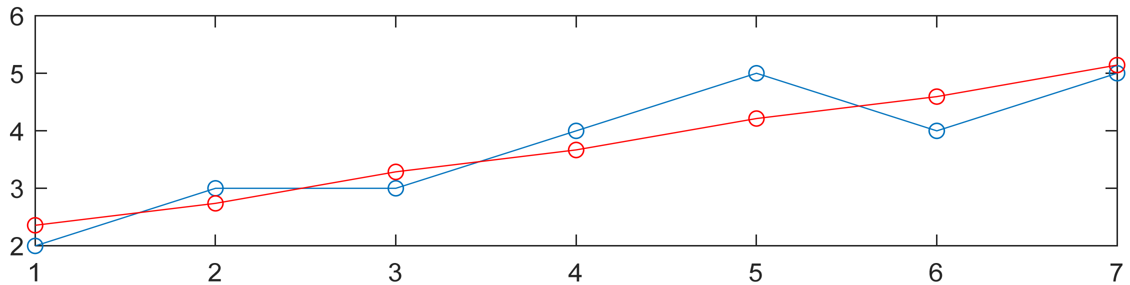
$$y[n] \cong \frac{26}{\sqrt{7}} a_1[n]$$



$$y[n] \cong \frac{26}{\sqrt{7}} a_1[n] + \frac{13}{2\sqrt{7}} a_2[n]$$



$$y[n] \cong \frac{26}{\sqrt{7}} a_1[n] + \frac{13}{2\sqrt{7}} a_2[n] + \frac{1}{2\sqrt{21}} a_3[n]$$



7.2.3 Method 3: Least Square Approximation

Suppose that $\{b_1[n], b_2[n], \ldots\ldots, b_M[n]\}$ are real and linearly independent but not orthogonal and incomplete. If we want to find x_m such that

$$E = \left\| \mathbf{y} - x_1\mathbf{b_1} - x_2\mathbf{b_2} - \cdots - x_M\mathbf{b}_M \right\|$$

is minimized, we can also apply the least square approximation method.

$$E^2 = \sum_{n=1}^{N}\left(y[n] - \sum_{k=1}^{M} x_k b_k[n] \right)^2$$

$$\frac{\partial}{\partial x_m}E^2 = \sum_{n=1}^{N}\left[\frac{\partial}{\partial x_m}\left(y[n] - \sum_{k=1}^{M} x_k b_k[n] \right)\right] 2\left(y[n] - \sum_{k=1}^{M} x_k b_k[n] \right)$$

$$= \sum_{n=1}^{N} -2b_m[n]\left(y[n] - \sum_{k=1}^{M} x_k b_k[n] \right)$$

$$= -2\sum_{n=1}^{N} b_m[n]y[n] + 2\sum_{k=1}^{M} x_k \sum_{n=1}^{N} b_m[n]b_k[n]$$

Therefore, if we want

$$\frac{\partial}{\partial x_m} E^2 = 0 \qquad \text{for } m = 1, 2, \ldots, M$$

then

$$\boxed{\sum_{k=1}^{M} x_k \sum_{n=1}^{N} b_m[n] b_k[n] = \sum_{n=1}^{N} b_m[n] y[n]} \qquad \text{for } m = 1, 2, \ldots, M$$

Therefore,

$$\mathbf{C}\mathbf{x} = \mathbf{z} \qquad\qquad \mathbf{x} = \mathbf{C}^{-1}\mathbf{z}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_M \end{bmatrix}^T \quad \mathbf{z} = \begin{bmatrix} \sum_{n=1}^{N} b_1[n] y[n] & \sum_{n=1}^{N} b_2[n] y[n] & \cdots & \sum_{n=1}^{N} b_M[n] y[n] \end{bmatrix}^T$$

$$\mathbf{C} = \begin{bmatrix} \sum_{n=1}^{N} b_1[n] b_1[n] & \sum_{n=1}^{N} b_1[n] b_2[n] & \cdots & \sum_{n=1}^{N} b_1[n] b_M[n] \\ \sum_{n=1}^{N} b_2[n] b_1[n] & \sum_{n=1}^{N} b_2[n] b_2[n] & \cdots & \sum_{n=1}^{N} b_2[n] b_M[n] \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{n=1}^{N} b_M[n] b_1[n] & \sum_{n=1}^{N} b_M[n] b_2[n] & \cdots & \sum_{n=1}^{N} b_M[n] b_M[n] \end{bmatrix}$$

