**Selected Topics in Engineering Mathematics Finals**

(2 pages)

1. Solve the following nonlinear DE: (7 %)

$$y^{-3}(x)\,y''(x)=1, \qquad y(1)=-\sqrt{2},\ y'(1)=\sqrt{2}$$

2. Solve the following PDEs: (27 %)

(a) $16\frac{\partial^2}{\partial x^2}u(x,t)=\frac{\partial}{\partial t}u(x,t), \qquad 0<x<2,\quad t>0, \qquad u(0,t)=u(2,t)=0,$

$u(x,0)=\sin(\pi x)$ for $0<x<1$, $u(x,0)=0$ for $1<x<2$

(b) $\frac{\partial}{\partial x}u(x,y)+y\frac{\partial}{\partial y}u(x,y)=\cos x+y$

(c) $\frac{\partial}{\partial x}u(x,y,z)+\tan y\frac{\partial}{\partial y}u(x,y,z)+\cot z\frac{\partial}{\partial z}u(x,y,z)=0$

3. Suppose that (6 %)

$$\phi_0(x)=1, \qquad \phi_1(x)=x+c_0, \qquad \phi_2(x)=x^2+d_1x+d_0, \qquad x\in[0,10]$$

Find $c_0$, $d_0$, and $d_1$ such that $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$ form an orthogonal set (unnecessary to be orthonormal) within $x\in[0,10]$ and with respect to the weight function of $w(x)=x(10-x)$.

4. Find the Fourier transforms of the following functions: (15 %)

(a) $g(x)=x \quad \textit{for}\ 10<|x|<20, \qquad g(x)=0 \quad \textit{otherwise}$

(Please express the result in terms of the sinc function)

(b) $g(x)=\exp(-x^2-2x-3)+\delta(4x)$

(c) $g(x,y)=1 \quad \textit{if}\ 1\le\sqrt{(x/2)^2+y^2}\le 2, \quad g(x,y)=0 \quad \textit{otherwise.}$

(Please determine the two-dimensional FT of $g(x, y)$)

5. Find the following convolutions: (10 %)

(a) $\mathrm{sinc}(3x)*\mathrm{sinc}(6x)*\mathrm{sinc}(12x)*\left(\cos(2\pi x)+\sin(4\pi x)+\cos(8\pi x)\right)$

(b) $\delta(x-1)*\delta''(x)*\delta(3x)*\left(x^3+x^2+x+1\right)$

*(Cont.)*

6. Suppose that the PMF of $X$ is (10 %)

$$P_X(1) = P_X(5) = 0.1, \quad P_X(2) = P_X(4) = 0.2, \quad P_X(3) = 0.4$$

Determine the variance, the skewness, the kurtosis, and the entropy of $X$.
Please express the entropy in terms of ln10 and ln2.

7. The matrix **A** and the vector **b** are

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & -2 \\ 2 & -2 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -17 \\ 9 \end{bmatrix}.$$

(a) Find the singular values of **A**. (4 %)

(b) Find the pseudo-inverse of **A**. (5 %)

(c) Determine the LS solution $\mathbf{x}_{LS}$ to $\mathbf{Ax} = \mathbf{b}$. (3 %)

(d) Find $\rho_{LS}$, which is the size of the minimum residual associated with $\mathbf{Ax} = \mathbf{b}$. (2 %)

8. Write **True** or **False** for these statements. There is no need to justify your answer. (6 %)

(a) If **J** is a Jordan block, then $\mathbf{J}^{1000}$ is also a Jordan block.

(b) Let $\mathbf{A} \in \mathbb{C}^{3\times3}$. The matrix 2-norm $||\mathbf{A}||_2$ and the nuclear norm $||\mathbf{A}||_*$ satisfy $||\mathbf{A}||_2 = |\mathbf{A}||_*$ if and only if rank(**A**) = 1.

(c) Let $\mathbf{A} \in \mathbb{C}^{M\times N}$. If $\sigma$ is a singular value of **A**, then $\sigma^2$ is an eigenvalue of $\mathbf{AA}^{\mathrm{H}}$.

9. The matrix **A** is (5%)

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

The integer $N$ satisfies $N \geq 1000$. Find the value of the entry-wise $L_\infty$ norm

$$\left\| \left( \mathbf{A} \otimes (\mathbf{A} - 3\mathbf{I}_5) \right)^N \right\|_\infty .$$

Simplify and express your answer in terms of $N$.