2. Partial Differential Equations

[1] D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017.

[2] http://djj.ee.ntu.edu.tw/DE.htm

Solving PDEs

- Separation of Variables
  - Introduction of the Method
    - Basic Process (Sec. 2.1)
    - Superposition (Sec. 2.1, Sec. 2.4)
    - Nonhomogeneous Cases (Sec. 2.5) (Two Methods)
  - Rectangular Coordinate Examples
    - Problem Classification (Sec. 2.2)
    - Heat Equation (Sec. 2.3)
    - Wave Equation
    - Laplace Equation (Sec. 2.4)
    - Higher-Dimensional Case (Sec. 2.6)
  - Polar and Spherical Coordinates (Sections 2.7, 2.8, 2.9)
- Integral Transform Methods (Sec. 2.10 and Sec. 2.11)

2.1 Boundary-Value Problem in Rectangular Coordinates

Use the methods of

(1) separation of variables ← Sections 2.1 ~ 2.9

(2) the Laplace / Fourier transforms ← Sections 2.10 and 2.11

to solve the PDE problem.

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.1.

linear second order partial differential equation for two independent variables

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

7 terms

not the form of solutions

$B^2-4AC>0$: hyperbolic (双曲), $B^2-4AC=0$: parabolic (拋物)

$B^2-4AC<0$: elliptic (橢圓)

幾何學 $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

$B^2-4AC>0$ → 双曲線 ex: $x^2-y^2=1$ $A=1, C=-1, B=0, B^2-4AC>0$

$B^2-4AC=0$ → 拋物線 $x^2-y=0$ $A=1, B=C=0, B^2-4AC=0$

$B^2-4AC<0$ → elliptic $x^2+y^2=1$ $A=C=1, B=0, B^2-4AC<0$

homogeneous : $G(x, y) = 0$, nonhomogeneous : $G(x, y) \neq 0$

Linear: A, B, C, D, E, F, and G are independent of u

2.1.1 Superposition Principle

【Theorem 2.1.1】 Superposition Principle

If u_1, u_2, …., u_k are solutions of a homogeneous linear partial differential equation, then

$$u = c_1 u_1 + c_2 u_2 + \cdots\cdots + c_k u_k$$

is also a solution of the homogeneous linear partial differential equation.

(Proof): If

$$A\frac{\partial^2 u_1}{\partial x^2} + B\frac{\partial^2 u_1}{\partial x \partial y} + C\frac{\partial^2 u_1}{\partial y^2} + D\frac{\partial u_1}{\partial x} + E\frac{\partial u_1}{\partial y} + F u_1 = 0$$

$$A\frac{\partial^2 u_2}{\partial x^2} + B\frac{\partial^2 u_2}{\partial x \partial y} + C\frac{\partial^2 u_2}{\partial y^2} + D\frac{\partial u_2}{\partial x} + E\frac{\partial u_2}{\partial y} + F u_2 = 0$$

then $$A\frac{\partial^2(c_1u_1+c_2u_2)}{\partial x^2}+B\frac{\partial^2(c_1u_1+c_2u_2)}{\partial x\partial y}+C\frac{\partial^2(c_1u_1+c_2u_2)}{\partial y^2}+$$

$$D\frac{\partial(c_1u_1+c_2u_2)}{\partial x}+E\frac{\partial(c_1u_1+c_2u_2)}{\partial y}+F(c_1u_1+c_2u_2)$$

$$=c_1\left[A\frac{\partial^2u_1}{\partial x^2}+B\frac{\partial^2u_1}{\partial x\partial y}+C\frac{\partial^2u_1}{\partial y^2}+D\frac{\partial u_1}{\partial x}+E\frac{\partial u_1}{\partial y}+Fu_1\right]$$

$$+c_2\left[A\frac{\partial^2u_2}{\partial x^2}+B\frac{\partial^2u_2}{\partial x\partial y}+C\frac{\partial^2u_2}{\partial y^2}+D\frac{\partial u_2}{\partial x}+E\frac{\partial u_2}{\partial y}+Fu_2\right]$$

$$=c_1 0+c_2 0=0$$

2.1.2 Method of Separation of Variables

解 PDE with BVP (or IVP) 的方法

(1) method of separation of variables

若 PDE 當中有對 x 及對 y 的偏微分，

假設解為 $u(x, y) = X(x)Y(y)$

(2) using the Fourier transform (or Fourier cosine transform, Fourier sine transform) (see Sections 2.10 and 2.11)

共通的精神： PDE ⟶ ODE

Method of Separation of Variables 的流程

(Step 1) 假設解為 $u(x, y) = X(x)Y(y)$

解法關鍵

(Step 2) 將 $u(x, y) = X(x)Y(y)$ 代入 PDE，把 PDE 變成

" function of X " = " function of Y " $= -\lambda$

的型態　　two ODEs

λ 被稱為 real separation constant

Steps 3, 4, 5 要分成不同的 Cases 來解

除了trivial 的情形外，所有可能的 cases 都要考慮

(Step 3) 將 function of $X = -\lambda$ 的解算出，即為 $X(x)$

註：(a) 如果有等於零的 boundary (initial) conditions，也要在這一步考慮

(See the Examples in Sections 2.3, 2.4, and 2.5)

(b) 有時，先解 $Y(y)$ 會比較容易

(視 boundary (initial) conditions 而定)

(c) 在這一步中，有的時候，會得出 λ 的限制

(Step 4) 將 function of $Y = -\lambda$ 的解算出，即為 $Y(y)$

需注意的地方和 Step 3 相同

(Step 5) $u(x, y) = X(x)Y(y)$

(Step 6) 將所有可能的解全部加起來

(Step 7) 用非零的 boundary (initial) conditions 將 coefficients 求出

註： 這一步經常會用到 Fourier series, Fourier cosine series 或 Fourier sine series

※ 若沒有 boundary (initial) conditions，Steps 6, 7 可以省略

Rules:

x 的 BVP (IVP) 簡單 ⟶ 先算 $X(x)$

y 的 BVP (IVP) 簡單 ⟶ 先算 $Y(y)$

沒有 BVP (IVP) ⟶ 先算 $X(x)$ 或 $Y(y)$ 皆可

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

$$u(0,y) = 0 \qquad u(L,y) = 0$$

$$u(x,0) = f(x) \qquad \left.\frac{\partial u}{\partial y}\right|_{y=0} = g(x)$$

先算 $X(x)$

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2}$$

$$u(0,y) = f(y) \qquad u(L,y) = 0$$

$$\left.\frac{\partial}{\partial y} u(x,y)\right|_{y=0} = 0 \qquad \left.\frac{\partial}{\partial y} u(x,y)\right|_{y=H} = 0$$

先算 $Y(y)$

Note: Separation of variables 的方法其實未必可以得出 PDE 所有的解

有些解無法用 $X(x)Y(y)$ 來表示

Separation of variables 的主要好處是比其他方法簡單

[Example 1]

$$\frac{\partial u^2}{\partial x^2} = 4\frac{\partial u}{\partial y}$$

linear
homogeneous

Step 1 假設解為 $u(x, y) = X(x)Y(y)$ (解法關鍵)

Step 2 將 $u(x, y) = X(x)Y(y)$ 代入 $\frac{\partial u^2}{\partial x^2} = 4\frac{\partial u}{\partial y}$

$$X''(x)Y(y) = 4X(x)Y'(y)$$

$$\frac{X''(x)}{4X(x)} = \frac{Y'(y)}{Y(y)}$$

real separation constant

令 $\frac{X''(x)}{4X(x)} = \frac{Y'(y)}{Y(y)} = -\lambda$ (解法關鍵)

$$X''(x) + 4\lambda X(x) = 0 \qquad Y'(y) + \lambda Y(y) = 0$$

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(The detail can be reviewed from the PowerPoint in DE1)

$$X''(x) + 4\lambda X(x) = 0 \qquad Y'(y) + \lambda Y(y) = 0$$

Case 1 for Steps 3, 4, 5 $\lambda = 0$

Step 3-1 $X''(x) = 0$

auxiliary function $m^2 = 0$ roots : 0, 0

$$X(x) = c_1 + c_2 x$$

Step 4-1 $Y'(y) = 0$ $Y(y) = c_3$

Step 5-1 $u(x,y) = X(x)Y(y) = (c_1 + c_2 x)c_3 = A_1 + B_1 x$

$$A_1 = c_1 c_3 \qquad B_1 = c_2 c_3$$

Case 2 for Steps 3, 4, 5 $\lambda < 0$

為了方便起見，令 $\lambda = -\alpha^2$

α is any real constant satisfying α>0

Step 3-2 $X''(x) - 4\alpha^2 X(x) = 0$ roots of the auxiliary function: 2α, -2α

$m^2 - 4\alpha^2 = 0 \quad m = \pm 2\alpha$

$$X(x) = d_1 e^{2\alpha x} + d_2 e^{-2\alpha x}$$

通常將解改寫成 $X(x) = c_4 \cosh(2\alpha x) + c_5 \sinh(2\alpha x)$

$= c_4 \dfrac{e^{2\alpha x} + e^{-2\alpha x}}{2} + c_5 \dfrac{e^{2\alpha x} - e^{-2\alpha x}}{2}$

$= \dfrac{c_4 + c_5}{2} e^{2\alpha x} + \dfrac{c_4 - c_5}{2} e^{-2\alpha x}$

Step 4-2 $\dfrac{Y'(y)}{Y(y)} = \alpha^2 \qquad Y'(y) - \alpha^2 Y(y) = 0$

$Y'(y) - \alpha^2 Y(y) = 0 \qquad Y(y) = c_6 e^{\alpha^2 y}$

Step 5-2 $u(x, y) = X(x)Y(y) = A_2 e^{\alpha^2 y} \cosh(2\alpha x) + B_2 e^{\alpha^2 y} \sinh(2\alpha x)$

$A_2 = c_4 c_6 \qquad B_2 = c_5 c_6$

Case 3 for Step 3 $\lambda > 0$

為了方便起見，令 $\lambda = \alpha^2$

$m^2+4\alpha^2=0 \quad m=\pm j2\alpha$

Step 3-3 $X''(x)+4\alpha^2 X(x)=0$ roots of the auxiliary function: $j2\alpha, -j2\alpha$

$$X(x)=c_7\cos(2\alpha x)+c_8\sin(2\alpha x)$$

Step 4-3 $\dfrac{Y'(y)}{Y(y)}=-\alpha^2 \quad Y'(y)+\alpha^2 Y(y)=0 \quad Y(y)=c_9 e^{-\alpha^2 y}$

Step 5-3 $u(x,y)=A_3 e^{-\alpha^2 y}\cos(2\alpha x)+B_3 e^{-\alpha^2 y}\sin(2\alpha x)$

Step 6

若要處理 boundary conditions，或著想得到 general solution，要將所有可能的解都加起來

Case 1, Case 2

$$u(x,y)=A_1+B_1 x+\sum_{\alpha>0}[A_{2,\alpha}e^{\alpha^2 y}\cosh(2\alpha x)+B_{2,\alpha}e^{\alpha^2 y}\sinh(2\alpha x)]$$

$$+\sum_{\alpha>0}[A_{3,\alpha}e^{-\alpha^2 y}\cos(2\alpha x)+B_{3,\alpha}e^{-\alpha^2 y}\sin(2\alpha x)]$$

α 是任意實數

Case 3

(註：nonseparable 的解在這一步得到)

附錄三： Hyperbolic Function

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

比較： $$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

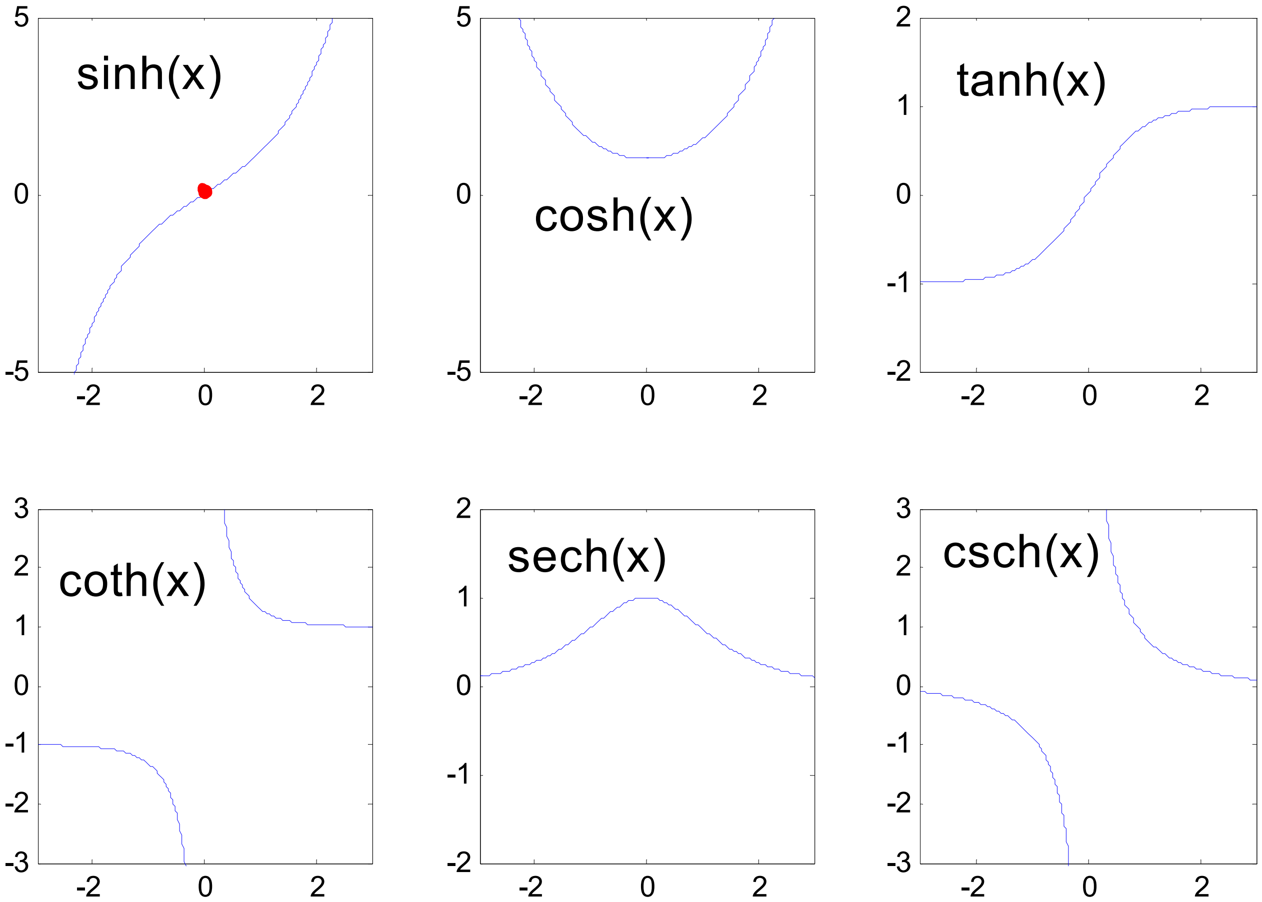
$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^{x} + e^{-x}}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^{x} - e^{-x}}$$


sinh(x)
cosh(x)
tanh(x)
coth(x)
sech(x)
csch(x)

$$\frac{d}{dx}\sinh(ax) = a\cosh(ax)$$

$$\sinh(0) = 0$$

$$\frac{d}{dx}\cosh(ax) = a\sinh(ax)$$

$$\cosh(0) = 1$$

$$\frac{d}{dx}\tanh(ax) = a\,\text{sech}^2(ax)$$

$$\sinh'(0) = 1$$

$$\frac{d}{dx}\coth(ax) = -a\,\text{csch}^2(ax)$$

$$\cosh'(0) = 0$$

$$\frac{d}{dx}\text{sech}(ax) = -a\,\text{sech}(ax)\tanh(ax)$$

$$\sin(ix) = i\sinh(x)$$

$$\frac{d}{dx}\text{csch}(ax) = -a\,\text{csch}(ax)\coth(ax)$$

$$\cos(ix) = \cosh(x)$$

Section 2.2 Classical PDEs and Boundary-Value Problems

2.2.1 本節綱要

(1) one-dimensional heat equation (或簡稱為 heat equation)

B=0, A=k, C=0
$B^2-4AC=0$
parabolic

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$$

$k>0$

Generally, $k\nabla^2 u=\frac{\partial u}{\partial t}$

(2) one-dimensional wave equation (或簡稱為 wave equation)

$A=a^2$, C=−1, B=0
$B^2-4AC>0$
hyperbolic

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

Generally, $a^2\nabla^2 u=\frac{\partial^2 u}{\partial t^2}$

(3) two-dimensional form of Laplace's equation (或簡稱為 Laplace's equation)

A=C=1, B=0
$B^2-4AC<0$
elliptic

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$\nabla^2 u=0$

Generally, $\nabla^2 u=0$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.2.