Also note that, if

$$\mathbf{A} = \begin{bmatrix} b_1[1] & b_2[1] & b_3[1] & \cdots & b_M[1] \\ b_1[2] & b_2[2] & b_3[2] & \cdots & b_M[2] \\ b_1[3] & b_2[3] & b_3[3] & \cdots & b_M[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_1[N] & b_2[N] & b_3[N] & \cdots & b_M[N] \end{bmatrix}$$

then

$$\mathbf{C} = \mathbf{A}^{\mathrm{T}}\mathbf{A}$$

$$\mathbf{z} = \mathbf{A}^{T}\mathbf{y} \qquad \text{where} \qquad \mathbf{y} = \begin{bmatrix} y[1] & y[2] & \cdots & y[M] \end{bmatrix}^{T}$$

Therefore, from $\mathbf{x} = \mathbf{C}^{-1}\mathbf{z}$, we have

$$\boxed{\mathbf{x} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{y}}$$

[Example 2] Suppose that

$$\mathbf{y} = \begin{bmatrix} 2 & 3 & 3 & 4 & 5 & 4 & 5 \end{bmatrix}^T$$

Try to express $\mathbf{y}$ as $x_1\mathbf{b_1} + x_2\mathbf{b_2} + x_3\mathbf{b_3}$ where

$$\mathbf{b_1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

$$\mathbf{b_2} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}^T$$

$$\mathbf{b_3} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}^T$$

such that

$$\left\| \mathbf{y} - x_1\mathbf{b_1} - x_2\mathbf{b_2} - x_3\mathbf{b_3} \right\| \quad \text{is minimized}$$

using the least square approximation method.

First, we construct the matrix

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccc} \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \end{array} \\ \left[\begin{array}{c:c:c} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \\ 1 & 4 & -1 \\ 1 & 5 & 1 \\ 1 & 6 & -1 \\ 1 & 7 & 1 \end{array}\right] \end{array}$$

Since

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 7 & 28 & 1 \\ 28 & 140 & 4 \\ 1 & 4 & 7 \end{bmatrix} \qquad (\mathbf{A}^T\mathbf{A})^{-1} = \frac{1}{336}\begin{bmatrix} 241 & -48 & -7 \\ -48 & 12 & 0 \\ -7 & 0 & 49 \end{bmatrix}$$

$$(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \frac{1}{168}\begin{bmatrix} 93 & 76 & 45 & 28 & -3 & -20 & -51 \\ -18 & -12 & -6 & 0 & 6 & 12 & 18 \\ 21 & -28 & 21 & -28 & 21 & -28 & 21 \end{bmatrix}$$

therefore, from $\mathbf{x} = (\mathbf{A}^{\mathbf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{y}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{168}\begin{bmatrix} 93 & 76 & 45 & 28 & -3 & -20 & -51 \\ -18 & -12 & -6 & 0 & 6 & 12 & 18 \\ 21 & -28 & 21 & -28 & 21 & -28 & 21 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \\ 5 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 311/168 \\ 13/28 \\ 1/24 \end{bmatrix}$$

$$y[n] \cong \frac{311}{168} b_1[n] + \frac{13}{28} b_2[n] + \frac{1}{24} b_3[n]$$

$$= \begin{bmatrix} \frac{99}{42} & \frac{115}{42} & \frac{138}{42} & \frac{154}{42} & \frac{177}{42} & \frac{193}{42} & \frac{216}{42} \end{bmatrix}$$

(the same as Example 1)

7.3 Generalized Inverse

Remember that, for the case where the vector sets are linearly independent and complete, one can use the matrix inverse method (pages 624, 625) to determine the linear combination coefficients:

If $\quad \mathbf{y} = \mathbf{A}\mathbf{x}$

then $\quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$

However, when

(1) The vector sets are not linearly independent (i.e., $\det(\mathbf{A}) = 0$)

(2) The number of vector sets is smaller than the vector length

(i.e., $\mathbf{A}$ is not a square matrix)

$\mathbf{A}^{-1}$ is hard to be determined.

[Definition] Generalized Inverse

For an matrix $\mathbf{A}$, if there is a matrix $\mathbf{A}^+$ such that

$$\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$$

then $\mathbf{A}^+$ is called the generalized inverse of $\mathbf{A}$.

We always use $\mathbf{A}^+$ to denote the generalized inverse of $\mathbf{A}$.

[Additional Definitions for Generalized Inverse]

(1) $\mathbf{A}\mathbf{A}^{+}\mathbf{A} = \mathbf{A}$

(2) $\mathbf{A}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{A}^{+}$

(3) $\left(\mathbf{A}\mathbf{A}^{+}\right)^{H} = \mathbf{A}\mathbf{A}^{+}$

(4) $\left(\mathbf{A}^{+}\mathbf{A}\right)^{H} = \mathbf{A}^{+}\mathbf{A}$

If (1) is satisfied, then $\mathbf{A}^{+}$ is called the generalized inverse of $\mathbf{A}$.

If (1) and (2) are satisfied, then $\mathbf{A}^{+}$ is called the reflexive generalized inverse of $\mathbf{A}$.

If (1), (2), (3), and (4) are all satisfied, then $\mathbf{A}^{+}$ is called the pseudo inverse of $\mathbf{A}$.

pseudo inverse $\subset$ reflexive generalized inverse $\subset$ generalized inverse

$$size(\mathbf{A}) = M \times N$$

- $M = N$
  - $\det(\mathbf{A}) \neq 0$ (Case 1)
    (columns are independent)
  - $\det(\mathbf{A}) = 0$ (Case 3)
    (columns are dependent)
- $M > N$
  - columns are independent (Case 2)
  - columns are dependent (Case 4)
- $M < N$ — columns must be dependent (Case 4)

[Case 1] If $\mathbf{A}$ is a square matrix and all the columns of $\mathbf{A}$ are linearly independent, then

$$\mathbf{A}^{+} = \mathbf{A}^{-1}$$

Note that, in this case,

$$\mathbf{A}\mathbf{A}^{+}\mathbf{A} = \mathbf{A}\mathbf{I} = \mathbf{A}$$

[Case 2] If $\mathbf{A}$ is an MxN matrix, $N < M$, and all the columns of $\mathbf{A}$ are linearly independent, then

$$\mathbf{A}^{+} = (\mathbf{A}^{\mathbf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}$$

Note that, in this case,

$$\mathbf{A}\mathbf{A}^{+}\mathbf{A} = \mathbf{A}(\mathbf{A}^{\mathbf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{A} = \mathbf{A}$$

Also note that it is the same as the least square approximation method introduced in subsection 7-2-3

[Case 3] Suppose that **A** is a square matrix and some columns of **A** are dependent. Then, in this case

$$\det(\mathbf{A}) = 0$$

and some of the eigenvalues of **A** are equal to zero.

[Case 3-1] Suppose that the eigenvector-eigenvalue decomposition of **A** exists

$$\mathbf{A} = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}$$

where **D** is a diagonal matrix where the diagonal entries are the eigenvalues of **A**.

$$D[m,n] = \begin{cases} \lambda_n & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

Then, the generalized inverse of **A** is

$$\mathbf{A}^{+} = \mathbf{E}\mathbf{D}^{+}\mathbf{E}^{-1} \qquad \text{where} \qquad D^{+}[m,n] = \begin{cases} 1/\lambda_n & \text{if } m = n \text{ and } \lambda_n \neq 0 \\ 0 & \text{if } m = n \text{ and } \lambda_n = 0 \\ 0 & \text{if } m \neq n \end{cases}$$

Note that, in this case,

$$\mathbf{AA^{+}A} = \mathbf{EDE^{-1}ED^{+}E^{-1}EDE^{-1}} = \mathbf{EDD^{+}DE^{-1}}$$

If

$$\mathbf{S} = \mathbf{DD^{+}D}$$

then

$$S[n,n] = \lambda_n \lambda_n^{-1} \lambda_n = \lambda_n \qquad \text{if } \lambda_n \neq 0$$

$$S[n,n] = \lambda_n 0 \lambda_n = 0 \qquad \text{if } \lambda_n = 0$$

$$S[m,n] = 0 \qquad \text{if } m \neq n$$

Therefore,

$$\mathbf{S} = \mathbf{DD^{+}D} = \mathbf{D}$$

$$\mathbf{AA^{+}A} = \mathbf{EDE^{-1}} = \mathbf{A}$$

[Example 1] Suppose that

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Determine the generalized inverse of **A**.

(Solution): The eigenvalues of **A** is $\lambda = 0, 1, 3$

The eigenvectors are

$[1 \quad -1 \quad 1]^T$ corresponding to $\lambda = 0$

$[1 \quad 0 \quad -1]^T$ corresponding to $\lambda = 1$

$[1 \quad 2 \quad 1]^T$ corresponding to $\lambda = 3$

Therefore, the eigenvector-eigenvalue decomposition of **A** is

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}^{-1}$$

Since

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 & 1/3 \\ 1/2 & 0 & -1/2 \\ 1/6 & 2/6 & 1/6 \end{bmatrix}$$

we have

$$\mathbf{A}^{+} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 & 1/3 \\ 1/2 & 0 & -1/2 \\ 1/6 & 2/6 & 1/6 \end{bmatrix}$$

$$\mathbf{A}^{+} = \begin{bmatrix} 5/9 & 1/9 & -4/9 \\ 1/9 & 2/9 & 1/9 \\ -4/9 & 1/9 & 5/9 \end{bmatrix}$$