名詞：

本節的重點：七大名詞，和它們所對應的公式

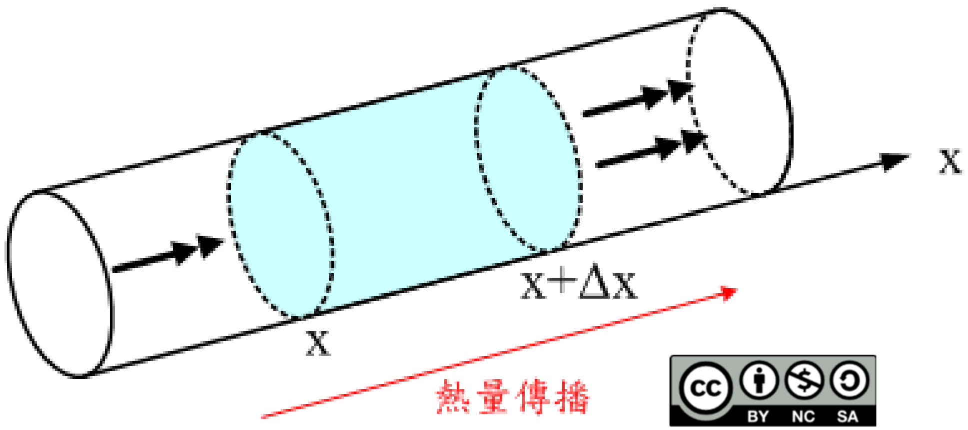
2.2.2 One-Dimensional Heat Equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

由來：熱傳導的理論

$u(x, t)$: temperature, t: time, x: location

Fig. 2.2.1



From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.2.

heat equation 別名：diffusion equation

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Example:

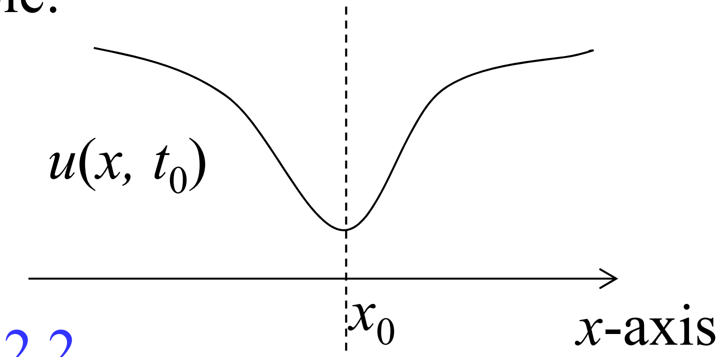


Fig. 2.2.2

$u(x, t)$: temperature,
t: time, x: location

x_0 的溫度將上升 $\left.\frac{\partial u(x,t)}{\partial t}\right|_{x=x_0} > 0$

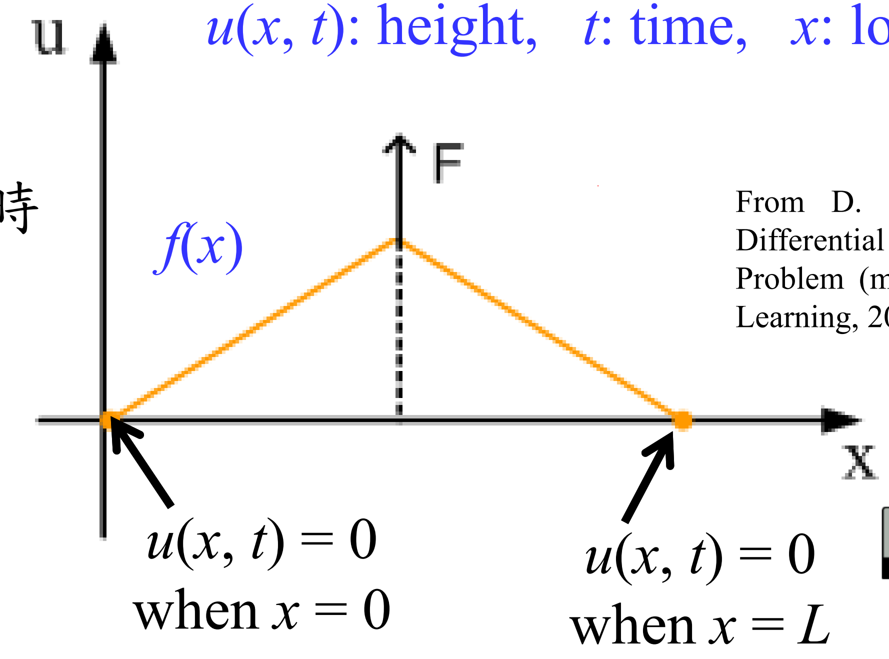
2.2.3 One-Dimensional Wave Equation

$$a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

「拉像皮筋」的模型

$u(x, t)$: height, t: time, x: location

(1) 剛開始時



From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.2.

Fig. 2.2.3

wave equation 別名： telegraph equation

(2) 手放開之後產生振動

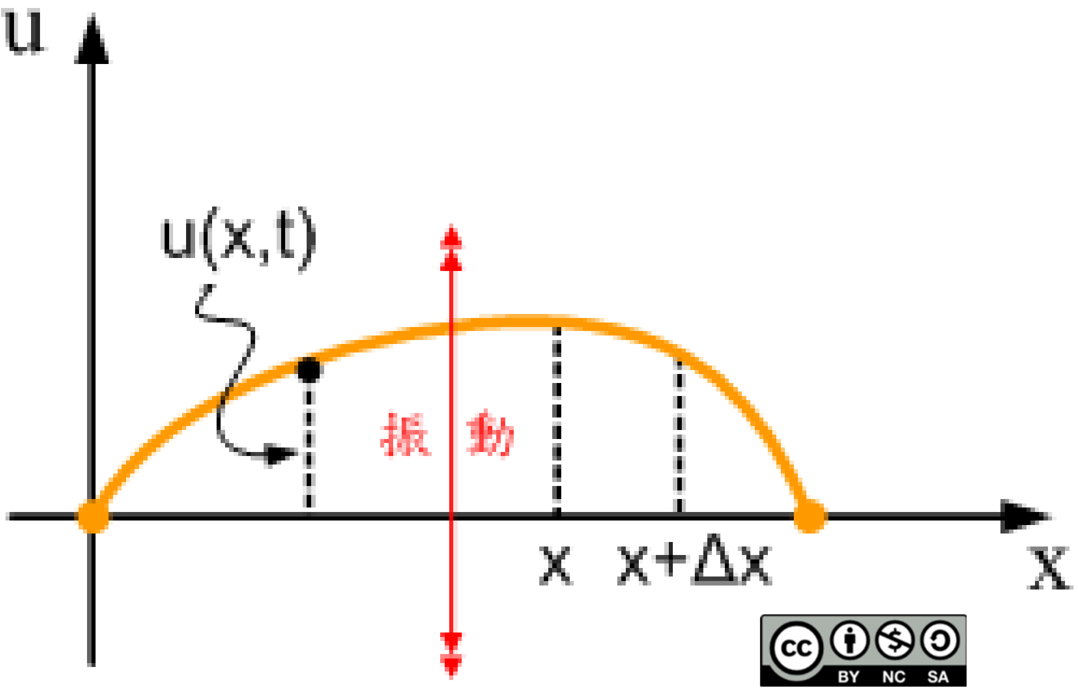


Fig. 2.2.4

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.2.

- Wave equation 其他的應用：

  Theory of high-frequency transmission line

  Fluid mechanics (流體力學)

  Acoustics (聲學)

  Elasticity (彈力學)

  Microwave engineering (電波工程)

2.2.4 Two-Dimensional Form of Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial u}{\partial t}$$

When $\frac{\partial u}{\partial t}=0$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y^2}=0$$

溫度隨著位置而變化的模型

$u(x, y)$: temperature,

x, y: location

Laplace's Equation 亦可用 Laplacian 表示, $\nabla^2 u(x, y) = 0$

Laplacian: ∇^2

$$\nabla^2 u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\nabla^2 u(x,y,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Modification

加上外力，或與外界的交互作用

例：heat equation 的 modification

$$k\frac{\partial^2 u}{\partial x^2} - h\left(u - u_m\right) = \frac{\partial u}{\partial t}$$

例：wave equation 的 modification

$$a^2\frac{\partial^2 u}{\partial x^2} + F\left(x, t, u, u_t\right) = \frac{\partial^2 u}{\partial t^2}$$

- Laplace's Equation 的其他應用

  Static displacement of membranes

  Edge detection (邊緣偵測)

  Microwave engineering (電波工程)

2.2.6 Boundary Conditions 或 Initial Conditions

Dirichlet condition $u = \ldots\ldots..$ (沒微分)

Neumann condition $\dfrac{\partial u}{\partial n} = \cdots\cdots$ (有微分)

Robin condition $\dfrac{\partial u}{\partial n} + hu = \cdots\cdots$ (混合)

h is a constant

2.3 Heat Equation

This section can also be viewed as an example of Section 2-1

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t},\quad 0<x<L,\quad t>0 \qquad (1)$$

x=0 → ← x=L

$$u(0,t)=0,\quad u(L,t)=0,\quad t>0 \qquad (2)$$

$$u(x,0)=f(x),\quad 0<x<L. \qquad (3)$$

only consider
$X(x)=c_\alpha \sin\alpha x = c_\alpha \sin\left(\frac{n\pi x}{L}\right)$
$\sin\alpha L=0 \quad \alpha L=n\pi,\ \alpha=\frac{n\pi}{L}$

Solution:

(Step 1) $\quad u(x,t)=X(x)T(t)$

$$kX''(x)T(t)=X(x)T'(t)$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.3.

$$kX''(x)T(t) = X(x)T'(t)$$

$X = \sin(\frac{n\pi}{L}x)$

$\frac{X''}{X} = -\frac{n^2\pi^2}{L^2}$, $\lambda = \frac{n^2\pi^2}{L^2}$

(Step 2) $$\frac{X''}{X} = \frac{T'}{kT} = -\lambda \quad (4)$$

PDE → 2 ODEs

$$X'' + \lambda X = 0 \quad (5)$$

$$T' + k\lambda T = 0. \quad (6)$$

(From Zero Boundary Conditions)

$u(x,t) = X(x)T(t)$

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad u(L,t) = X(L)T(t) = 0.$$

Since for a nontrivial solution, $T(t)$ cannot be zero,

$X(0) = 0$ and $X(L) = 0$.

If $T(t) = 0$, $u(x,t) = 0$ trivial solution

We have

$$\begin{cases} X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0. \\ T' + k\lambda T = 0. \end{cases} \quad (7)$$

(i) $X''+\lambda X=0,\ X(0)=0,\ X(L)=0.$ (ii) $T'+k\lambda T=0.$

Case 1 for Steps 3, 4, 5 $\lambda = 0$

$X''=0 \Longrightarrow X(x)=c_1+c_2x$

From $X(0)=0,\ X(L)=0 \Longrightarrow c_1=c_2=0 \Longrightarrow X(x)=0$

$u(x,t)=X(x)T(t)=0$

(trivial solution)

(i) $X'' + \lambda X = 0, \ X(0) = 0, \ X(L) = 0.$ (ii) $T' + k\lambda T = 0.$

X **Case 2 for Steps 3, 4, 5 $\lambda < 0$**

Set $\lambda = -\alpha^2$

$$X'' - \alpha^2 X = 0 \Longrightarrow X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

$$\Longrightarrow X(x) = c_3 \cosh(\alpha x) + c_4 \sinh(\alpha x)$$

Note:

$$c_3 \cosh(\alpha x) + c_4 \sinh(\alpha x) = c_3 \frac{e^{\alpha x} + e^{-\alpha x}}{2} + c_4 \frac{e^{\alpha x} - e^{-\alpha x}}{2}$$

cosh(0)=1, sinh(0)=0

$c_3 \cdot 1 + c_4 \cdot 0 = 0$

$$= \frac{c_3 + c_4}{2} e^{\alpha x} + \frac{c_3 - c_4}{2} e^{-\alpha x}$$

From $X(0) = 0$, $c_3 = 0$

From page 111

From $X(L) = 0$, $c_4 \sinh(\alpha L) = 0$, $c_4 = 0$

$\sinh(x) \neq 0$ if $x \neq 0$

$$\Longrightarrow X(x) = 0 \Longrightarrow u(x,t) = X(x)T(t) = 0$$

(trivial solution)

(i) $X''+\lambda X=0,\ \ X(0)=0,\ \ X(L)=0.$ (ii) $T'+k\lambda T=0.$

Case 3 for Steps 3, 4, 5 $\lambda>0$

Set $\lambda=\alpha^2$

$X''+\alpha^2 X=0 \Longrightarrow X(x)=c_1\cos\alpha x+c_2\sin\alpha x.$

sin nπ=0 for all integer n

$X(0)=0, \Longrightarrow c_1=0,$

$X(L)=0. \Longrightarrow c_2\sin\alpha L=0 \Longrightarrow \alpha=n\pi/L,\quad \lambda=n^2\pi^2/L^2.$

αL= nπ

$X(x)=c_2\sin(\pi n x/L),\quad \lambda=n^2\pi^2/L^2.\qquad n=1,2,3,\ldots.$

$T'+k\dfrac{n^2\pi^2}{L^2}T=0. \Longrightarrow T(t)=c_3e^{-k(n^2\pi^2/L^2)t}$

$u_n(x,t)=X(x)T(t)=A_n e^{-k(n^2\pi^2/L^2)t}\sin\dfrac{n\pi}{L}x,\qquad n=1,2,3,\ldots.$

n=0 λ=0

n<0 has the same λ as n>0

(Step 6) $u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}A_n e^{-k(n^2\pi^2/L^2)t}\sin\frac{n\pi}{L}x,$

(Step 7) From the boundary condition, $u(x,0)=f(x)$

$$u(x,0)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{L}x=f(x)$$

From Fourier sine series (page 131 in 附錄四)

$$g(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{p}x \qquad b_n=\frac{2}{p}\int_0^p g(x)\sin\frac{n\pi}{p}x\,dx$$

p is replaced by L

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx.$$

Therefore,

$$u(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\right)e^{-k(n^2\pi^2/L^2)t}\sin\frac{n\pi}{L}x.$$

附錄四 Review for Fourier Series and Fourier Cosine / Sine Series

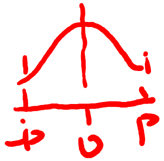
Fourier Series

$f(x) = f(x+p)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \qquad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$



Fourier Series

- $f(x)$ is even

  $b_n = 0$

  Fourier cosine series (或 cosine series)

  $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x$$

  $$a_0 = \frac{2}{p}\int_0^p f(x)\,dx \qquad a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

  $\int_{-p}^{p} \text{odd} = 0 \qquad \int_{-p}^{p} \text{even} = 2\int_0^p \text{even}$

- $f(x)$ is odd

  $a_0 = a_n = 0$

  Fourier sine series (或 sine series)

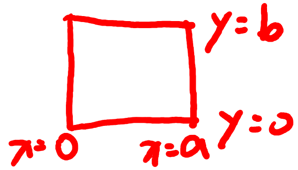
  $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x$$

  $$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

Section 2.4 Laplace's Equation

2.4.1 Section 2.4 綱要

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$



(使用method of separation of variables 來解)

「問題 1」 $\left.\frac{\partial u}{\partial x}\right|_{x=0}=0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x=a}=0 \qquad$ for $0<y<b$,

$u(x,0)=0 \qquad u(x,b)=f(x) \qquad$ for $0<x<a$

「問題 2」 $u(0,y)=0 \qquad u(a,y)=0 \qquad$ for $0<y<b$,

$X(a)=0$

$X(0)=0$

$u(x,0)=0 \qquad u(x,b)=f(x) \qquad$ for $0<x<a$

$u(x,y)=X(x)Y(y)$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.5.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$

「問題 3」 $u(0,y)=F(y)$ $\quad u(a,y)=G(y)$ $\quad$ for $0<x<a$

$u(x,0)=f(x)$ $\quad u(x,b)=g(x)$ $\quad$ for $0<y<b,$

※ **特別注意** **“superposition principle”**

2.4.2 Solutions for Laplace's Equations (挑戰解解看)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x=a}=0 \qquad \text{for } 0<y<b,$$

$$u(x,0)=0 \qquad u(x,b)=f(x) \qquad \text{for } 0<x<a$$

$Y(0)=0$

$X'(0)=X'(a)=0$

$X(x)=C$

or $C_\alpha \cos\alpha x$

$\frac{d}{dx}(C_\alpha \cos(\alpha x))$

$=C_\alpha \alpha \sin(\alpha x)$

$C_\alpha \alpha \sin(\alpha a)=0$

$\alpha a=n\pi$, $\alpha=\frac{n\pi}{a}$

$\frac{X''}{X}=-\frac{n^2\pi^2}{a^2}$, $\lambda=\frac{n^2\pi^2}{a^2}$

Step 1 假設解為 $u(x,y)=X(x)Y(y)$

Step 2 代入 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ 得出

$$X''(x)Y(y)+X(x)Y''(y)=0 \qquad \frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}$$

令 $\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=-\lambda$

得出2 個 ODEs $X''(x)+\lambda X(x)=0 \qquad Y''(y)-\lambda Y(y)=0$

Steps 3, 4, 5 的前處理

(1) 因為 x 的 boundary condition 較簡單，所以先解 $X(x)$

(2) 分成 $\lambda=0,\ \lambda<0,\ \lambda>0$ 三個 cases

(3) 由於 $\left.\frac{\partial u}{\partial x}\right|_{x=0}=0$ for all $0<y<b$,

$$\left.\frac{\partial X(x)Y(y)}{\partial x}\right|_{x=0}=X'(0)Y(y)=0$$

→ $X'(x)Y(y)$

$Y(y)$ 不可為 0 (否則 $u(x,y)=X(x)Y(y)=0$)

所以 $X'(0)=0$

同理，由 $\left.\frac{\partial u}{\partial x}\right|_{x=a}=0 \longrightarrow X'(a)=0$

→ $X'(a)Y(y)=0$

同理，由 $u(x,0)=0 \longrightarrow Y(0)=0$

$X(x)Y(0)=0$

$X''(x)+\lambda X(x)=0$ $X'(0)=0$ $X'(a)=0$

$Y''(y)-\lambda Y(y)=0$ $Y(0)=0$

Case 1 of Steps 3, 4, 5: $\lambda = 0$

Step 3-1 $X''(x)=0$ solution: $X(x)=c_1+c_2x$ $X'(x)=c_2$

由 boundary conditions $X'(0)=0$ $X'(a)=0$ $c_2=0$

$X(x)=c_1$

Step 4-1 $Y''(y)=0$ $Y(0)=0$

solution: $Y(y)=c_3+c_4y$

根據 boundary condition $Y(0)=0$, $c_3=0$

$Y(y)=c_4y$

Step 5-1

$u(x,y)=X(x)Y(y)=c_1c_4y=A_0y$ $A_0=c_1c_4$

✗ Case 2 of Steps 3, 4, 5: $\lambda < 0$

令 $\lambda = -\alpha^2$

Step 3-2 $X''(x) - \alpha^2 X(x) = 0$ $X'(0) = 0$ $X'(a) = 0$

solution: $X(x) = d_2 e^{\alpha x} + d_3 e^{-\alpha x}$

可改寫成 $X(x) = d_4 \cosh(\alpha x) + d_5 \sinh(\alpha x)$

from page 112
$X'(x) = \alpha d_4 \sinh(\alpha x) + \alpha d_5 \cosh(\alpha x)$

由 boundary conditions $X'(0) = 0$ $X'(a) = 0$

以及 $\frac{d}{dx}\cosh(\alpha x) = \alpha \sinh(\alpha x)$, $\frac{d}{dx}\sinh(\alpha x) = \alpha \cosh(\alpha x)$

$$\begin{cases} d_5 \alpha = 0 \\ d_4 \alpha \sinh(\alpha a) + d_5 \alpha \cosh(\alpha a) = 0 \end{cases} \Longrightarrow \begin{cases} d_5 = 0 \\ d_4 = 0 \end{cases} \Longrightarrow X(x) = 0$$