One can show that

$$\mathbf{A}\mathbf{A}^{+}\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \mathbf{A}$$

[Case 3-2]

[Generalized Inverse when the Eigenvectors are not Complete]

If $\mathbf{A} = \mathbf{EDE}^{-1}$ where $\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}_\mathbf{K} \end{bmatrix}$

$$\mathbf{D}_\mathbf{k} = \lambda_k, \quad \begin{bmatrix} \lambda_k & 0 & \cdots & 0 \\ 0 & \lambda_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}$$

then

$\mathbf{A}^+ = \mathbf{ED}^+\mathbf{E}^{-1}$ where $\mathbf{D}^+ = \begin{bmatrix} \mathbf{D}_1^+ & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2^+ & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{D}_\mathbf{K}^+ \end{bmatrix}$

When $\lambda_k \neq 0$

if $\mathbf{D_k} = \lambda_k$, then $\mathbf{D_k^+} = 1/\lambda_k$,

(1) If $\lambda_k \neq 0$

$$\mathbf{D_k} = \begin{bmatrix} \lambda_k & 0 & \cdots & 0 \\ 0 & \lambda_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix}, \text{ then } \mathbf{D_k^+} = \begin{bmatrix} 1/\lambda_k & 0 & \cdots & 0 \\ 0 & 1/\lambda_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\lambda_k \end{bmatrix},$$

(2) If

$$\mathbf{D_k} = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & \cdots & 0 & \lambda_k \end{bmatrix}, \mathbf{D_k^+} = \begin{bmatrix} \lambda_k^{-1} & -\lambda_k^{-2} & \lambda_k^{-3} & \cdots & (-1)^M \lambda_k^{-M} \\ 0 & \lambda_k^{-1} & -\lambda_k^{-2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \lambda_k^{-3} \\ 0 & 0 & \cdots & \lambda_k^{-1} & -\lambda_k^{-2} \\ 0 & 0 & \cdots & 0 & \lambda_k^{-1} \end{bmatrix}$$

(suppose that the size of $\mathbf{D_k}$ is MxM)

One can show that $\mathbf{D_k}\mathbf{D_k^+} = \mathbf{I}$

When $\lambda_k = 0$

if $\mathbf{D_k} = \lambda_k$, then $\mathbf{D_k^+} = 0$,

(3) If $\lambda_k = 0$ $\mathbf{D_k} = \begin{bmatrix} \lambda_k & 0 & \cdots & 0 \\ 0 & \lambda_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix}$, then $\mathbf{D_k^+} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$,

(4) If $\mathbf{D_k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$, then $\mathbf{D_k^+} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$

Note that if

$$\mathbf{D_k} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad \mathbf{D_k^+} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

then $$\mathbf{D_k^+ D_k} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{D_k D_k^+ D_k} = \mathbf{D_k}$$

[Case 4] Suppose that **A** is an MxN matrix, when
(i) $M < N$ or
(ii) $N < M$ but some column vectors are not linearly independent,
the methods introduced in this chapter cannot be applied.

We can use the singular value decomposition (SVD) method introduced in Section 8.1 to solve the generalized inverse problem in Cases 1, 2, 3, and 4.

7.4 Discrete Orthogonal Polynomials

(只教不考)

[Definition of Discrete Orthogonal Polynomials]

Suppose that there is a set of discrete functions as follows

$$P_m[n] = \sum_{k=0}^{m} c_{m,k}(n)_k \qquad m = 0, 1, 2, \ldots\ldots$$

where $(n)_k$ is called the falling factorial function:

$$(n)_0 = 1, \qquad (n)_1 = n, \qquad (n)_2 = n(n-1),$$

$$(n)_k = n(n-1)(n-2)\cdots\cdots(n-k+1)$$

If

$$\boxed{\sum_{n=n_0}^{n_1} w[n]P_m[n]P_s[n] = 0 \quad when\ m \neq s}$$

then we call $\{P_0[n], P_1[n], P_2[n], \ldots\ldots\}$ a discrete orthogonal polynomial set within $n \in [n_0, n_1]$ with the weight $w[n]$

Note that since

$$span\{(n)_0, (n)_1, (n)_2, \cdots, (n)_m\} = span\{1, n, n^2, \cdots, n^m\}$$

therefore, $P_m[n]$ can also be expressed as a linear combination of 1, n, n^2, …, n^m.

[Discrete Legendre Polynomials]

$$w[n] = 1 \qquad n \in [0, N]$$

The Discrete Legendre Polynomial of Order m

$$\boxed{P_m[n] = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{m+k}{k} \frac{(n)_k}{(N)_k}}$$

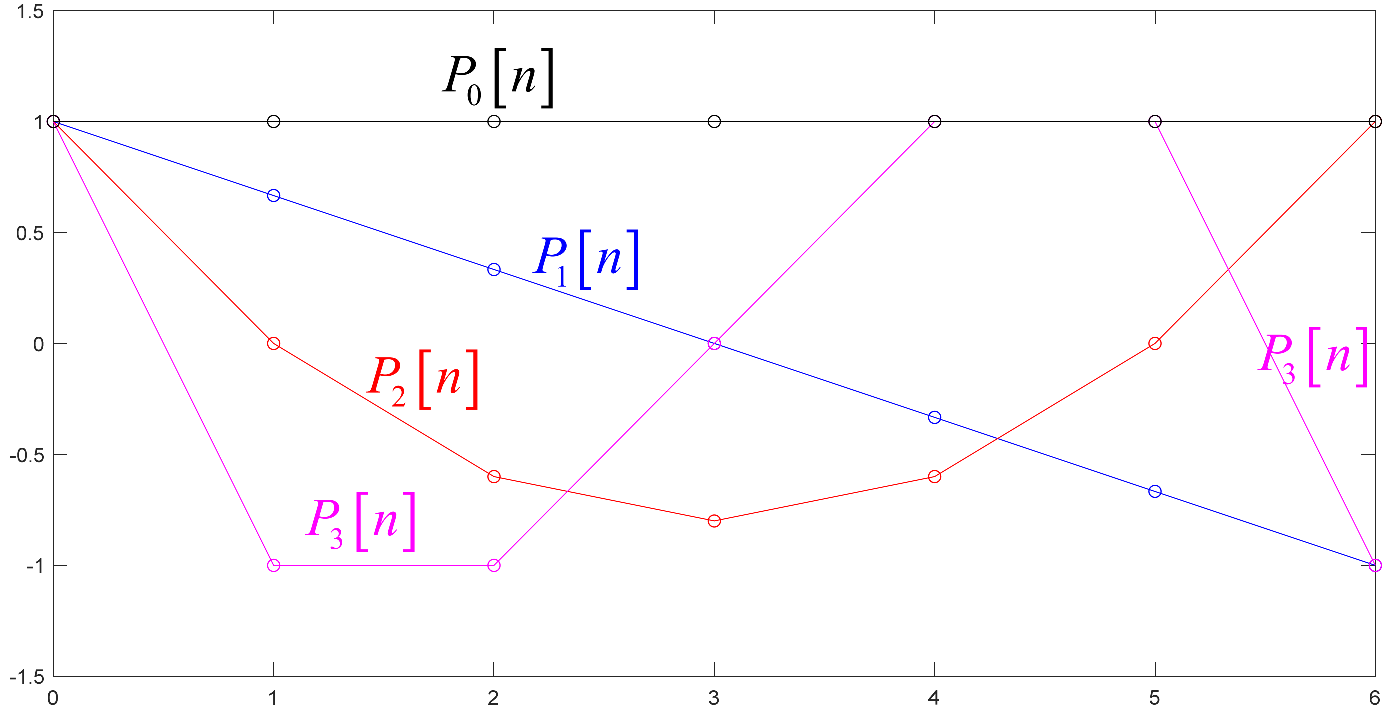
$$\sum_{n=0}^{N} P_m[n] P_s[n] = \begin{cases} \dfrac{(N+m+1)!(N-m)!}{(2m+1)(N!)^2} & if \ m = s \\ 0 & if \ m \neq s \end{cases}$$

$$P_0[n] = 1 \qquad P_1[n] = 1 - 2\frac{n}{N}$$

$$P_2[n] = 1 - 6\frac{n}{N} + 6\frac{(n)_2}{(N)_2} \qquad P_3[n] = 1 - 12\frac{n}{N} + 30\frac{(n)_2}{(N)_2} - 20\frac{(n)_3}{(N)_3}$$

$$P_4[n] = 1 - 20\frac{n}{N} + 90\frac{(n)_2}{(N)_2} - 140\frac{(n)_3}{(N)_3} + 70\frac{(n)_4}{(N)_4}$$

$N = 6$



[Hahn Polynomials]

When $\alpha = \beta =$ -1/2, it is analogous to the continuous Chebyshev polynomial on page 315.