$\alpha \neq 0$

$\alpha a > 0$ $\sinh(\alpha a) \neq 0$

因此， case 2 得出 trivial solution $u(x, y) = X(x)Y(y) = 0$

$u(x,b) = f(x)$ 將無法滿足 $\lambda < 0$ 時無解

(不需再算 Steps 4-2, 5-2)

Case 3 of Steps 3, 4, 5: $\lambda > 0$

令 $\lambda = \alpha^2$

Step 3-3 $X''(x) + \alpha^2 X(x) = 0$ $X'(0) = 0$ $X'(a) = 0$

solution: $X(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

$X'(x) = -c_1 \alpha \sin(\alpha x) + c_2 \alpha \cos(\alpha x)$

由 boundary conditions $X'(0) = 0$ $X'(a) = 0$

$$\begin{cases} c_2 \alpha = 0 \\ -c_1 \alpha \sin(\alpha a) + c_2 \alpha \cos(\alpha a) = 0 \end{cases} \Rightarrow \begin{cases} c_1 = \text{any nonzero constant} \\ \alpha = \dfrac{n\pi}{a} \quad n \text{ 是任意整數} \\ c_2 = 0 \end{cases}$$

$\sin(n\pi) = 0$, $\alpha a = n\pi$, $\alpha = \frac{n\pi}{a}$

再次注意：不可直接判斷成 $c_1 = 0$ and $c_2 = 0$

應該看看是否有適當的 α, 讓第二個式子等於零

$$X_n(x) = c_1 \cos \frac{n\pi}{a} x$$

n 是任意正整數 $\lambda = \alpha^2 = \dfrac{n^2\pi^2}{a^2}$

only consider $n > 0$

$n \neq 0$

Step 4-3 $Y''(y) - \frac{n^2\pi^2}{a^2} Y(y) = 0$ since $\lambda = \frac{n^2\pi^2}{a^2}$

$Y(0) = 0$

solution: $Y_n(y) = d_3 e^{\frac{n\pi}{a}y} + d_4 e^{-\frac{n\pi}{a}y}$

經常改寫為 $Y_n(y) = c_3 \cosh\left(\frac{n\pi}{a}y\right) + c_4 \sinh\left(\frac{n\pi}{a}y\right)$

根據 boundary condition $Y(0) = 0$ $c_3 = 0$

$$Y_n(y) = c_4 \sinh\left(\frac{n\pi}{a}y\right)$$

Step 5-3

$$u(x,y) = X(x)Y(y)$$

$$= c_1 \cos\left(\frac{n\pi}{a}x\right) c_4 \sinh\left(\frac{n\pi}{a}y\right) = A_n \cos\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right)$$

n 是任意正整數 $A_n = c_1 c_4$

Step 6 把所有可能的解，全部加起來

$$u(x,y)=A_0y+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

case1 case3

Q: 為什麼 n 是從 1 加到 ∞，而非由 $-\infty$ 加到 ∞？

討論：既然 n 是任意整數，那為什麼 n 是從1 加到 ∞，

而非由 $-\infty$ 加到 ∞？

因為 $\cos\left(\frac{n\pi}{a}x\right)=\cos\left(\frac{-n\pi}{a}x\right)$, $\sinh\left(\frac{n\pi}{a}y\right)=-\sinh\left(\frac{-n\pi}{a}y\right)$,

$\sinh(0)=0$

可證明

$$\sum_{n=-\infty}^{\infty} B_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

$$=\sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{a}x\right)\left[B_n \sinh\left(\frac{n\pi}{a}y\right)-B_{-n}\sinh\left(\frac{n\pi}{a}y\right)\right]$$

$$=\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

$$A_n = B_n - B_{-n}$$

Step 7 $$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right)$$

nonzero boundary condition: $u(x,b) = f(x)$

$$f(x) = A_0 b + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}b\right)$$

也就是說，$2A_0b$ 和 $A_n \sinh\left(\frac{n\pi}{a}b\right)$ $(n = 1, 2, \ldots, \infty)$ ↔ a_n

是 $f(x)$ 的 Fourier cosine series 的 coefficients page 131

$A_0 b \leftrightarrow \frac{a_0}{2}$ p is replaced by a

$$2A_0 b = \frac{2}{a}\int_0^a f(x)dx \qquad A_n \sinh\left(\frac{n\pi}{a}b\right) = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi}{a}x dx$$

$$A_0 = \frac{1}{ab}\int_0^a f(x)dx \qquad A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a f(x)\cos\frac{n\pi}{a}x dx$$

2.4.3 Laplace's Equations with Dirichlet Problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$

$X(0)=X(a)=0$ $X(x)=c_\alpha \sin\alpha x$

$\sin\alpha a=0$
$\alpha a=n\pi$
$\alpha=\frac{n\pi}{a}$

$$u(0,y)=0 \qquad u(a,y)=0 \qquad 0<y<b,$$

$$u(x,0)=0 \qquad u(x,b)=f(x) \qquad 0<x<a,$$

$X(x)=C_n \sin\left(\frac{n\pi}{a}x\right)$

用 method of separation of variables，經過計算得出

$u=XY$

$X''Y+Y''X=0$ 除以 XY

$\frac{X''}{X}+\frac{Y''}{Y}=0$

$\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda$

$$u(x,y)=\sum_{n=1}^{\infty}A_n \sinh\frac{n\pi}{a}y\sin\frac{n\pi}{a}x$$

$$A_n=\frac{2}{a\sinh\frac{n\pi}{a}b}\int_0^a f(x)\sin\frac{n\pi}{a}x\,dx$$

可自行練習解解看

$\lambda=\frac{n^2\pi^2}{a^2}$

$Y''-\lambda Y=0$

$Y''-\frac{n^2\pi^2}{a^2}Y=0$

$m^2-\frac{n^2\pi^2}{a^2}=0,\ m=\pm\frac{n\pi}{a}$

$Y(y)=c_1 e^{\frac{n\pi}{a}y}+c_2 e^{-\frac{n\pi}{a}y}=c_3\cosh\frac{n\pi}{a}y+c_4\sinh\frac{n\pi}{a}y$

2.4.4 Superposition Principle

Dirichlet Problem 可分解成兩個子問題

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$$

$$u(0,y)=F(y) \qquad u(a,y)=G(y) \qquad \text{for } 0<y<b,$$

$$u(x,0)=f(x) \qquad u(x,b)=g(x) \qquad \text{for } 0<x<a,$$

當四個邊界都不為零時，很難直接用 separation of variable 的方法解出來

子問題 1 $\dfrac{\partial^2 u_1}{\partial x^2}+\dfrac{\partial^2 u_1}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$

$$u_1(0,y)=0 \qquad u_1(a,y)=0 \qquad \text{for } 0<y<b,$$

$$u_1(x,0)=f(x) \qquad u_1(x,b)=g(x) \qquad \text{for } 0<x<a,$$

子問題 2 $\dfrac{\partial^2 u_2}{\partial x^2}+\dfrac{\partial^2 u_2}{\partial y^2}=0 \qquad 0<x<a, \qquad 0<y<b,$

$$u_2(0,y)=F(y) \qquad u_2(a,y)=G(y) \qquad \text{for } 0<y<b,$$

$$u_2(x,0)=0 \qquad u_2(x,b)=0 \qquad \text{for } 0<x<a,$$

假設 $u_1(x,y)$, $u_2(x,y)$ 分別是子問題 1, 子問題 2 的解

則 $\mathbf{u(x,y)=u_1(x,y)+u_2(x,y)}$ 是原來問題的解

當 $u(x, y) = u_1(x, y) + u_2(x, y)$

$$u(0, y) = u_1(0, y) + u_2(0, y) = 0 + F(y) = F(y)$$
$$u(a, y) = u_1(a, y) + u_2(a, y) = 0 + G(y) = G(y)$$
$$u(x, 0) = u_1(x, 0) + u_2(x, 0) = f(x) + 0 = f(x)$$
$$u(x, b) = u_1(x, b) + u_2(x, b) = g(x) + 0 = g(x)$$

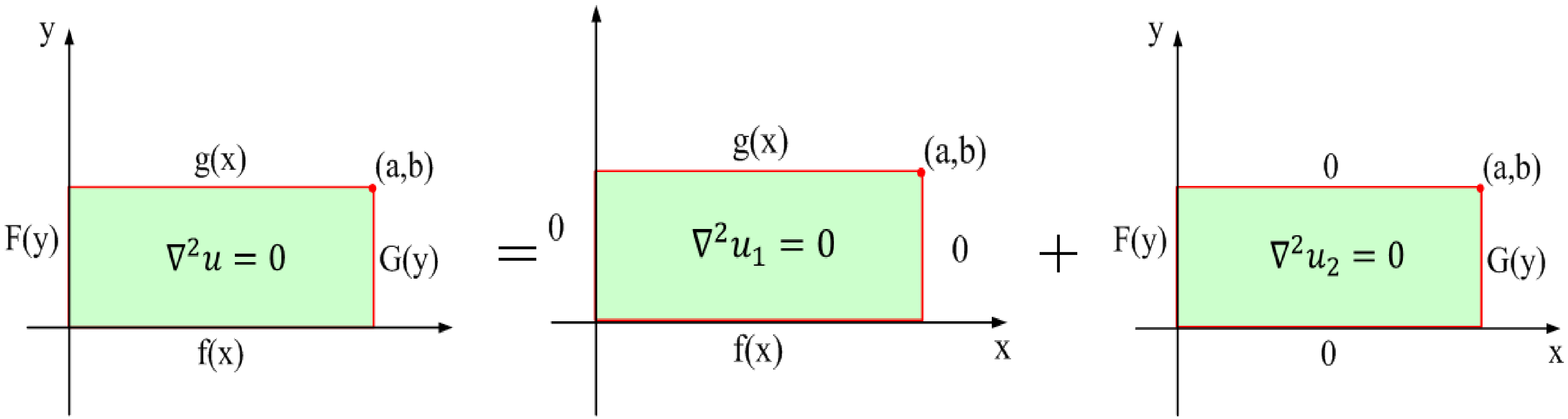


Fig. 2.5.1

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.5.

子問題 1 的解 $u_1(x,y)=\sum_{n=1}^{\infty}\left\{A_n\cosh\frac{n\pi}{a}y+B_n\sinh\frac{n\pi}{a}y\right\}\sin\frac{n\pi}{a}x$

$$A_n=\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

$$B_n=\frac{1}{\sinh\left(\frac{n\pi}{a}b\right)}\left[\frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi}{a}x\right)dx-A_n\cosh\left(\frac{n\pi}{a}b\right)\right]$$

子問題 2 的解 $u_2(x,y)=\sum_{n=1}^{\infty}\left\{A_n\cosh\frac{n\pi}{b}x+B_n\sinh\frac{n\pi}{b}x\right\}\sin\frac{n\pi}{b}y$

$$A_n=\frac{2}{b}\int_0^b F(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

$$B_n=\frac{1}{\sinh\left(\frac{n\pi}{b}a\right)}\left[\frac{2}{b}\int_0^b G(y)\sin\left(\frac{n\pi}{b}y\right)dy-A_n\cosh\left(\frac{n\pi}{b}a\right)\right]$$

原來問題的解 $u_1(x,y)+u_2(x,y)$

2.4.5 Sections 2.1~2.4需要注意的地方

(1) Method of separation of variables 解 PDE 的過程雖然長，但是把握住講義 pages 101-103 的 7 個 steps，就大致上沒問題。

(2) 注意，

若 boundary conditions 出現 $u(0, y) = 0$, $u(L, y) = 0$,

最後的解總是和 sine 有關 $X(x) = c_2 \sin \frac{n\pi}{L} x$ 週期為 $2L/n$

若 boundary conditions 出現 $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$ $\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$

最後的解總是和 cosine 或 constant 有關

$X(x) = c_1$ or $X_n(x) = c_1 \cos \frac{n\pi}{L} x$ 週期也為 $2L/n$

(3) 經驗足夠後，看到 $u(x, y)$ 的 boundary conditions

出現 $u(a, y) = 0$ $\longrightarrow$ 就知道 $X(a) = 0$ ，

看到 $u(x, b) = 0$ $\longrightarrow$ 就知道 $Y(b) = 0$ 。

看到 $\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0$ $\longrightarrow$ 就知道 $X'(a) = 0$，

看到 $\left.\frac{\partial u}{\partial y}\right|_{y=b} = 0$ $\longrightarrow$ 就知道 $Y'(b) = 0$

(4) 要熟悉 $\cosh(x)$, $\sinh(x)$ 的性質

(5) Method of separation of variables 在計算上容易出錯的地方

(以 講義 pages 134-142 Laplace equations 為例)

(a) $$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

(b) Steps 3, 4, 5 要考慮所有 cases

(c) 不可直接由 $c_1 = 0$ 及 $c_1 \cos\alpha x + c_2 \sin\alpha x = 0$ 判斷 $c_1 = c_2 = 0$

因為 α 可以是 $\pi n/L$, 如講義 page 138 所述

(d) 在 Step 6，要將所有可能的解加起來，才是 $u(x, t)$ 的一般解

如講義 page 140 所述

2.5 Nonhomogeneous Boundary-Value Problems

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

Nonhomogeneous: $G \neq 0$

Key ideas: Separate the original problem into two or more problems

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.6.

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

ψ /saɪ/

Method 1 $u(x,y)=v(x,y)+\psi(x)$

Method 2 $u(x,y)=v(x,y)+\psi(y)$

Method 3 $u(x,y)=v(x,y)+\psi_1(x)+\psi_2(y)$

Method 4 $u(x,y)=v(x,y)+\psi(x,y)$

(Method 4 只教不考)

Extra Methods: Expansion by Fourier series, Fourier cosine series, or Fourier sine series

2.6.1 Method 1

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

Method 1 $u(x,y)=v(x,y)+\psi(x)$

[Constraint]: G is independent of y

$$A\frac{\partial^2 v(x,y)}{\partial x^2}+B\frac{\partial^2 v(x,y)}{\partial x\partial y}+C\frac{\partial^2 v(x,y)}{\partial y^2}+D\frac{\partial v(x,y)}{\partial x}+E\frac{\partial v(x,y)}{\partial y}+Fv(x,y)$$

$$+A\psi''(x)+D\psi'(x)+F\psi(x)=G(x)$$

Problem A: (ODE for $\psi(x)$)

$$A\psi''(x)+D\psi'(x)+F\psi(x)=G(x)$$

Problem B: (homogeneous PDE for $v(x, y)$)

$$A\frac{\partial^2 v(x,y)}{\partial x^2}+B\frac{\partial^2 v(x,y)}{\partial x\partial y}+C\frac{\partial^2 v(x,y)}{\partial y^2}+D\frac{\partial v(x,y)}{\partial x}+E\frac{\partial v(x,y)}{\partial y}+Fv(x,y)=0$$

[Example 1] nonhomogeneous

Solve
$$k\frac{\partial^2 u}{\partial x^2}+r=\frac{\partial u}{\partial t},\quad 0<x<1,\quad t>0$$
$$u(0,t)=0,\quad u(1,t)=u_1,\quad t>0$$
$$u(x,0)=f(x),\quad 0<x<1$$

r and u_1 are nonzero constants

(Solution): Since $G=r$, (independent of t) $\quad u(0,t)=0,\ u(1,t)=u_1$ (constants)

Method 1 can be applied.

$u(1,t)=v(1,t)+\psi(1)=u_1$

Let $v(1,t)=0,\ \psi(1)=u_1$

$$u(x,t)=v(x,t)+\psi(x),$$

$u(x,0)=v(x,0)+\psi(x)=f(x)$

$v(x,0)=f(x)-\psi(x)$

$$k\frac{\partial^2 v(x,t)}{\partial x^2}+k\frac{d^2\psi(x)}{dx^2}+r=\frac{\partial v(x,t)}{\partial t}+\frac{d\psi(x)}{dt}$$

$$k\frac{\partial^2 v(x,t)}{\partial x^2}+k\frac{d^2\psi(x)}{dx^2}+r=\frac{\partial v(x,t)}{\partial t}$$

Problem A (for $k\frac{d^2\psi(x)}{dx^2}+r$)

Problem A $k\psi''(x)+r=0,\quad \psi(0)=0,\quad \psi(1)=u_1$

Problem B $k\dfrac{\partial^2 v}{\partial x^2}=\dfrac{\partial v}{\partial t},\quad 0<x<1,\quad t>0$

$v(0,t)=0,\quad v(1,t)=0,\quad t>0$

$v(x,0)=f(x)-\psi(x),\quad 0<x<1$

(i) For Problem A

$$k\psi''(x)+r=0,\qquad \psi(0)=0,\quad \psi(1)=u_1$$

$$\psi''(x)=-r/k \qquad \psi(x)=-\frac{r}{2k}x^2+c_1x+c_0$$

$\psi(x)=\iint -\frac{r}{k}\,dx$

$$\psi(0)=0,\quad \psi(1)=u_1 \Longrightarrow c_0=0,\quad -\frac{r}{2k}+c_1=u_1$$

$$\psi(x)=-\frac{r}{2k}x^2+\left(\frac{r}{2k}+u_1\right)x$$

(ii) For Problem B, from Section 2-3 page 129

$$v(x,t)=\sum_{n=1}^{\infty}A_n e^{-kn^2\pi^2 t}\sin(n\pi x)$$

where $A_n = 2\int_0^1\left[f(x)+\frac{r}{2k}x^2-\left(\frac{r}{2k}+u_1\right)x\right]\sin(n\pi x)\,dx$

Therefore,

$$u(x,t)=\underbrace{-\frac{r}{2k}x^2+\left(\frac{r}{2k}+u_1\right)x}_{\psi(x)}+\underbrace{\sum_{n=1}^{\infty}A_n e^{-kn^2\pi^2 t}\sin(n\pi x)}_{v(x,t)}$$

2.6.2 Methods 2 and 3

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

Problem A

$$C\psi''(y)+E\psi'(y)+F\psi(y)=G(y)$$

Method 2 $u(x,y)=v(x,y)+\psi(y)$

G(y)

$$A\frac{\partial^2 v}{\partial x^2}+B\frac{\partial^2}{\partial x\partial y}v+C\frac{\partial^2}{\partial y^2}v+D\frac{\partial v}{\partial x}+E\frac{\partial v}{\partial y}+Fv=0$$

Problem B

[Constraint]: G is independent of x

Method 3 $u(x,y)=v(x,y)+\psi_1(x)+\psi_2(y)$

[Constraint]: $G = G_1(x) + G_2(y)$

$G_1(x)$ is independent of y

$G_2(y)$ is independent of x

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

$$u(x,y)=v(x,y)+\psi_1(x)+\psi_2(y)$$

$$A\frac{\partial^2 v(x,y)}{\partial x^2}+B\frac{\partial^2 v(x,y)}{\partial x\partial y}+C\frac{\partial^2 v(x,y)}{\partial y^2}+D\frac{\partial v(x,y)}{\partial x}+E\frac{\partial v(x,y)}{\partial y}+Fv(x,y)$$

$$+A\psi_1''(x)+D\psi_1'(x)+F\psi_1(x)+C\psi_2''(y)+E\psi_2'(y)+F\psi_2(y)=G_1(x)+G_2(y)$$

Problem A: (ODE for $\psi_1(x)$)

$$A\psi_1''(x)+D\psi_1'(x)+F\psi_1(x)=G_1(x)$$

Problem B: (ODE for $\psi_2(x)$)

$$C\psi_2''(y)+E\psi_2'(y)+F\psi_2(y)=G_2(y)$$

Problem C: (homogenous PDE for $v(x, y)$)

$$A\frac{\partial^2 v(x,y)}{\partial x^2}+B\frac{\partial^2 v(x,y)}{\partial x\partial y}+C\frac{\partial^2 v(x,y)}{\partial y^2}+D\frac{\partial v(x,y)}{\partial x}+E\frac{\partial v(x,y)}{\partial y}+Fv(x,y)=0$$

2.6.3 Method 4 (只教不考)

Method 4 $u(x,t)=v(x,t)+\psi(x,t)$

Constraint of Method 4: Not applicable for Laplace's equation.