Two extra parameters: α, β

$$w[n] = \binom{n+\alpha}{n}\binom{N-n+\beta}{N-n} \qquad n \in [0, N]$$

If α or β is not an integer, it can still be defined:

$$w[n] = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \frac{\Gamma(N-n+\beta+1)}{\Gamma(N-n+1)\Gamma(\beta+1)}$$

The Hahn Polynomial of Order m

$$\boxed{P_m[n] = {}_3F_2\left(\begin{matrix} -m, -n, m+\alpha+\beta+1; \\ \alpha+1, -N; 1 \end{matrix}\right)}$$

$${}_pF_q\begin{pmatrix} a_1, a_2, \cdots, a_p; \\ b_1, b_2, \cdots, b_q; z \end{pmatrix}:$$ hypergeometric function

$${}_pF_q\begin{pmatrix} a_1, a_2, \cdots, a_p; \\ b_1, b_2, \cdots, b_q; z \end{pmatrix} = \sum_{k=0}^{\infty} \frac{a_1^{(k)} a_2^{(k)} \cdots a_p^{(k)}}{b_1^{(k)} b_2^{(k)} \cdots b_q^{(k)}} \frac{z^k}{k!}$$

where $a^{(k)}$ is called the rising factorial function:

$$a^{(0)} = 1$$
$$a^{(k)} = a(a+1)(a+2)\cdots\cdots(a+k-1)$$

discrete		continuous
Hahn polynomials	$\xrightarrow{\text{analogous}}$	Jacobi polynomials
Meixner polynomials	$\xrightarrow{\text{analogous}}$	Laguerre polynomials
Krawtchouk polynomials	$\xrightarrow{\text{analogous}}$	Hermite polynomials (refer to page 318)

Hahn polynomials α, β (discrete Jacobi polynomials) $\xrightarrow{\alpha = \beta}$ discrete ultraspherical polynomials

discrete ultraspherical polynomials:

- $\alpha = 0$: discrete Legendre polynomials
- $\alpha = -1/2$: discrete Chebyshev polynomials (I)
- $\alpha = 1/2$: discrete Chebyshev polynomials (II)

[Meixner Polynomials]

When $A = e^{-\lambda}$, $b = 1$, it is analogous to the continuous Laguerre polynomial on page 316.

Two extra parameters: A, b

$$w[n] = A^n \frac{b^{(n)}}{n!} \qquad n \in [0, \infty)$$

The Meixner Polynomial of Order m

$$P_m[n] = {}_2F_1\left(\begin{matrix} -m, -n; \\ b; 1 - \frac{1}{A} \end{matrix}\right)$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

Note: When

$$A = e^{-\lambda}, \quad b = 1$$

then

$$w[n] = e^{-\lambda n}$$

(the same weight function as the continuous Laguarre polynomial)

[Krawtchouk Polynomials]

As shown on the next page, when p = 1/2, it is analogous to the continuous Hermite polynomial on page 318.

One extra parameter: p

$$w[n] = p^n (1-p)^{N-n} \binom{N}{n} \qquad n \in [0, N]$$

(Similar to the Binomial distribution)

The Krawtchouk Polynomial of Order m

$$P_m[n] = {}_2F_1\left(\begin{matrix} -m, -n; \\ -N; \dfrac{1}{p} \end{matrix}\right)$$

$$w[n] = p^n (1-p)^{N-n} \binom{N}{n}$$

Note: When

$$p = 1/2$$

then

$$w[n] = \binom{N}{n}$$

Moreover, when $N \to \infty$

$$\lim_{N\to\infty} \binom{N}{n} \cong \frac{2^N}{\sqrt{N\pi/2}} \exp\left(-\frac{(n-N/2)^2}{N/2}\right)$$

which is near to the weight function of the continuous Hermite polynomial. Therefore, the Krawtchouk polynomial is also called the discrete Hermite polynomial.

附錄十 Approximation Using Other Norms

Until now, we discuss the approximation problem based on the L_2 norm, that is, to find $\mathbf{x}$ that can minimize

$$\|\mathbf{y}-\mathbf{Ax}\|$$

$$\|\mathbf{y}-\mathbf{Ax}\| = \sqrt{\sum_{n=1}^{N}\left(y[n]-\sum_{m=1}^{M}A[n,m]x_m\right)^2}$$

However, how do we minimize the approximation problem based on the L_α norm, that is, to find $\mathbf{x}$ that can minimize

$$\|\mathbf{y}-\mathbf{Ax}\|_\alpha$$

$$\|\mathbf{y}-\mathbf{Ax}\| = \sqrt[\alpha]{\sum_{n=1}^{N}\left|y[n]-\sum_{m=1}^{M}A[n,m]x_m\right|^\alpha}$$

The problem of minimizing

$$\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_\alpha$$

is always hard to solve if $\alpha \neq 2$.

However, when $\alpha \geq 1$, $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_\alpha$ is convex, which means that $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_\alpha$ has only one local minimum (i.e., local minimum = global minimum). Therefore, many numerical methods (the simplex algorithm, Golden search, gradient descent, Newton's method, …..) can be applied to minimize $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_\alpha$. We describe the general method to minimize $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_\alpha$ when $\alpha \geq 1$ as follows.

It is even harder to minimize $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_\alpha$ when $\alpha < 1$.

(Problem): Determine

$$\mathbf{x} = \arg\min_{\mathbf{x}} \left\| \mathbf{y} - \mathbf{A}\mathbf{x} \right\|_{\alpha}$$

It means that to find $\mathbf{x}$ that can minimize $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\alpha}$

Suppose that

size($\mathbf{A}$) = $N \times M$, length($\mathbf{y}$) = N, length($\mathbf{x}$) = M

$M < N$

(Step 1): Initial: $\mathbf{x} = \mathbf{0}$, $E_0 = \|\mathbf{y}\|_{\alpha}$, $c = 1$, $try = 0$

Set Δ (the threshold for error convergence)

Set T (the upper bound of times for no error reduction)

(Step 2): Choose the feasible direction as follows.

(Method 1): Assign the feasible direction **b** as the projection of $\mathbf{y} - \mathbf{Ax}$ on

$$span(\mathbf{A_1}, \mathbf{A_2}, \cdots\cdots, \mathbf{A_M})$$

where $\mathbf{A_1}$, $\mathbf{A_2}$, …, $\mathbf{A_M}$ are columns of **A**.

(Method 2): If the projection is 0 or **c** = 0 (i.e., the adjusting step in the previous iteration is zero)
Generate d_m randomly.
Then, set the feasible direction **b** as

$$\mathbf{b} = \sum_{m=1}^{M} d_m \mathbf{A_m} / \|\mathbf{A_m}\|$$

(Step 3): Find c to minimize $\|\mathbf{y} - \mathbf{A}(\mathbf{x} + c\mathbf{b})\|_\alpha$

$$c = \arg\min_{c} \|\mathbf{y} - \mathbf{A}(\mathbf{x} + c\mathbf{b})\|_\alpha$$

Then, update **x** as

$$\mathbf{x} \Leftarrow \mathbf{x} + c[e_1, e_2, \cdots, e_M] \qquad \text{if} \quad \mathbf{b} = e_1\mathbf{A_1} + e_2\mathbf{A_2} + \cdots e_M\mathbf{A_M}$$

(Step 4): Determine $E_1 = \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_{\alpha}$. If

$$E_0 - E_1 < \Delta$$

then set

$$try \Leftarrow try + 1$$

Otherwise, set $try = 0$.

(Step 5): If $try \leq T$:

Set $E_0 = E_1$ and return to (Step 2)

If $try > T$:

The process is terminated and the solution is obtained.

Flowchart

Initial: $\mathbf{x} = \mathbf{0}$

Assign the feasible direction $\mathbf{b}$ as the projection of $\mathbf{y} - \mathbf{A}\mathbf{x}$ on *span*(Columns of $\mathbf{A}$)
(If the projection is 0, assign $\mathbf{b}$ randomly)

Find c iteratively to minimize $\|\mathbf{y} - \mathbf{A}(\mathbf{x} + c\mathbf{b})\|_\alpha$

Update $\mathbf{x}$ by $\mathbf{x} \Leftarrow \mathbf{x} + c\left[e_1, e_2, \cdots, e_M\right]$

if $\mathbf{b} = e_1\mathbf{A_1} + e_2\mathbf{A_2} + \cdots e_M\mathbf{A_M}$

Whether the error does not decrease for T times

No

Yes

Obtain the solution of $\mathbf{x}$

[Example 1] Suppose that

$$\mathbf{y} = \begin{bmatrix} 2 & 3 & 3 & 4 & 5 & 4 & 5 \end{bmatrix}$$

Try to express **y** as $x_1\mathbf{b_1} + x_2\mathbf{b_2} + x_3\mathbf{b_3}$ where

$$\mathbf{b_1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{b_2} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}$$

$$\mathbf{b_3} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

such that

$$\left\| \mathbf{y} - x_1\mathbf{b_1} - x_2\mathbf{b_2} - x_3\mathbf{b_3} \right\|_1 \quad \text{is minimized}$$

(Solution): (Step 1): Initially, set

$$[x_1, x_2, x_3] = [0, 0, 0]$$

$$E_0 = \left\| \mathbf{y} - x_1\mathbf{b_1} - x_2\mathbf{b_2} - x_3\mathbf{b_3} \right\|_1 = 26$$