Method 1 can be applied to the wave equation and Laplace's equation, but Method 4 cannot.

Example:

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\quad 0<x<L,\quad t>0$$

$$u(0,t)=u_0(t),\quad u(L,t)=u_1(t),\quad t>0$$

$$u(x,0)=f(x),\quad 0<x<L,$$

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\quad 0<x<L,\quad t>0$$

$$u(0,t)=u_0(t),\quad u(L,t)=u_1(t),\quad t>0$$

$$u(x,0)=f(x),\quad 0<x<L,$$

Set $u(x,t)=v(x,t)+\psi(x,t)$

Since $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 \psi}{\partial x^2}$ $\quad\dfrac{\partial u}{\partial t}=\dfrac{\partial v}{\partial t}+\dfrac{\partial \psi}{\partial t}$

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t}\Longrightarrow k\frac{\partial^2 v}{\partial x^2}+k\frac{\partial^2 \psi}{\partial x^2}+F(x,t)=\frac{\partial v}{\partial t}+\frac{\partial \psi}{\partial t}$$

$$u(0,t)=u_0(t)\Longrightarrow v(0,t)+\psi(0,t)=u_0(t)$$

$$u(L,t)=u_1(t)\Longrightarrow v(L,t)+\psi(L,t)=u_1(t)$$

$$u(x,0)=f(x)\Longrightarrow v(x,0)+\psi(x,0)=f(x)$$

Therefore, after setting $u(x,t)=v(x,t)+\psi(x,t)$, we separate

$$k\frac{\partial^2 u}{\partial x^2}+F(x,t)=\frac{\partial u}{\partial t},\quad 0<x<L,\quad t>0$$

$$u(0,t)=u_0(t),\quad u(L,t)=u_1(t),\quad t>0$$

$$u(x,0)=f(x),\quad 0<x<L,$$

$\psi(x,t)= C_1(t)x+C_0(t)$

into two sub-problems:

ψ is dependent on both x and t

Problem A: $k\frac{\partial^2 \psi}{\partial x^2}=0,\quad \psi(0,t)=u_0(t),\quad \psi(L,t)=u_1(t)$

Problem B: $k\frac{\partial^2 v}{\partial x^2}+G(x,t)=\frac{\partial v}{\partial t},\quad 0<x<L,\quad t>0$

$$v(0,t)=0,\quad v(L,t)=0,\quad t>0$$

$$v(x,0)=f(x)-\psi(x,0),\quad 0<x<L$$

where $G(x,t)=F(x,t)-\frac{\partial \psi}{\partial t}$

Guess: The solution of Problem B $v(x,t)=\sum_{n=1}^{\infty}v_n(t)\sin\frac{n\pi x}{L}$

Problem A: $k\dfrac{\partial^2\psi}{\partial x^2}=0,\quad \psi(0,t)=u_0(t),\qquad \psi(L,t)=u_1(t)$

$$\psi(x,t)=c_1(t)x+c_0(t)$$

$$\psi(0,t)=u_0(t)\Longrightarrow c_0(t)=u_0(t)$$

$$\psi(L,t)=u_1(t)\Longrightarrow c_1(t)L+u_0(t)=u_1(t)$$

$$\psi(x,t)=u_0(t)+\frac{x}{L}\big(u_1(t)-u_0(t)\big)$$

To Solve Problem B:

$$k\frac{\partial^2 v}{\partial x^2}+G(x,t)=\frac{\partial v}{\partial t},\quad 0<x<L,\quad t>0$$

$$v(0,t)=0,\quad v(L,t)=0,\quad t>0$$

$$v(x,0)=f(x)-\psi(x,0),\quad 0<x<L$$

where $G(x,t)=F(x,t)-\frac{\partial \psi}{\partial t}$

$= F(x,t)+u_0'(t)+\frac{x}{L}\left(u_1'(t)-u_0'(t)\right)$

An assumption can be applied
(from the associated homogeneous PDE).

$$v(x,t)=\sum_{n=1}^{\infty} v_n(t)\sin\frac{n\pi}{L}x \qquad G(x,t)=\sum_{n=1}^{\infty} G_n(t)\sin\frac{n\pi}{L}x$$

from page 131

$G_n(t)=\frac{2}{L}\int_0^L G(x,t)\sin\frac{n\pi}{L}x\,dx$

Try to solve $v_n(t)$ and $G_n(t)$.

$$\sum_{n=1}^{\infty}\left(-k\frac{n^2\pi^2}{L^2}v_n(t)+G_n(t)\right)\sin\frac{n\pi}{L}x=\sum_{n=1}^{\infty} v_n'(t)\sin\frac{n\pi}{L}x$$

Summary for the Process of Method 4

(Step 1) Use $u(x,t) = v(x,t) + \psi(x,t)$ to separate the original problem into two sub-problems.

(Step 2) Solve Problem A

(Step 3) Use the associated homogeneous PDE to express the solution of Problem B by Fourier sine series

(Step 4) Expand $G_n(t)$ to solve $v_n(t)$

(Step 5) Use $u(x, 0)$ to solve the unknowns of $v_n(t)$

(Step 6) Add the solutions of Problems A and B and obtain $u(x, t)$.

[Example 2]

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$u(0,t) = \cos t, \quad u(1,t) = 0, \quad t > 0$$

$$u(x,0) = 0, \quad 0 < x < 1.$$

(Solution): $u(x,0) = v(x,0) + \psi(x,0) = 0 \qquad v(x,0) = -\psi(x,0)$

(Step 1) $\quad u(x,t) = v(x,t) + \psi(x,t)$

Problem A: $\quad \frac{\partial^2 \psi}{\partial x^2} = 0, \qquad \psi(0,t) = \cos t, \quad \psi(1,t) = 0$

Problem B:

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial \psi}{\partial t} = \frac{\partial v}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0$$

$$v(x,0) = -\psi(x,0), \quad 0 < x < 1$$

(Step 2)

Problem A: $\dfrac{\partial^2 \psi}{\partial x^2} = 0, \qquad \psi(0,t) = \cos t, \quad \psi(1,t) = 0$

$$\psi(x,t) = c_1(t)\,x + c_0(t)$$

Solution: $\psi(x,t) = [0 - \cos t]\,x + \cos t = (1-x)\cos t$

Problem B: $\dfrac{\partial^2 v}{\partial x^2} + (1-x)\sin t = \dfrac{\partial v}{\partial t}, \quad 0 < x < 1, \quad t > 0$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0$$

$$v(x,0) = x - 1, \quad 0 < x < 1$$

We can guess that the solution of Problem B is

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin n\pi x$$

Problem B:
$$\frac{\partial^2 v}{\partial x^2}+(1-x)\sin t=\frac{\partial v}{\partial t},\quad 0<x<1,\quad t>0$$
$$v(0,t)=0,\quad v(1,t)=0,\quad t>0$$
$$v(x,0)=x-1,\quad 0<x<1$$

(Step 3) From the associated homogeneous PDE
$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial v}{\partial t},\quad 0<x<1,\quad t>0$$
$$v(0,t)=0,\quad v(1,t)=0,\quad t>0$$
$$v(x,0)=x-1,\quad 0<x<1$$

$$v(x,t)=X(x)T(t)$$
$$X''(x)T(t)=X(x)T'(t) \qquad \frac{X''(x)}{X(x)}=\frac{T'(t)}{T(t)}=-\lambda$$
$$X''(x)+\lambda X(x)=0 \qquad X(0)=0 \qquad X(1)=0$$
$$T'(t)+\lambda T(t)=0$$

$$X''(x)+\lambda X(x)=0 \qquad X(0)=0 \qquad X(1)=0$$

After checking the three cases, the non-trivial solution exists only when

$$\lambda = n^2\pi^2 > 0$$

In this case,

$$X''(x)+n^2\pi^2 X(x)=0 \qquad X(x)=c\sin n\pi x$$

Therefore, the solution of Problem B should has the following form:

$$v(x,t)=\sum_{n=1}^{\infty} v_n(t)\sin n\pi x$$

to be solved

$$\frac{\partial^2 v}{\partial x^2} + (1-x)\sin t = \frac{\partial v}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0$$

$$v(x,0) = x-1, \quad 0 < x < 1$$

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin n\pi x$$

$v_n(t)$ ← to be solved

(Step 4) First, express the non-homogeneous term $(1-x)\sin t$ as

$$(1-x)\sin t = \sum_{n=1}^{\infty} G_n(t) \sin n\pi x$$

From the Fourier sine series (附錄四)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

$$G_n(t) = \frac{2}{1} \int_0^1 (1-x) \sin t \sin \frac{n\pi}{1} x dx = \frac{2}{n\pi} \sin t$$

$$(1-x)\sin t = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin t \sin n\pi x$$

$$\frac{\partial^2 v}{\partial x^2}+(1-x)\sin t=\frac{\partial v}{\partial t} \qquad v(x,t)=\sum_{n=1}^{\infty} v_n(t)\sin n\pi x$$

Since $\quad (1-x)\sin t=\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin t\sin n\pi x$

$$\frac{\partial^2}{\partial x^2}v(x,t)=\sum_{n=1}^{\infty}v_n(t)\left(-n^2\pi^2\right)\sin n\pi x \qquad \frac{\partial}{\partial t}v(x,t)=\sum_{n=1}^{\infty}v_n'(t)\sin n\pi x$$

we have

$$\sum_{n=1}^{\infty}\left[v_n(t)\left(-n^2\pi^2\right)+\frac{2}{n\pi}\sin t\right]\sin n\pi x=\sum_{n=1}^{\infty}v_n'(t)\sin n\pi x$$

$$v_n'(t)+n^2\pi^2 v_n(t)=\frac{2\sin t}{n\pi}$$

particular solution
A sint + B cost

$$v_n(t)=2\frac{n^2\pi^2\sin t-\cos t}{n\pi\left(n^4\pi^4+1\right)}+C_n e^{-n^2\pi^2 t}$$

$$v(x,t)=\sum_{n=1}^{\infty}\left(2\frac{n^2\pi^2\sin t-\cos t}{n\pi\left(n^4\pi^4+1\right)}+C_n e^{-n^2\pi^2 t}\right)\sin n\pi x$$

$$\frac{\partial^2 v}{\partial x^2} + (1-x)\sin t = \frac{\partial v}{\partial t}, \quad 0 < x < 1, \quad t > 0$$

$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0$$

$$v(x,0) = x-1, \quad 0 < x < 1$$

$$v(x,t) = \sum_{n=1}^{\infty} \left(2\frac{n^2\pi^2 \sin t - \cos t}{n\pi\left(n^4\pi^4+1\right)} + C_n e^{-n^2\pi^2 t} \right) \sin n\pi x$$

(Step 5) To determine C_n, we can apply $v(x, 0) = x-1$

$$x - 1 = \sum_{n=1}^{\infty} \left(\frac{-2}{n\pi\left(n^4\pi^4+1\right)} + C_n \right) \sin n\pi x$$

page 131
p = 1

From the Fourier sine series

$$\frac{-2}{n\pi\left(n^4\pi^4+1\right)} + C_n = 2\int_0^1 (x-1)\sin n\pi x dx = \frac{-2}{n\pi}$$

$$C_n = \frac{2}{n\pi\left(n^4\pi^4+1\right)} - \frac{2}{n\pi}$$

$$v(x,t)=\sum_{n=1}^{\infty}\left(2\frac{n^2\pi^2\sin t-\cos t}{n\pi\left(n^4\pi^4+1\right)}+C_n e^{-n^2\pi^2 t}\right)\sin n\pi x$$

$$C_n=\frac{2}{n\pi\left(n^4\pi^4+1\right)}-\frac{2}{n\pi}$$

$$v(x,t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{n^2\pi^2\sin t-\cos t+e^{-n^2\pi^2 t}}{n\left(n^4\pi^4+1\right)}-\frac{e^{-n^2\pi^2 t}}{n}\right)\sin n\pi x$$

(Step 6) $u(x,t)=v(x,t)+\psi(x,t)$

$$u(x,t)=(1-x)\cos t+\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{n^2\pi^2\sin t-\cos t+e^{-n^2\pi^2 t}}{n\left(n^4\pi^4+1\right)}-\frac{e^{-n^2\pi^2 t}}{n}\right)\sin n\pi x$$

2.6 Higher-Dimensional Problems

Modifying the method in Section 2-1 just a little.

page 121 Laplacian

heat $\nabla^2 u = \frac{\partial u}{\partial t}$

Two-dimensional heat equation

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}.$$

3D heat equation

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial u}{\partial t}$$

wave $\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$

Two-dimensional wave equation

$$a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial^2 u}{\partial t^2}.$$

3D wave equation

$$a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial^2 u}{\partial t^2}$$

3D Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

$$u(x,y,t)=X(x)Y(y)T(t)$$

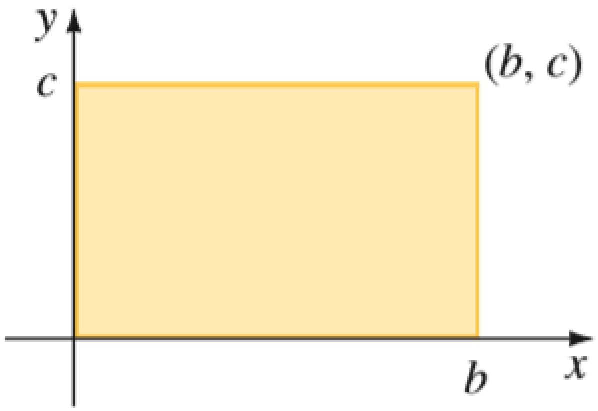
$$\frac{\partial^2 u}{\partial x^2}=X''YT,\quad \frac{\partial^2 u}{\partial y^2}=XY''T,\quad \text{and}\quad \frac{\partial u}{\partial t}=XYT'.$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.8.

[Example 1] Temperatures in a Plate

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t},\quad 0<x<b,\quad 0<y<c,\quad t>0$$

$$u(0,y,t)=0,\quad u(b,y,t)=0,\quad 0<y<c,\quad t>0$$
$$u(x,0,t)=0,\quad u(x,c,t)=0,\quad 0<x<b,\quad t>0$$
$$u(x,y,0)=f(x,y),\quad 0<x<b,\quad 0<y<c.$$



From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 12.8.

$$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t} \qquad u(x,y,t)=X(x)Y(y)T(t),$$

$$k(X''YT+XY''T)=XYT'$$

Divided by XYT $(u=XYT)$

$$k\left(\frac{X''}{X}+\frac{Y''}{Y}\right)=\frac{T'}{T} \qquad \frac{X''}{X}=-\frac{Y''}{Y}+\frac{T'}{kT}$$

$-\lambda \quad -\mu \quad -k(\lambda+\mu)$

Set

$$\frac{X''}{X}=-\frac{Y''}{Y}+\frac{T'}{kT}=-\lambda$$

$$X''+\lambda X=0$$

3 ODEs

$$\frac{Y''}{Y}=\frac{T'}{kT}+\lambda. = -\mu$$

$$\frac{Y''}{Y}=-\mu \qquad \frac{T'}{kT}+\lambda=-\mu$$

$$Y''+\mu Y=0 \qquad T'+k(\lambda+\mu)T=0.$$

$u(x,y,t) = X(x)Y(y)T(t) \quad u(0,y,t) = X(0)Y(y)T(t) = 0$

$u(0,y,t)=0, \;\; u(b,y,t)=0, \Longrightarrow X(0)=0, \;\; X(b)=0,$

$u(x,0,t)=0, \;\; u(x,c,t)=0 \Longrightarrow Y(0)=0, \;\; Y(c)=0$

$$X''+\lambda X=0, \;\; X(0)=0, \;\; X(b)=0$$

$$Y''+\mu Y=0, \;\; Y(0)=0, \;\; Y(c)=0.$$

$$T'+k(\lambda+\mu)T=0.$$

There are 3 cases for X: $\lambda=0$, $\lambda<0$, and $\lambda>0$.

There is non-trivial solution for X only when $\lambda_m = \dfrac{m^2\pi^2}{b^2} > 0$

In this case, $X(x) = c_2 \sin \dfrac{m\pi}{b} x$

$X''(x) + \lambda X(x) = 0$

$\left(-c_2 \dfrac{m^2\pi^2}{b^2} + c_2\lambda\right) \sin \dfrac{m\pi}{b} x = 0$

$\lambda = \dfrac{m^2\pi^2}{b^2}$

There are 3 cases for Y: $\mu=0$, $\mu<0$, and $\mu>0$.