(Step 2):

Then, we find the projection of $\mathbf{y} - 0\mathbf{b}_1 - 0\mathbf{b}_2 - 0\mathbf{b}_3 = \mathbf{y}$ on Span($\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$):

$$\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \xrightarrow[\text{Gram-Schmidt}]{} \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$$

$$\mathbf{a}_1 = \frac{1}{\sqrt{7}}\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix} \qquad \mathbf{a}_2 = \frac{1}{2\sqrt{7}}\begin{bmatrix}-3 & -2 & -1 & 0 & 1 & 2 & 3\end{bmatrix}$$

$$\mathbf{a}_3 = \frac{1}{2\sqrt{21}}\begin{bmatrix}3 & -4 & 3 & -4 & 3 & -4 & 3\end{bmatrix}$$

Since

$$\sum_n \mathbf{y}[n]\mathbf{a}_1[n] = 9.2871 \qquad \sum_n \mathbf{y}[n]\mathbf{a}_2[n] = 2.4568$$

$$\sum_n \mathbf{y}[n]\mathbf{a}_3[n] = 0.1091$$

the projection of $\mathbf{y}$ on Span($\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$) is

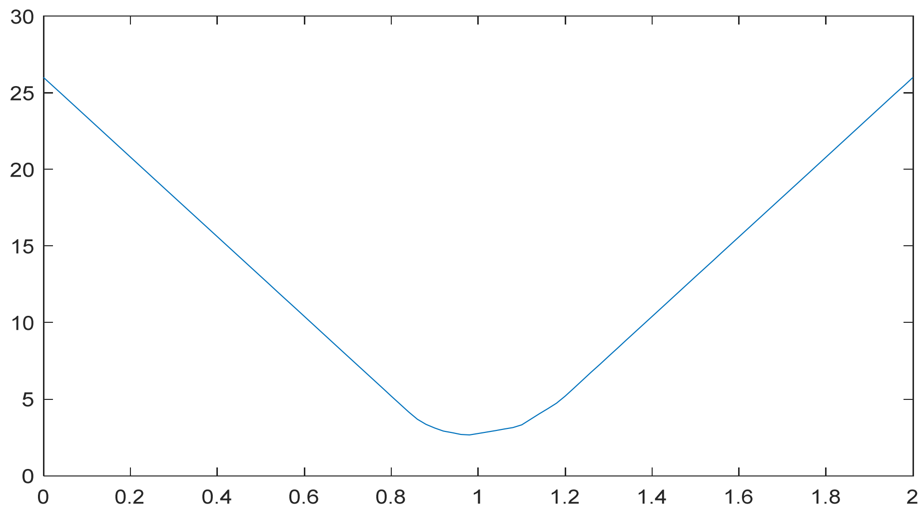
$$9.2871\mathbf{a}_1 + 2.4568\mathbf{a}_2 + 0.1091\mathbf{a}_3 = 1.8512\mathbf{b}_1 + 0.4643\mathbf{b}_2 + 0.0417\mathbf{b}_3$$

Therefore, we choose the feasible direction **b** as

$$\mathbf{b} = 1.8512\,\mathbf{b_1} + 0.4643\,\mathbf{b_2} + 0.0417\,\mathbf{b_3}$$
$$= [2.3571,\ 2.7381,\ 3.2857,\ 3.6667,\ 4.2143,\ 4.5952,\ 5.1429]$$

(Step 3): Find c to minimize $||\mathbf{y} - \mathbf{A}(\mathbf{x} + c\mathbf{b})||_1$

$||\mathbf{y} - \mathbf{A}(\mathbf{x} + c\mathbf{b})||_1$



c

The solution is $c = 0.9722$. Then, update **x** as

$$\mathbf{x} \Leftarrow \mathbf{x} + 0.9722\mathbf{b} = [1.7998,\ 0.4514,\ 0.0405]$$

(Step 4): Determine the residue

$$\mathbf{y} - \mathbf{Ax} = [-0.2917,\ 0.338,\ -0.1944,\ 0.4352,\ 0.9028,\ -0.4676,\ 0]$$

and calculate the error

$$E_1 = \|\mathbf{y} - \mathbf{Ax}\|_1 = 2.6296$$

(Step 5): Return to (Step 2)

$$\vdots$$

$$\vdots$$

$$\vdots$$

After 60-110 times of iterations, we obtain

$$\mathbf{x} = [1.75,\ 0.5,\ -0.25]$$

$$\mathbf{y} - \mathbf{Ax} = [0, 0, 0, 0, 1, -1, 0]$$

$$\|\mathbf{y} - \mathbf{Ax}\|_1 = 2$$