There is non-trivial solution for Y only when $\mu_n = \dfrac{n^2\pi^2}{c^2} > 0$

In this case, $Y(y) = c_4 \sin \dfrac{n\pi}{c} y$

$$\lambda_m = \frac{m^2\pi^2}{b^2} \qquad \mu_n = \frac{n^2\pi^2}{c^2}.$$

$$T' + k(\lambda+\mu)T = 0 \Longrightarrow T' + k\left(\frac{m^2\pi^2}{b^2} + \frac{n^2\pi^2}{c^2}\right)T = 0$$

$$\Longrightarrow T(t) = c_5 e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t}.$$

$T' + aT = 0$

$T(t) = ce^{-at}$

auxiliary

$m + a = 0$

$m = -a$

$$u(x,y,t) = X(x)Y(y)T(t),$$

$$u_{mn}(x,y,t) = A_{mn} e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y,$$

$m, n = 1, 2, 3, \ldots\ldots$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} e^{-k[(m\pi/b)^2 + (n\pi/c)^2]t} \sin\frac{m\pi}{b}x \sin\frac{n\pi}{c}y.$$

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}e^{-k[(m\pi/b)^2+(n\pi/c)^2]t}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y.$$

$$u(x,y,0)=f(x,y) \qquad 0<x<b,\quad 0<y<c.$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y=f(x,y)$$

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi}{c}y\right)\sin\frac{m\pi}{b}x=f(x,y)$$

replace p by b
replace b_n by $\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi}{c}y$
replace $f(x,y)$ by $g(x)$

From the Fourier sine series along the *x*-axis

$$g(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{p}x \qquad b_n=\frac{2}{p}\int_0^p g(x)\sin\frac{n\pi}{p}x\,dx$$

$$\left(\sum_{n=1}^{\infty}A_{mn}\sin\frac{n\pi}{c}y\right)=\frac{2}{b}\int_0^b f(x,y)\sin\frac{m\pi}{b}x\,dx$$

$p\to c,\ x\to y$

From the Fourier sine series along the *y*-axis

$$A_{mn}=\frac{2}{c}\int_0^c\frac{2}{b}\int_0^b f(x,y)\sin\left(\frac{m\pi}{b}x\right)dx\sin\left(\frac{n\pi}{c}y\right)dy$$

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}e^{-k[(m\pi/b)^2+(n\pi/c)^2]t}\sin\frac{m\pi}{b}x\sin\frac{n\pi}{c}y.$$

where

$$A_{mn}=\frac{4}{bc}\int_0^c\int_0^b f(x,y)\sin\left(\frac{m\pi}{b}x\right)dx\sin\left(\frac{n\pi}{c}y\right)dy$$

[Example 2] $$k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial u}{\partial t}.$$

$u(x,y,z,t)=X(x)Y(y)Z(z)T(t)$

$k(X''YZT+XY''ZT+XYZ''T)=XYZT'$

divided by $u=XYZT$

$$k\left(\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}\right)=\frac{T'}{T}$$

$-\lambda \quad -\tau \quad -\rho \quad -k(\lambda+\tau+\rho)$

$\frac{X''}{X}=-\lambda,\ \frac{Y''}{Y}=-\tau,\ \frac{Z''}{Z}=-\rho,\ \frac{T'}{T}=-k(\lambda+\tau+\rho)$

4 ODEs

$X''+\lambda X=0$

$Y''+\tau Y=0$

$Z''+\rho Y=0$

$T'+k(\lambda+\tau+\rho)=0$

Double Sine Series (Two Dimensional Sine Series)

$$f(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{m,n}\sin\left(\frac{m\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

where

$$B_{m,n}=\frac{4}{bc}\int_0^c\int_0^b f(x,y)\sin\left(\frac{m\pi}{b}x\right)\sin\left(\frac{n\pi}{c}y\right)dxdy$$

2.7 Polar Coordinates

Sections 2.7, 2.8, 2.9 are extended from Sec. 2.1, but the polar, cylindrical, and spherical coordinates are adopted.

- 2D — Polar Coordinate (Sec. 2.7) r, θ
- 3D
  - Cylindrical Coordinate (Sec. 2.8) (r, θ, z)
  - Spherical Coordinate (Sec. 2.9) (r, θ, ϕ)

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.1

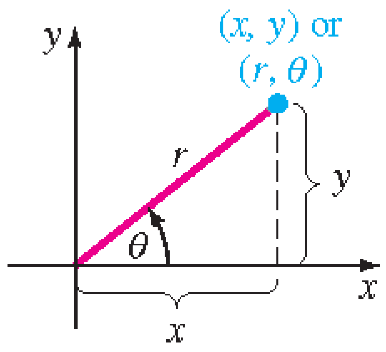


Fig. 2.7.1 Polar coordinates of a point (x, y) are (r, θ)

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.1.

(x, y) ⟶ (r, θ)

original coordinate → polar coordinate

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{and} \quad r^2 = x^2 + y^2$$

the Laplacian of u in x-y coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

the Laplacian of u in polar coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

In this section we focus only on boundary-value problems involving Laplace's equation $\nabla^2 u = 0$ in polar coordinates:

The key points of this section.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

heat ...... $= \frac{\partial u}{\partial t}$

wave ...... $= \frac{\partial^2 u}{\partial t^2}$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \Longrightarrow \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

(Proof): Since

From page 184 to page 188

$$x = r\cos\theta, \quad y = r\sin\theta$$

we have

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan\frac{y}{x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial \theta} \qquad \frac{\partial u}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial \theta}$$

page 11

$$\frac{\partial u}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{1+(y/x)^2}\frac{-y}{x^2}\frac{\partial u}{\partial \theta}$$

from $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

$$= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{-y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial \theta} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan \frac{y}{x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial \theta} = \frac{2y}{2\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{1+(y/x)^2}\frac{1}{x}\frac{\partial u}{\partial \theta}$$

$$= \frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{1}{\sqrt{x^2+y^2}}\frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\boxed{\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}} \qquad \boxed{\frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\frac{\partial u}{\partial x} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}\frac{\partial u}{\partial x}$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial \theta\partial r}$$

$$+ \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial \theta\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r}\frac{\partial u}{\partial y} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}\frac{\partial u}{\partial y}$$

$$= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$= \sin^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}$$

$$+ \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

The proof is completed.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

In this section, we focus on the Laplace's equation with steady temperature, i.e.,

$$\nabla^2 u = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$\frac{\partial u}{\partial t} = 0$ Laplace's equation

[Example 1] Steady Temperatures in a Circular Plate

Solve Laplace's equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to $u(c,\theta) = f(\theta),\ 0 < \theta < 2\pi$

$0 \le r \le c$

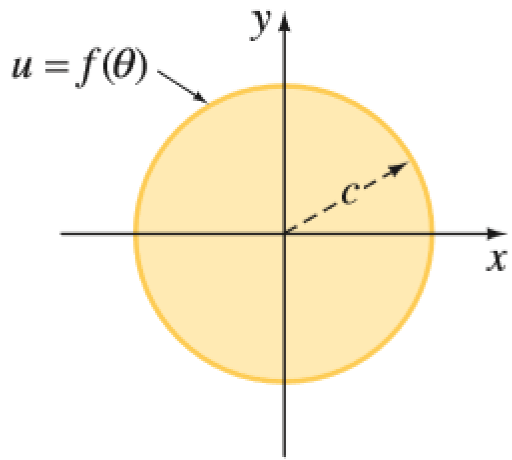


Fig. 2.8.2

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.2.

SOLUTION

(Step 1) Suppose that $u(r,\theta) = R(r)\Theta(\theta)$

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

divided by $u = R\Theta$

then multiplied by r^2

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

(Step 2) $$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda.$$

Cauchy-Euler

$$r^2R''+rR'-\lambda R=0 \qquad \Theta''+\lambda\Theta=0.$$

There is no zero boundary condition.

But note that $\Theta(\theta)$ should be periodic: $\Theta(\theta)=\Theta(\theta+2\pi)$

It must be satisfied.

(Step 3) Then, we try to solve

$$\Theta''(\theta)+\lambda\Theta(\theta)=0, \qquad \Theta(\theta)=\Theta(\theta+2\pi)$$

Case 1: $\lambda=0, \Longrightarrow \Theta''(\theta)=0, \Longrightarrow \Theta(\theta)=c_1+c_2\theta$

from $\Theta(\theta)=\Theta(\theta+2\pi) \Longrightarrow \Theta(\theta)=c_1$

Case 2: $\lambda<0,$ set $\lambda=-\alpha^2 \Longrightarrow \Theta''(\theta)-\alpha^2\Theta(\theta)=0,$

$\Longrightarrow \Theta(\theta)=c_1\cosh\alpha\theta+c_2\sinh\alpha\theta \Longrightarrow \Theta(\theta)=0$

(trivial)

$$\Theta''(\theta)+\lambda\Theta(\theta)=0, \qquad \Theta(\theta)=\Theta(\theta+2\pi)$$

Case 3: $\lambda>0$, set $\lambda=\alpha^2 \Longrightarrow \Theta''(\theta)+\alpha^2\Theta(\theta)=0,$

$$\Longrightarrow \Theta(\theta)=c_1\cos\alpha\theta+c_2\sin\alpha\theta$$

From $\Theta(\theta)=\Theta(\theta+2\pi)$

$$c_1\cos\alpha\theta+c_2\sin\alpha\theta=c_1\cos(\alpha\theta+\alpha 2\pi)+c_2\sin(\alpha\theta+\alpha 2\pi)$$

$\alpha 2\pi=n2\pi$ where n is a positive integer,

$\alpha=n,\qquad (\lambda=n^2)$

$\Theta(\theta)=c_1\cos n\theta+c_2\sin n\theta$ where n is a positive integer,

when $n=0$, $\lambda=0$

$\Theta(\theta)=c_1$ (case 1)

Combine the results of Cases 1 and 3

$$\Theta(\theta)=c_1\cos n\theta+c_2\sin n\theta \qquad (\lambda=n^2)$$

where n is a nonnegative integer

$n=0, 1, 2, 3 \cdots$

case 1 (for $n=0$), case 3 (for $n=1, 2, 3 \cdots$)

(Step 4)

$$r^2R''+rR'-\lambda R=0$$

This is an important application of the Cauchy-Euler equation on page 37

Since $\lambda_n=n^2,\ n=0,\ 1,\ 2,\ \ldots\ldots.$

$$r^2R''+rR'-n^2R=0$$ ← Cauchy-Euler

Auxiliary: $m(m-1)+m-n^2=0,\quad m=\pm n$

$m^2-n^2=0$

the solutions are

$$R(r)=c_3+c_4\ln r,\quad n=0$$

$$R(r)=c_5r^n+c_6r^{-n},\quad n=1,\ 2,\ \ldots.$$

Since $\ln 0\to-\infty$ $\quad 0^{-n}\to\infty$ $\quad$ but $R(0)$ should not be infinite

$c_4=c_6=0$ should be satisfied

⟹ $R(r)=c_3,\quad n=0,\qquad R(r)=c_5r^n,\quad n=1,\ 2,\ \ldots.$

$\Theta(\theta) = c_1 \cos n\theta + c_2 \sin n\theta$ where n is a nonnegative integer

$R(r) = c_3, \quad n = 0, \qquad R(r) = c_5 r^n, \quad n = 1,\ 2,\ \ldots.$

(Step 5) $u_n(r,\theta) = R(r)\Theta(\theta)$

$u_0(r,\theta) = A_0$ when $n = 0$,

$u_n(r,\theta) = r^n(A_n \cos n\theta + B_n \sin n\theta)$ when $n = 1,\ 2,\ \ldots.$

(Step 6)

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

(Step 7)

By applying the boundary condition $u(c,\theta) = f(\theta),\ 0 < \theta < 2\pi.$

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} c^n (A_n \cos n\theta + B_n \sin n\theta)$$

Next, solve the unknowns from the formula of the Fourier series

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} c^n (A_n \cos n\theta + B_n \sin n\theta)$$

From the formula of the Fourier series

$p \to \pi, \; x \to \theta$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$-p < x < p$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

$\Rightarrow$

$$\begin{cases} A_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\,d\theta \\ A_n = \dfrac{1}{c^n\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta \\ B_n = \dfrac{1}{c^n\pi}\displaystyle\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta. \end{cases}$$

Since $f(\theta) = f(\theta + 2\pi)$ $\Rightarrow$

$$\begin{cases} A_0 = \dfrac{1}{2\pi}\displaystyle\int_{0}^{2\pi} f(\theta)\,d\theta \\ A_n = \dfrac{1}{c^n\pi}\displaystyle\int_{0}^{2\pi} f(\theta)\cos n\theta\,d\theta \\ B_n = \dfrac{1}{c^n\pi}\displaystyle\int_{0}^{2\pi} f(\theta)\sin n\theta\,d\theta. \end{cases}$$

$\frac{\partial u}{\partial t} = 0$

EXAMPLE 2 Steady Temperatures in a Semicircular Plate

Find the steady-state temperature $u(r, \theta)$ in

$\nabla^2 u = 0$

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.2.

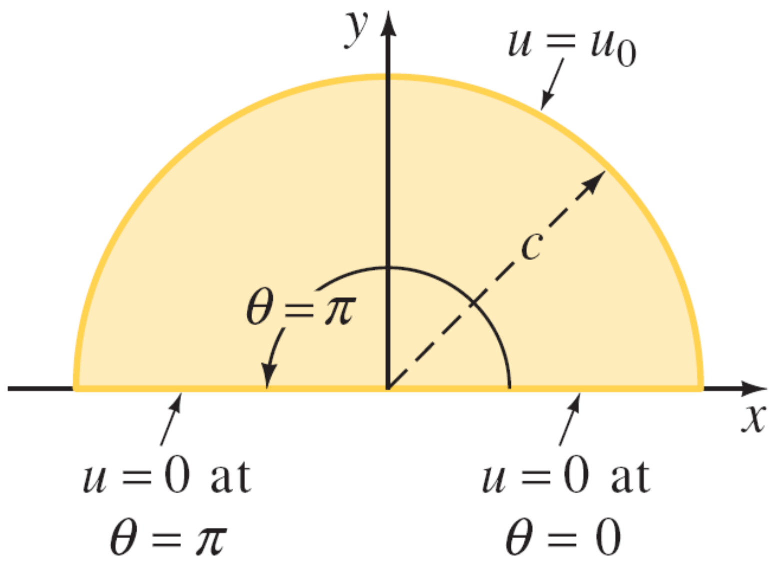


Fig. 2.8.3 Semicircular plate in Example 2

SOLUTION The problem can be formulated as

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$$

$$u(c,\theta)=u_0,\quad 0<\theta<\pi$$

$$u(r,0)=0,\quad u(r,\pi)=0,\quad 0<r<c.$$

(Step 1) Suppose that $u(r,\theta)=R(r)\Theta(\theta)$

$$R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)=0$$

$$r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\frac{\Theta''(\theta)}{\Theta(\theta)}=0$$

(Step 2)

$$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda$$

$$r^2R''+rR'-\lambda R=0$$

$$\Theta''+\lambda\Theta=0.$$

(Step 3)

From $u(r,0)=0, \quad u(r,\pi)=0,$

$$R(r)\Theta(0)=0, \quad R(r)\Theta(\pi)=0,$$

$$\Theta(0)=0 \text{ and } \Theta(\pi)=0.$$

We then try to solve

$$\Theta''+\lambda\Theta=0, \quad \Theta(0)=0, \quad \Theta(\pi)=0.$$

Case 1: $\lambda=0 \Rightarrow \Theta''(\theta)=0 \Rightarrow \Theta(\theta)=c_1\theta+c_2$ ✗

From $\Theta(0)=0, \quad \Theta(\pi)=0 \Rightarrow \Theta(\theta)=0$

Case 2: $\lambda<0, \quad \text{set } \lambda=-\alpha^2 \Rightarrow \Theta''(\theta)-\alpha^2\Theta(\theta)=0$ ✗

$$\Rightarrow \Theta(\theta)=c_3\cosh\alpha\theta+c_4\sinh\alpha\theta$$

From $\Theta(0)=0, \quad \Theta(\pi)=0 \Rightarrow \Theta(\theta)=0$

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(0) = 0, \quad \Theta(\pi) = 0.$$

Case 3: $\lambda > 0$, set $\lambda = \alpha^2 \Rightarrow \Theta''(\theta) + \alpha^2\Theta(\theta) = 0$

$$\Rightarrow \Theta(\theta) = c_5 \cos\alpha\theta + c_6 \sin\alpha\theta$$

$\sin\alpha\pi = 0$

$\alpha\pi = n\pi$

$\alpha = n$

From $\Theta(0) = 0$, $\Theta(\pi) = 0 \Rightarrow c_5 = 0$, $\alpha = n$

$$\Rightarrow \Theta(\theta) = c_6 \sin n\theta \qquad n = 1, 2, 3, \ldots.$$

The only nontrivial solution for $\Theta(\theta)$ is

$$\boxed{\Theta(\theta) = c_6 \sin n\theta} \qquad n = 1, 2, 3, \ldots.$$

In this case, $\lambda = n^2$

$m(m-1) + m - n^2 = 0$

$m^2 = n^2$

$m = \pm n$

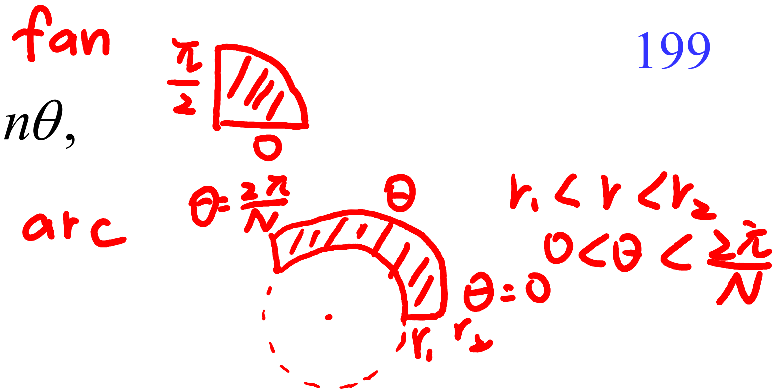
(Step 4) To solve $R(r)$ page 37

$$r^2 R'' + rR' - \lambda R = 0 \Rightarrow r^2 R'' + rR' - n^2 R = 0$$

$$R(r) = c_7 r^n + c_8 r^{-n}$$

To be bounded at $r = 0$, c_8 must be 0

$$\boxed{R(r) = c_7 r^n} \qquad n = 1, 2, 3, \ldots.$$



(Step 5) $u_n(r,\theta) = R(r)\Theta(\theta) = A_n r^n \sin n\theta,$

(Step 6) $u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$

(Step 7) From $u(c,\theta) = u_0$

$$u_0 = \sum_{n=1}^{\infty} A_n c^n \sin n\theta$$

Using Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x$$

$$0 < x < p$$

$$b_n = \frac{2}{p}\int_0^p f(x) \sin \frac{n\pi}{p} x dx$$

$p=\pi$, $x \to \theta$, $b_n = A_n c^n$

$$\int_0^{\pi} \sin n\theta \, d\theta = -\frac{1}{n}\cos n\theta \Big|_0^{\pi}$$

$$A_n c^n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin n\theta d\theta$$

$= \frac{-1}{n}((-1)^n - 1)$

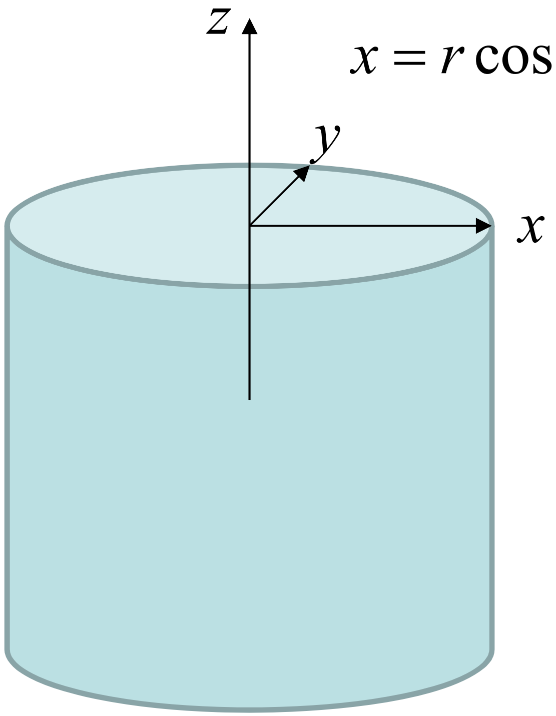
$\cos n\pi = (-1)^n$

$$A_n = \frac{2u_0}{\pi c^n} \frac{1-(-1)^n}{n}$$

Solution:

$$u(r,\theta) = \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \left(\frac{r}{c}\right)^n \sin n\theta$$

2.8 CYLINDRICAL COORDINATES



$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.2

2.8.1 Review for Special Functions

- **Bessel's equation of order *v***

$$x^2 y'' + xy' + (x^2 - v^2) y = 0$$

Solution: $c_1 J_v(x) + c_2 Y_v(x)$

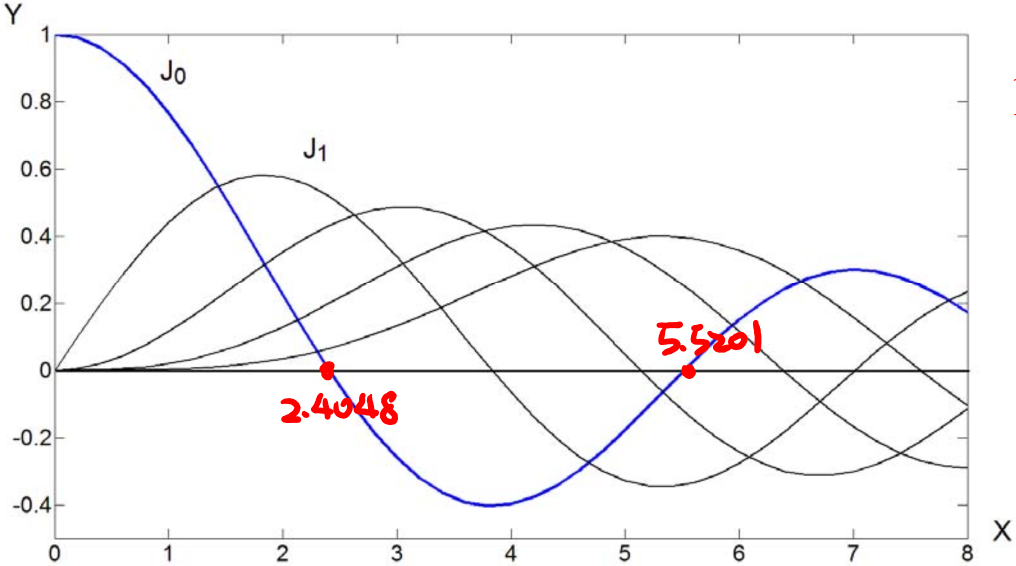
$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}$$: 1st kind Bessel function

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$: 2nd kind Bessel function

where $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt$ (gamma function)

$$\Gamma(n+1) = n!$$

$$\Gamma(x+1) = x\Gamma(x)$$

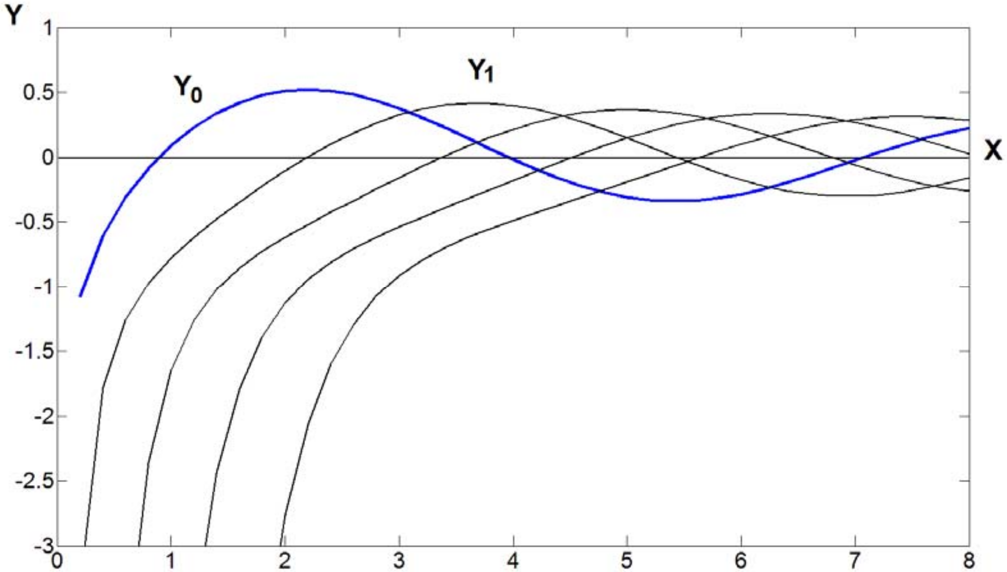


1st kind Bessel function

$$J_0(0) = 1$$

$$J_v(0) = 0 \quad for\ v \neq 0$$

$$\lim_{x \to \infty} J_v(x) = 0$$



2nd kind Bessel function

$$Y_v(0) \to -\infty$$

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

Zeros of J_0, J_1, Y_0, and Y_1

	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
x_1	2.4048	0.0000	0.8936	2.1971
x_2	5.5201	3.8317	3.9577	5.4297
x_3	8.6537	7.0156	7.0861	8.5960
x_4	11.7915	10.1735	10.2223	11.7492
x_5	14.9309	13.3237	13.3611	14.8974

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

- **Generalization of Bessel's equation of order *v***

$$x^2y'' + xy' + (x^2 - v^2)y = 0 \qquad 解：c_1J_v(x) + c_2Y_v(x)$$

$$x^2y'' + xy' + (\alpha^2x^2 - v^2)y = 0 \qquad 解：c_1J_v(\alpha x) + c_2Y_v(\alpha x)$$

Proof: Set $t = \alpha x$

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \alpha\frac{dy}{dt}$$

Similarly, $\displaystyle \frac{d^2y}{dx^2} = \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right) = \alpha\frac{d}{dt}\left(\alpha\frac{dy}{dt}\right) = \alpha^2\frac{d^2y}{dt^2}$

$$原式 = x^2y'' + xy' + (\alpha^2x^2 - v^2)y = \frac{t^2}{\alpha^2}\alpha^2\frac{d^2y}{dt^2} + \frac{t}{\alpha}\alpha\frac{dy}{dt} + (\alpha^2\frac{t^2}{\alpha^2} - v^2)y$$

$$= t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - v^2)y = 0$$ → 對 t 而言是 Bessel equation

$$y = c_1J_v(t) + c_2Y_v(t) = c_1J_v(\alpha x) + c_2Y_v(\alpha x)$$

• Modified Bessel's equation of order *v*

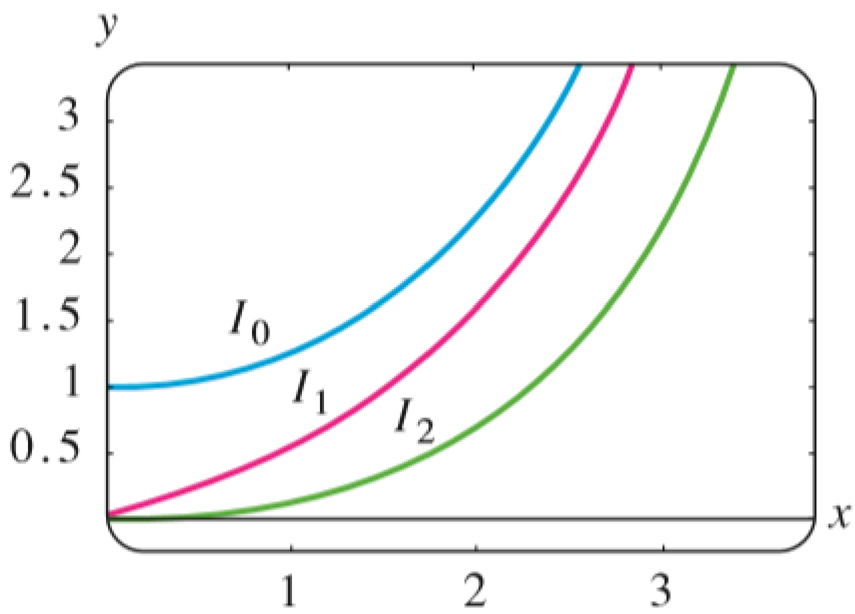
$$x^2y'' + xy' + (-x^2 - v^2)y = 0$$ 解：$c_1 I_v(x) + c_2 K_v(x)$

$$x^2y'' + xy' + (-\alpha^2 x^2 - v^2)y = 0$$ 解：$c_1 I_v(\alpha x) + c_2 K_v(\alpha x)$

其中 $I_v(x) = i^{-v} J_v(ix)$ 稱作是 modified Bessel function of the first kind of order *v*

$K_v(x) = \dfrac{\pi}{2} \dfrac{I_{-v}(x) - I_v(x)}{\sin v\pi}$ 稱作是 modified Bessel function of the second kind of order *v*

當 *v* 為整數時，也是取 limit

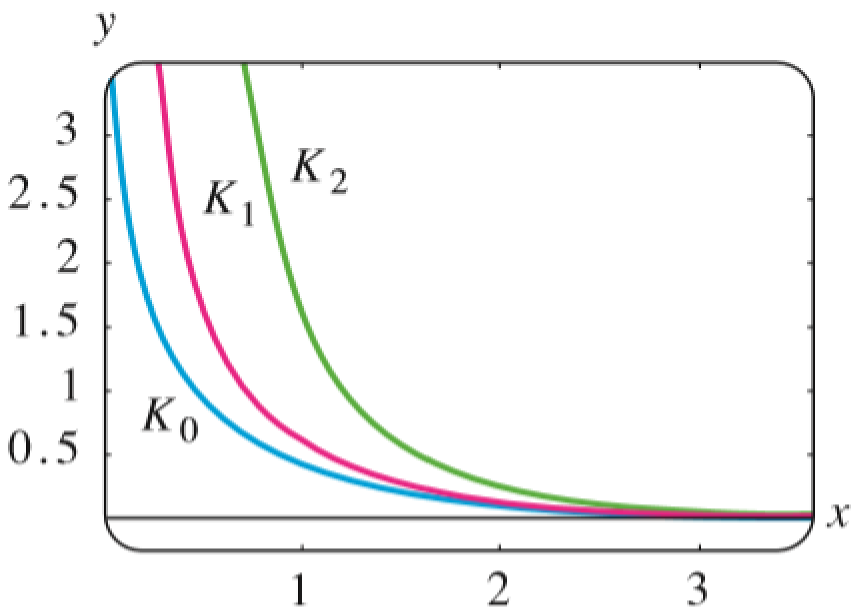


Modified Bessel functions of the first kind for $n = 0, 1, 2$

$$I_0(0) = 1$$

$$I_v(0) = 0 \quad \textit{for } v \neq 0$$

$$I_v(x) \neq 0 \quad \textit{for } x \neq 0$$



Modified Bessel functions of the second kind for $n = 0, 1, 2$

$$K_v(0) \to \infty$$

$$K_v(x) \neq 0 \quad \textit{for } x \neq 0$$

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

Similar to Cauchy-Euler but the coefficient of y'' is $1-x^2$

• Legendre's equation of order *n*

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

One of the solution: Legendre polynomials $P_n(x)$

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x)=\frac{1}{2}\left(3x^2-1\right)$$

$$P_3(x)=\frac{1}{2}\left(5x^3-3x\right)$$

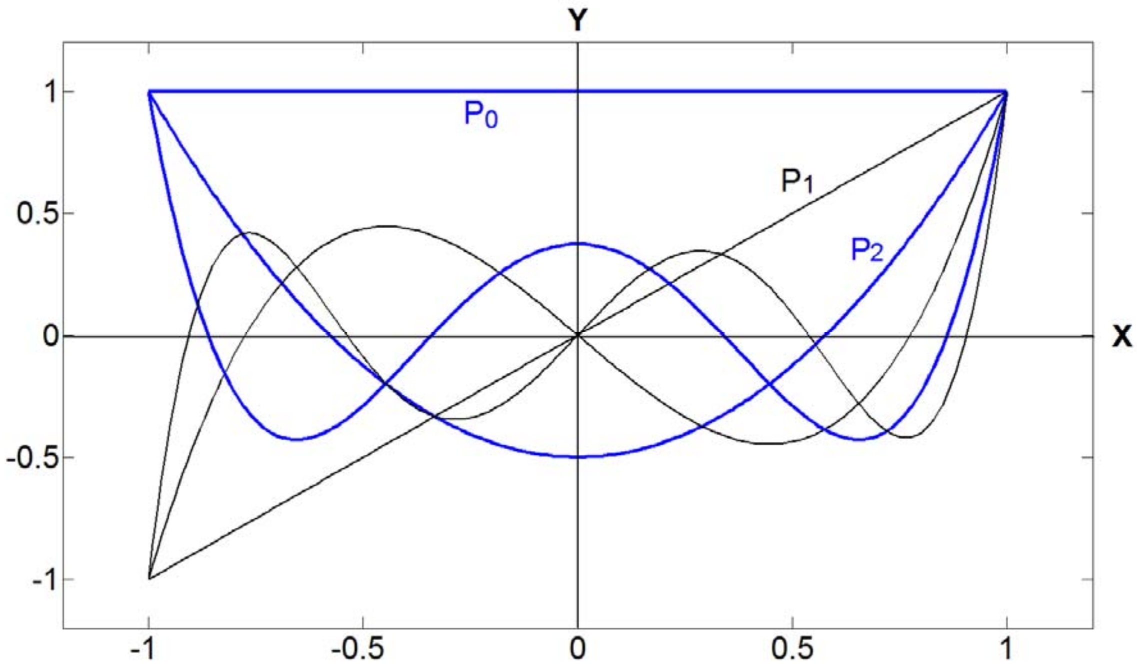
$$P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right)$$

$$P_5(x)=\frac{1}{8}\left(63x^5-70x^3+15x\right)$$

$$\int_{-1}^{1}P_n(x)P_m(x)dx=\begin{cases}0 & \text{if } m\neq n\\ \frac{1}{2n+1} & \text{if } m=n\end{cases}$$

$$\int_{-1}^{1}P_n(x)P_m(x)\,dx=\frac{2}{2n+1}\delta_{n,m}$$

Legendre polynomials



Interval:

$x \in [-1, 1]$

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

2.8.2 PDE for Cylindrical Coordinates

Laplacian in Cylindrical Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$

For the case of **radial symmetry** $\frac{\partial u}{\partial \theta} = 0$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}.$$

[Example 1] Steady Temperatures in a Circular Cylinder

$\nabla^2 u = 0$

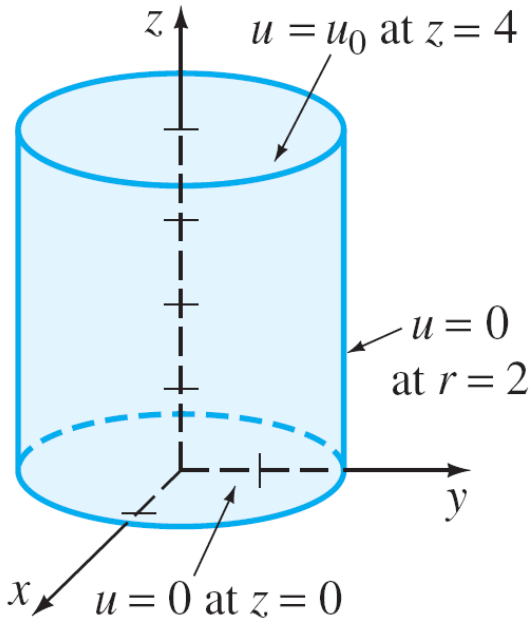


FIGURE 13.2.5 Circular cylinder in Example 2

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.2.

$$u(2, z) = 0, \quad 0 < z < 4$$

$$u(r, 0) = 0, \quad u(r, 4) = u_0, \quad 0 < r < 2.$$

It is a problem of the 2D heat equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} = 0$$

steady temperatures ↓

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

radial symmetry ↓ (since $u = 0$ when $r = 2$)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$u(2, z) = 0, \quad 0 < z < 4$$

$$u(r, 0) = 0, \quad u(r, 4) = u_0, \quad 0 < r < 2$$

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$$

(Step 1) $u(r,z)=R(r)Z(z)$

$$R''(r)Z(z)+\frac{1}{r}R'(r)Z(z)+R(r)Z''(z)=0$$

$$\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}+\frac{Z''(z)}{Z(z)}=0$$

(Step 2) $$\frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z}=-\lambda$$

$$rR''+R'+\lambda rR=0 \qquad Z''-\lambda Z=0$$

From $u(2,z)=0,\quad u(r,0)=0,$

$$R(2)=0 \qquad Z(0)=0$$

$$rR'' + R' + \lambda r R = 0 \qquad R(2) = 0$$

$$Z'' - \lambda Z = 0 \qquad Z(0) = 0$$

(Step 3)

auxiliary

Case 1: $\lambda = 0$ $\quad rR'' + R' = 0$ $\quad r^2R'' + rR' = 0$ $\quad m(m-1) + m = 0$

✗

auxiliary: $m(m\text{-}1) + m = 0$, $\quad m = 0, 0$

$$R(r) = c_1 + c_2 \ln r$$

Since $\ln 0 \to -\infty$, $c_2 = 0$

Since $R(2) = 0 \to c_1 + c_2 \ln 2 = 0 \to c_1 = 0$

$R(r) = 0$ (trivial)

Case 2: $\lambda = -\alpha^2 < 0$ $\quad rR'' + R' - \alpha^2 rR = 0$

$$r^2R'' + rR' - \alpha^2 r^2 R = 0$$

compared to modified Bessel function (page 205)

$\nu = 0,\ r \to x$

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0 \qquad \text{solution}: c_1 I_\nu(x) + c_2 K_\nu(x)$$

$$x^2y'' + xy' - (\alpha^2 x^2 + \nu^2)y = 0 \qquad \text{solution}: c_1 I_\nu(\alpha x) + c_2 K_\nu(\alpha x)$$

$I_\nu(x)$: modified Bessel function of the 1st kind

$K_\nu(x)$: modified Bessel function of the 2nd kind

The solution of $rR'' + R' - \alpha^2 rR = 0$ is

$$R(r) = c_1 I_0(\alpha r) + c_2 K_0(\alpha r)$$

$I_0(x) \neq 0$ for all x

(page 206) Since $K_0(0) \to \infty$, $c_2 = 0$

Since $R(2) = 0$, $c_1 I_0(2\alpha) = 0$ $\Rightarrow$ $R(r) = 0$ (trivial)

(page 206) From $I_0(x) \neq 0$ for all $x \neq 0 \to c_1 = 0$

Case 3: $\lambda = \alpha^2 > 0$

$$rR'' + R' + \alpha^2 rR = 0$$

$$r^2R'' + rR' + \alpha^2 r^2 R = 0$$

$\nu = 0,\ r \to x$

compared to the Bessel function on page 204

$$x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2) y = 0$$

solution：$c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$

$$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$$

(page 202)

Since $Y_0(0) \to -\infty$, $c_2 = 0$ $R(r) = c_1 J_0(\alpha r)$

Since $R(2) = 0$ $c_1 J_0(2\alpha) = 0$

Therefore,

$$\boxed{R(r) = c_1 J_0(\alpha_n r)}$$ where $\alpha_n = x_n / 2$

x_n are the zeros of $J_0(x)$, i.e., $J_0(x_n) = 0$

(defined on page 203) $\lambda_n = \alpha_n^2$, $\lambda_n = x_n^2 / 4$

(Step 4) Try to solve

$$Z''-\lambda Z=0 \qquad\qquad Z(0)=0$$

Since $\lambda_n=\alpha_n^2,$

$$Z''-\alpha_n^2 Z=0$$

$$Z(z)=c_3\cosh(\alpha_n z)+c_4\sinh(\alpha_n z)$$

From $Z(0)=0, \quad c_3=0$

$$\boxed{Z(z)=c_4\sinh(\alpha_n z)} \qquad \text{where} \quad \alpha_n=x_n/2 \qquad J_0(x_n)=0$$

(Step 5) $u(r,z)=R(r)Z(z)=A_n J_0(\alpha_n r)\sinh(\alpha_n z)$

(Step 6) $u(r,z)=\sum_{n=1}^{\infty}A_n\sinh\alpha_n z J_0(\alpha_n r).$

where $\alpha_n=x_n/2 \qquad J_0(x_n)=0$

(Step 7) From $u(r,4)=u_0,$

$$u_0=\sum_{n=1}^{\infty}A_n\sinh(4\alpha_n)J_0(\alpha_n r).$$

From Eq. (9) in Section 11.5 of D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017

$$\int_0^2 rJ_0(\alpha_n r)J_0(\alpha_m r)dr=\begin{cases}0 & if\ m\neq n\\ 2J_1^2(2\alpha_n) & if\ m=n\end{cases}$$

$$\int_0^2 u_0 rJ_0(\alpha_m r)dr=\sum_{n=1}^{\infty}A_n\sinh(4\alpha_n)\int_0^2 J_0(\alpha_n r)rJ_0(\alpha_m r)dr$$

only preserve n=m

$$=A_m\sinh(4\alpha_m)2J_1^2(2\alpha_m)$$

$\alpha_n=\frac{x_n}{2}$

$$A_n=\frac{u_0}{2\sinh(4\alpha_n)J_1^2(2\alpha_n)}\int_0^2 rJ_0(\alpha_n r)dr=\frac{u_0}{\alpha_n\sinh(4\alpha_n)J_1(2\alpha_n)}$$

(Solution):

$$u(r,z) = u_0 \sum_{n=1}^{\infty} \frac{1}{\alpha_n \sinh(4\alpha_n) J_1(2\alpha_n)} \sinh(\alpha_n z) J_0(\alpha_n r)$$

where $\alpha_n = x_n / 2$ $\quad J_0(x_n) = 0$

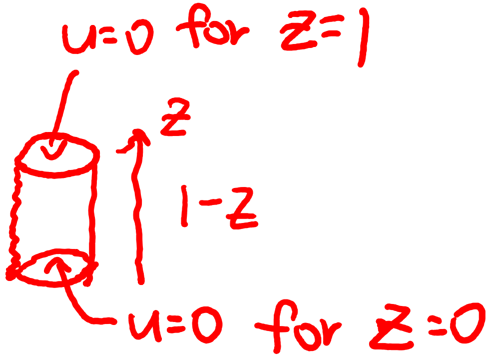
[Example 2] Steady Temperatures in a Circular Cylinder

$\nabla^2 u = 0$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$u(1,z) = 1 - z, \quad 0 < z < 1$$

$$u(r,0) = 0, \quad u(r,1) = 0 \quad 0 < r < 1$$



(Solution):

(Step 1) $u(r,z) = R(r)Z(z)$

$$R''(r)Z(z) + \frac{1}{r}R'(r)Z(z) + R(r)Z''(z) = 0$$

divided by $u = RZ$

(Step 2) $$\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -\lambda$$

$$rR'' + R' + \lambda r R = 0 \qquad Z'' - \lambda Z = 0$$

From $u(r,0) = 0, \quad u(r,1) = 0$

$$Z(0) = 0, \quad Z(1) = 0$$

(Step 3): Solve

$$Z'' - \lambda Z = 0 \qquad Z(0) = 0, \quad Z(1) = 0$$

There is no non-trivial solution for $\lambda = 0$ and $\lambda > 0$

When $\lambda < 0$, set $\lambda = -\alpha^2$,

from $Z(0)=0$, $c_1=0$
from $Z(1)=0$
$c_n \cos\alpha = 0$, $\alpha = n\pi$

$$Z'' + \alpha^2 Z = 0$$

$$Z(z) = c_1 \cos\alpha z + c_2 \sin\alpha z$$

From $Z(0) = 0, \quad Z(1) = 0$

$$Z(z) = c_2 \sin\alpha_n z$$ where $\alpha_n = n\pi$ $\qquad \lambda_n = -\alpha_n^2 = -n^2\pi^2$

(Step 4): Solve

$$rR'' + R' + \lambda rR = 0$$

$$rR'' + R' - n^2\pi^2 rR = 0$$

$rR''+R'-n^2\pi^2 rR=0$ $\quad r^2R''+rR'-n^2\pi^2r^2R=0$

$\nu=0,\ \alpha=n\pi$

Since $x^2y''+xy'-(\alpha^2x^2+\nu^2)y=0$ solution：$c_1I_\nu(\alpha x)+c_2K_\nu(\alpha x)$

the solution of $r^2R''+rR'-n^2\pi^2r^2R=0$ is

$R(r) = c_3I_0(n\pi r) + c_4K_0(n\pi r)$ $\quad 0<r<1$ (page 205)

$I_\nu(x)$: modified Bessel function of the 1st kind

$K_\nu(x)$: modified Bessel function of the 2nd kind

Since $K_0(0)\to\infty$

$R(r) = c_3I_0(n\pi r)$

(Step 5): $u(r,z)=R(r)Z(z)$ $\quad u_n(r,z)=A_nI_0(n\pi r)\sin(n\pi z)$

(Step 6): $u(r,z)=\sum_{n=1}^{\infty}A_nI_0(n\pi r)\sin(n\pi z)$

$$u(r,z)=\sum_{n=1}^{\infty}A_n I_0(n\pi r)\sin(n\pi z)$$

(Step 7): From $u(1,z)=1-z$

$$\sum_{n=1}^{\infty}A_n I_0(n\pi)\sin(n\pi z)=1-z$$

From the Fourier sine series

$p=1,\ x\to z,\ b_n=A_n I_0(n\pi)$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{p}x \qquad b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

we have

$\int u'v = uv-\int uv'$ $\quad u=-\frac{\cos(n\pi z)}{n\pi}$ $\quad v=1-z$

$$A_n I_0(n\pi)=2\int_0^1(1-z)\sin(n\pi z)dz=\frac{2}{n\pi}(z-1)\cos(n\pi z)\Big|_0^1-\int_0^1\frac{2}{n\pi}\cos(n\pi z)dx$$

$$A_n=\frac{2}{n\pi I_0(n\pi)}$$

$$u(r,z)=2\sum_{n=1}^{\infty}\frac{I_0(n\pi r)}{n\pi I_0(n\pi)}\sin(n\pi z)$$

2.9 SPHERICAL COORDINATES

Spherical Coordinates

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta.$$

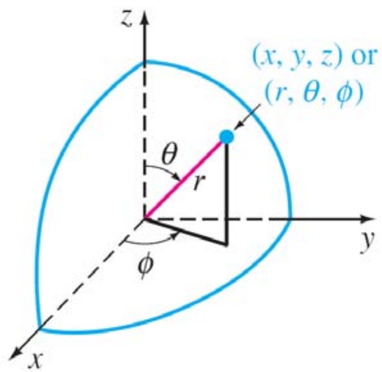


FIGURE 13.3.1 Spherical coordinates of a point (x, y, z) are (r, θ, ϕ).

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.3.

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.3

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta.$$

Laplacian in Spherical Coordinates

The 3D Laplacian is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

⇓ refer to pages 184-188

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}.$$

⇓ Suppose that u is independent of ϕ.

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}.$$

[Example 1] Steady Temperatures in a Sphere

$\nabla^2 u = 0$

Find the steady-state temperature $u(r, \theta)$ within the sphere shown in Figure 13.3.2.

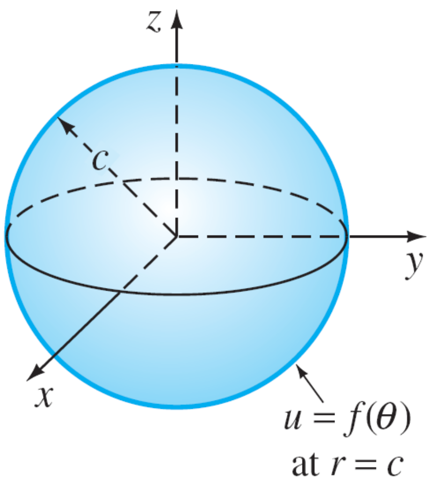


FIGURE 13.3.2 Dirichlet problem for a sphere in Example 1

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.3.

$u(c, \theta) = f(\theta)$

$0 < r < c$

SOLUTION

Steady Temperatures

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

⇓ Analogous to page 184-188

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0$$

With $u(c,\theta) = f(\theta), \quad 0 < \theta < \pi$ $\quad 0<r<c$

(Step 1): $u = R(r)\Theta(\theta)$,

$$R''(r)\Theta(\theta) + \frac{2}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) + \frac{\cot\theta}{r^2}R(r)\Theta'(\theta) = 0$$

$$\frac{R''(r)}{R(r)} + \frac{2}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} + \frac{\cot\theta}{r^2}\frac{\Theta'(\theta)}{\Theta(\theta)} = 0$$

divided by $u = R\Theta$

$$r^2\frac{R''(r)}{R(r)} + 2r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} + \cot\theta\frac{\Theta'(\theta)}{\Theta(\theta)} = 0$$

multiplied by r^2

(Step 2): $$\frac{r^2R''+2rR'}{R} = -\frac{\Theta''+\cot\theta\Theta'}{\Theta} = \lambda.$$

$$r^2R''+2rR'-\lambda R = 0 \qquad \sin\theta\Theta''+\cos\theta\Theta'+\lambda\sin\theta\Theta = 0.$$

Cauchy-Euler   in fact Legendre

(Step 3): Try to solve

$$\sin\theta\Theta''(\theta)+\cos\theta\Theta'(\theta)+\lambda\sin\theta\Theta(\theta) = 0. \qquad 0\le\theta\le\pi$$

Set $x=\cos(\theta)$ $\quad -1\le x\le 1$

$$\frac{d}{d\theta}\Theta = \frac{dx}{d\theta}\frac{d}{dx}\Theta = -\sin\theta\frac{d}{dx}\Theta$$

chain rule

$$\frac{d^2}{d\theta^2}\Theta = \frac{d}{d\theta}\frac{d}{d\theta}\Theta = -\frac{d}{d\theta}\left(\sin\theta\frac{d}{dx}\Theta\right) = -\cos\theta\frac{d}{dx}\Theta - \sin\theta\frac{d}{d\theta}\frac{d}{dx}\Theta$$

$$= -\cos\theta\frac{d}{dx}\Theta - \sin\theta\frac{dx}{d\theta}\frac{d^2}{dx^2}\Theta = -\cos\theta\frac{d}{dx}\Theta + \sin^2\theta\frac{d^2}{dx^2}\Theta$$

$$\sin^3\theta\frac{d^2\Theta}{dx^2} - 2\sin\theta\cos\theta\frac{d\Theta}{dx} + \lambda\sin\theta\Theta = 0$$

$$\sin^3\theta \frac{d^2\Theta}{dx^2} - 2\sin\theta\cos\theta \frac{d\Theta}{dx} + \lambda \sin\theta\Theta = 0$$

$x = \cos(\theta)$

$$\sin^2\theta \frac{d^2\Theta}{dx^2} - 2\cos\theta \frac{d\Theta}{dx} + \lambda\Theta = 0$$

page 207

$$\left(1 - x^2\right)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \lambda\Theta = 0$$

$-1 \le x \le 1$

page 207

When $\lambda = n(n+1)$, it becomes Legendre's equation of order n

This is one of the applications of the Legendre equation

The solution is

$$\Theta(x) = cP_n(x)$$

$P_n(x)$: the Legendre polynomial of order n

$$\Theta(\theta) = cP_n(\cos\theta)$$

$n = 0, 1, 2, \ldots.$

Note: The other linearly independent solution may not have finite derivatives at $x = \pm 1$.

(Step 4): Try to solve

$$r^2R''+2rR'-\lambda R=0$$

Since $\lambda = n(n+1)$,

$$r^2R''+2rR'-n(n+1)R=0$$

Cauchy-Euler page 37

Auxiliary: $m(m-1)+2m-n(n+1)=0$

$m^2+m-n(n+1)=0$

$(m+n+1)(m-n)=0$

$$m=n, \quad -(n+1)$$

$$R(r)=c_1r^n+c_2r^{-(n+1)}$$

Since $0^{-(n+1)} \to \infty$ $\qquad c_2=0$

$$R(r)=c_1r^n$$

(Step 5): $u_n(r,\theta)=A_nr^nP_n(\cos\theta) \qquad n=0, 1, 2, \ldots.$

(Step 6): $$u(r,\theta)=\sum_{n=0}^{\infty}A_nr^nP_n(\cos\theta)$$

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta)$$

(Step 7): From $u(c,\theta)=f(\theta)$

$$\sum_{n=0}^{\infty}A_n c^n P_n(\cos\theta)=f(\theta)$$

We have known that from page 207

$$\int_{-1}^{1}P_n(x)P_m(x)dx=\frac{2}{2n+1}\delta_{n,m} \qquad \delta_{n,n}=1$$

$$\delta_{n,m}=0 \quad \text{if } n\neq m$$

Set $x=\cos(\theta)$ $\qquad \frac{dx}{d\theta}=-\sin\theta$ $\qquad dx=-\sin\theta\, d\theta$

$$\int_{\pi}^{0}P_n(\cos\theta)P_m(\cos\theta)(-\sin\theta)d\theta=\frac{2}{2n+1}\delta_{n,m}$$

$$\int_{0}^{\pi}P_n(\cos\theta)P_m(\cos\theta)\sin\theta d\theta=\frac{2}{2n+1}\delta_{n,m}$$

(orthogonal with the weight function $\sin\theta$)

$u(c,\theta)=f(\theta)$

$$f(\theta)=\sum_{m=0}^{\infty}A_m c^m P_m(\cos\theta)$$

$$\int_0^{\pi}P_n(\cos\theta)P_m(\cos\theta)\sin\theta d\theta=\frac{2}{2n+1}\delta_{n,m}$$

$$\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta=\sum_{m=0}^{\infty}A_m c^m\int_0^{\pi}P_m(\cos\theta)P_n(\cos\theta)\sin\theta d\theta$$

$$\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta=A_n c^n\frac{2}{2n+1}$$

only preserve $m=n$

$$A_n=\frac{2n+1}{2c^n}\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta$$

$$u(r,\theta)=\sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta\right)\left(\frac{r}{c}\right)^n P_n(\cos\theta)$$

Another Method for Solving PDEs: Method of Characteristics (只教不考)

Method of Characteristics

The method is suitable for the 1st Order PDE.

Suppose that there is a 1st Order PDE as follows:

$$a_1(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_1}+a_2(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_2}$$

$$+\cdots\cdots+a_k(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_k}=g(x_1,x_2,\cdots,x_k,u)$$

Method of Characteristics

$$a_1(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_1}+a_2(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_2}$$

$$+\cdots\cdots+a_k(x_1,x_2,\cdots,x_k,u)\frac{\partial u(x_1,x_2,\cdots,x_k)}{\partial x_k}=g(x_1,x_2,\cdots,x_k,u)$$

We set a variable s and suppose that

$$\frac{\partial x_1}{\partial s}=a_1(x_1,x_2,\cdots,x_k,u),\quad \frac{\partial x_2}{\partial s}=a_2(x_1,x_2,\cdots,x_k,u),$$

$$\cdots\cdots,\quad \frac{\partial x_k}{\partial s}=a_k(x_1,x_2,\cdots,x_k,u).$$

Then, the original equation can be expressed as

$$\frac{\partial x_1}{\partial s}\frac{\partial u}{\partial x_1}+\frac{\partial x_2}{\partial s}\frac{\partial u}{\partial x_2}+\cdots\cdots+\frac{\partial x_k}{\partial s}\frac{\partial u}{\partial x_k}=g(x_1,x_2,\cdots,x_k,u)$$

⇓

$$\frac{du}{ds}=g(x_1,x_2,\cdots,x_k,u)$$ ⟹ Solve the 1st order ODE for u

Then, we apply the process as follows to solve u:

(Step 1) Solve the relation between x_1, x_2, …, x_k and s from each of the following ODEs:

$$\frac{\partial x_1}{\partial s} = a_1(x_1, x_2, \cdots, x_k, u) \qquad \Rightarrow \qquad x_1 = a_1(x_1, x_2, \cdots, x_k, u)s + c_1$$

$$\frac{\partial x_2}{\partial s} = a_2(x_1, x_2, \cdots, x_k, u), \qquad x_2 = a_2(x_1, x_2, \cdots, x_k, u)s + c_2$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\frac{\partial x_k}{\partial s} = a_k(x_1, x_2, \cdots, x_k, u). \qquad x_k = a_k(x_1, x_2, \cdots, x_k, u)s + c_k$$

(Step 2) Try to find the general equation b such that

$$b(x_1, x_2, \cdots, x_k) = c_b$$ where c_b is a constant

(Step 3) Solve the following ODE

$$\frac{du}{ds} = g\left(x_1(s), x_2(s), \cdots, x_k(s), u\right)$$ Solve the 1st order ODE for u

The solution has the form of $u = u_1\left(s, c_g\right)$ $u = \int g\, ds$

where c_g is some unknown constant.

(Specially, for case where $g = 0$, $u = c_g$)

(Step 4) Replace c_g by

$$c_g = f\left(c_b\right) = f\left(b\left(x_1, x_2, \cdots, x_k, u\right)\right)$$

where f is any function, then the solution is

$$u = u_1\left(s\left(x_1\right), f\left(c_b\right)\right)$$ Here, we express s as a function of x_1.

(Specially, for case where $g = 0$, $u = f(c_b)$)

[Example 1] Solve

$$\frac{\partial u(x,y)}{\partial x} = 2\frac{\partial u(x,y)}{\partial y}$$

(Solution): $$\frac{\partial u(x,y)}{\partial x} - 2\frac{\partial u(x,y)}{\partial y} = 0$$

(1) $$\frac{\partial x}{\partial s} = 1, \quad x = s + c_x$$

$$\frac{\partial y}{\partial s} = -2, \quad y = -2s + c_y$$

(2) $$2x + y = c_b$$ $b(x,y) = 2x+y$

(3) $$\frac{du}{ds} = 0, \quad u = c_g = f(c_b)$$

$$u = f(2x+y)$$ where f is any function.

[Example 2] Solve

$$\frac{\partial u(x,y,z)}{\partial x}=\frac{\partial u(x,y,z)}{\partial y}-\frac{\partial u(x,y,z)}{\partial z}+u(x,y,z)$$

$u = XYZ$

$X'YZ = XY'Z - XYZ' + XYZ$

$\frac{X'}{X} = \frac{Y'}{Y} - \frac{Z'}{Z} + 1$

$\lambda \quad \tau \quad \tau-\lambda+1$

(Solution):

$$\frac{\partial u(x,y,z)}{\partial x}-\frac{\partial u(x,y,z)}{\partial y}+\frac{\partial u(x,y,z)}{\partial z}=u(x,y,z)$$

$X' - \lambda X = 0 \qquad X = e^{\lambda x}$

$Y' - \tau Y = 0 \qquad Y = e^{\tau y}$

$Z' - (\tau-\lambda+1) = 0 \qquad Z = e^{(\tau-\lambda+1)z}$

(1) $\frac{\partial x}{\partial s}=1, \quad \frac{\partial y}{\partial s}=-1, \quad \frac{\partial z}{\partial s}=1$

$$x=s+c_x, \quad y=-s+c_y, \quad z=s+c_z$$

$u_{\lambda\tau} = XYZ = e^{\lambda(x-z)} e^{\tau(y+z)} e^{z}$

(2) Note that

$$x+y=c_x+c_y \quad \text{and} \quad y+z=c_y+c_z$$

$b_1(x,y,z)=x+y$

$b_2(x,y,z)=y+z$

can all generate a constant. Although x-z can also generate a constant, it is dependent on x+y and y+z. Therefore, a general way to obtain a constant is

$$f(x+y,\, y+z)=c$$

$u = \left(\sum_{\lambda,\tau} C_{\lambda,\tau} e^{(\lambda-1)(x+y)} e^{(\tau-\lambda+1)(y+z)}\right) e^{x}$

$= e^{x} f(x+y,\, y+z)$

where f is any function with two independent variables.

(3) $$\frac{\partial x}{\partial s}\frac{\partial u(x,y,z)}{\partial x}+\frac{\partial y}{\partial s}\frac{\partial u(x,y,z)}{\partial y}+\frac{\partial z}{\partial s}\frac{\partial u(x,y,z)}{\partial z}=u(x,y,z)$$

$\frac{du}{ds}=u,$ Since $x=s+c_x$

$u=c_1e^s=c_1e^{x-c_x}=ce^x$ Here we set $c=c_1e^{-c_x}$

$u=f(x+y,\ y+z)e^x$

附錄五 Review for Laplace Transforms

比較: 2-sided form Laplace transform $\int_{-\infty}^{\infty} e^{-st} f(t)dt$

Laplace transform (1-sided form) $F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$

1-sided Laplace

Differentiation Property for the Laplace transform

if $s = j2\pi f$

Laplace → Fourier

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots\cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\int_0^{\infty} e^{-st} f'(t) dt$$

$$= e^{-st} f(t)\Big|_0^{\infty} - \int (-s) e^{-st} f(t) dt$$

$$= -f(0) + sF(s)$$

$s^a f^{(b)}(0), \quad a + b = n - 1$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^{n}	$\frac{n!}{s^{n+1}}$
$\exp(at)$	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$

Seven Important Properties of the Laplace Transform

input	Laplace transform
(1) Differentiation $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots\cdots$ $-sf^{(n-2)}(0) - f^{(n-1)}(0)$
(2) Multiplication by t $t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
(3) Integration $\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$

(續)

input	Laplace transform
(4) Multiplication by exp $e^{at} f(t)$	$F(s-a)$
(5.1) Translation $u(t)$: unit step $f(t-a)u(t-a)$	$e^{-as}F(s)$
(5.2) Translation $g(t)u(t-a)$	$e^{-as}\mathcal{L}\{g(t+a)\}$
(6) Convolution $y(t)=\int_0^t f(\tau)g(t-\tau)d\tau$	$Y(s)=F(s)G(s)$
(7) Periodic Input $f(t)=f(t+T)$	$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$

2.10 Solving PDEs by Laplace Transforms

(只教不考)

In this section, we see that a linear PDE with constant coefficients is transformed into an ODE using the 1-sided Laplace transform.

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 14.2

Transform of a Function of Two Variables

$$U(x,s) = \mathcal{L}\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t)dt$$

To avoid confusing, we denote it as

$$U(x,s) = \mathcal{L}_{t\to s}\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t)dt$$

Transform of Partial Derivatives

$$\mathcal{L}_{t\to s}\left\{\frac{\partial u}{\partial t}\right\} = sU(x,s) - u(x,0),$$

$$\mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2U(x,s) - su(x,0) - u_t(x,0).$$

where

$$u_t(x,t) = \frac{\partial}{\partial t} u(x,t)$$

Because we are transforming with respect to t,

(not to x)

Be careful

$$\mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2 U}{dx^2}.$$

[Example 1] Laplace Transform of a PDE

Find the Laplace transform of the wave equation $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},\ t > 0.$
(respect to t)

$\mathcal{L}(u''(t)) = s^2 U(s) - s u(0) - u'(0)$

SOLUTION

$$\mathcal{L}_{t\to s}\left\{a^2 \frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$

where $u_t = \frac{\partial}{\partial t} u$

$$a^2 \frac{d^2}{dx^2} \mathcal{L}_{t\to s}\{u(x,t)\} = s^2 \mathcal{L}_{t\to s}\{u(x,t)\} - su(x,0) - u_t(x,0)$$

$U(x,s) = \mathcal{L}_{t\to s}(u(x,t))$

$$a^2 \frac{d^2 U(x,s)}{dx^2} - s^2 U(x,s) = -su(x,0) - u_t(x,0).$$

The process for solving the BVP or IVP of a partial differential equation (PDE) by the Laplace transform

The range of the independent variable should be $[0, \infty)$

(Step 1) Apply the Laplace transform for one independent variable to change the PDE into an ordinary differential equation (ODE) with another independent variable.

(Step 2) Solve the ODE obtain in Step 1.

(Step 3) Solution in Step 2 contains some constants. Find these constants by transforming the initial conditions.

(Step 4) Inverse transform.

[Example 2] Using the Laplace Transform to Solve a BVP

Solve $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0$$

Subject to $$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0$$

$$u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin \pi x, \quad 0 < x < 1.$$

SOLUTION

(Step 1) From Example 1 and the given initial conditions,

$$\frac{d^2 U(x,s)}{dx^2} - s^2 U(x,s) = -\sin \pi x,$$ (from page 245)

where $U(x,s) = \mathcal{L}_{t \to s}\{u(x,t)\}$.

$$\mathcal{L}_{t \to s}\{u(0,t)\} = U(0,s) = 0 \qquad \mathcal{L}_{t \to s}\{u(1,t)\} = U(1,s) = 0.$$

auxiliary $m^2 - s^2 = 0$ $m = \pm s$

$U_c(x,s) = C_3 e^{sx} + C_4 e^{-sx}$

(Step 2) $$\frac{d^2U(x,s)}{dx^2} - s^2U(x,s) = -\sin \pi x,$$

Complementary Function

Particular Solution

$$U_c(x,s) = c_1 \cosh sx + c_2 \sinh sx.$$

$$U_p(x,s) = a \sin \pi x + b \cos \pi x.$$

$$a = \frac{1}{s^2 + \pi^2}, \quad b = 0$$

$-a\pi^2 \sin \pi x - b\pi^2 \cos \pi x$

$-s^2 a \sin \pi x - s^2 b \cos \pi x = -\sin \pi x$

$$U(x,s) = U_c(x,s) + U_p(x,s)$$

$$= c_1 \cosh sx + c_2 \sinh sx + \frac{1}{s^2 + \pi^2} \sin \pi x.$$

$$U(x,s) = c_1 \cosh sx + c_2 \sinh sx + \frac{1}{s^2+\pi^2}\sin \pi x.$$

(Step 3) From

$$\mathcal{L}_{t\to s}\{u(0,t)\} = U(0,s) = 0 \qquad \mathcal{L}_{t\to s}\{u(1,t)\} = U(1,s) = 0.$$

$$c_1 = 0. \qquad c_1 \cosh s + c_2 \sinh s = 0$$

We have $c_1 = 0$ and $c_2 = 0$.

$$\mathcal{L}^{-1}\left(\frac{\pi}{s^2+\pi^2}\right) = \sin \pi t,$$

from page 240, $\pi = k$

$$U(x,s) = \frac{1}{s^2+\pi^2}\sin \pi x$$

(Step 4) $$u(x,t) = \mathcal{L}^{-1}_{s\to t}\left\{\frac{1}{s^2+\pi^2}\sin \pi x\right\} = \frac{1}{\pi}\sin \pi x \mathcal{L}^{-1}_{s\to t}\left\{\frac{\pi}{s^2+\pi^2}\right\}.$$

$$u(x,t) = \frac{1}{\pi}\sin \pi x \sin \pi t.$$

[Example 3] The Wave Equation with Gravity

Solve $$a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \quad x > 0, \quad t > 0$$

Subject to $$u(0,t) = 0, \quad \lim_{x\to\infty} \frac{\partial u}{\partial x} = 0, \quad t > 0$$

$$u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad x > 0$$

$U(0,s) = \mathcal{L}_{t\to s}(u(0,t)) = 0$

$\lim_{x\to\infty} U_x(x,s) = \mathcal{L}_{t\to s}(u_x(x,t)) = 0$

$\mathcal{L}(1) = \frac{1}{s}$

Solution:

(Step 1) $$\mathcal{L}_{t\to s}\left\{a^2 \frac{\partial^2 u}{\partial x^2}\right\} - \mathcal{L}_{t\to s}\{g\} = \mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$

$$a^2 \frac{\partial^2 U(x,s)}{\partial x^2} - \frac{g}{s} = s^2 U(x,s) - su(x,0) - u_t(x,0) = s^2 U(x,s)$$

$$\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s^2}{a^2} U(x,s) = \frac{g}{a^2 s}$$

(Step 2) $$\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s^2}{a^2}U(x,s) = \frac{g}{a^2 s}$$

auxiliary

$m^2 - \frac{s^2}{a^2} = 0$, $m = \pm\frac{s}{a}$

constant for x

Complementary Function

Particular Solution

$$U_c(x,s) = c_1 \cosh\left(\frac{s}{a}x\right) + c_2 \sinh\left(\frac{s}{a}x\right)$$

$$U_p(x,y) = c_3$$

$$c_3 = -\frac{g}{s^3}$$

$$U(x,s) = c_1 \cosh\left(\frac{s}{a}x\right) + c_2 \sinh\left(\frac{s}{a}x\right) - \frac{g}{s^3}$$

$$U(x,s) = c_1 \cosh\left(\frac{s}{a}x\right) + c_2 \sinh\left(\frac{s}{a}x\right) - \frac{g}{s^3}$$

sinh(x) = (e^x − e^{−x})/2
cosh(x) = (e^x + e^{−x})/2
$\lim_{x\to\infty}$ sinh(x) = $\lim_{x\to\infty}$ cosh(x) = e^x/2

(Step 3) From initial conditions

$$\mathcal{L}_{t\to s}\{u(0,t)\} = U(0,s) = 0$$

$$c_1 - \frac{g}{s^3} = 0$$

$$c_1 = \frac{g}{s^3}$$

$$\mathcal{L}_{t\to s}\{\lim_{x\to\infty} \frac{\partial u(x,t)}{\partial x}\} = \lim_{x\to\infty} \frac{\partial}{\partial x} U(x,s) = 0.$$

$$\lim_{x\to\infty} c_1 \frac{s}{a} \sinh\left(\frac{s}{a}x\right) + c_2 \frac{s}{a} \cosh\left(\frac{s}{a}x\right) = 0$$

$$\lim_{x\to\infty} c_1 \frac{s}{2a} \exp\left(\frac{s}{a}x\right) + c_2 \frac{s}{2a} \exp\left(\frac{s}{a}x\right) = 0$$

$$c_2 = -c_1 = -\frac{g}{s^3}$$

Note:

$$\lim_{x\to\infty} \sinh(x) = \lim_{x\to\infty} \cosh(x) = \exp(x)/2$$

$$U(x,s) = \frac{g}{s^3}\left(\cosh\left(\frac{s}{a}x\right) - \sinh\left(\frac{s}{a}x\right)\right) - \frac{g}{s^3} = \frac{g}{s^3}\exp\left(-\frac{s}{a}x\right) - \frac{g}{s^3}$$

$$U(x,s)=\frac{g}{s^3}\exp\left(-\frac{s}{a}x\right)-\frac{g}{s^3}$$

$\mathcal{L}(t^2)=\frac{2}{s^3}$ (from page 240)

(Step 4)

$$u(x,t)=\mathcal{L}^{-1}_{s\to t}\left\{\frac{g}{s^3}\exp\left(-\frac{s}{a}x\right)-\frac{g}{s^3}\right\}$$

$$u(x,t)=\frac{1}{2}g\left(t-\frac{x}{a}\right)^2\mathcal{U}\left(t-\frac{x}{a}\right)-\frac{1}{2}gt^2$$

from page 242 (5.1)

where $\mathcal{U}(t)$ is the unit step function

We have applied the translation property:

$$\mathcal{L}\{f(t-a)u(t-a)\}=\exp(-as)F(s)$$

附錄六 Review for Fourier Transforms

Fourier transform

$$\Im\left[g(x)\right]=\int_{-\infty}^{\infty}g(x)e^{-j2\pi fx}dx=G(f)$$

$\Im$ 代表 Fourier transform

inverse Fourier transform

$$\Im^{-1}\left[G(f)\right]=\int_{-\infty}^{\infty}G(f)e^{j2\pi fx}df=g(x)$$

文獻上其他 Fourier transform 的定義

$$\Im\left[g(x)\right] = \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx = G(\omega)$$

$$\Im^{-1}\left[G(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) e^{j\omega x} d\omega = g(x)$$

或者

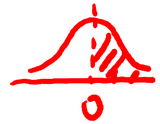
$$\Im\left[g(x)\right] = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx = G(\omega)$$

$$\Im^{-1}\left[G(\omega)\right] = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} G(\omega) e^{j\omega x} d\omega = g(x)$$

或者

$$\Im\left[g(x)\right] = \int_{-\infty}^{\infty} g(x) e^{j\alpha x} dx = G(\alpha)$$

$$\Im^{-1}\left[G(\alpha)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\alpha) e^{-j\alpha x} d\alpha = g(x)$$



When $g(x)$ is even ($g(x) = g(-x)$)

Fourier transform ⟶ Fourier cosine transform

$$\Im_c\left[g(x)\right] = \int_0^{\infty} g(x)\cos(2\pi fx)dx = G_c(f)$$

$$\Im_c^{-1}\left[G_c(f)\right] = 4\int_0^{\infty} G_c(f)\cos(2\pi fx)df = g(x)$$

$$\int_{-\infty}^{\infty} \text{even}\, dx = 2\int_0^{\infty} \text{even}\, dx$$

$$\int_{-\infty}^{\infty} \text{odd}\, dx = 0$$

Remember that if $k(x)$ is even

$$\int_{-\infty}^{\infty} k(x)dx = 2\int_0^{\infty} k(x)dx$$

If $k(x)$ is odd

$$\int_{-\infty}^{\infty} k(x)dx = 0$$

$$\int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}dx$$

$$= \int_{-\infty}^{\infty} g(x)\cos(2\pi fx)dx - j\int_{-\infty}^{\infty} g(x)\sin(2\pi fx)dx$$

$$= 2\int_0^{\infty} g(x)\cos(2\pi fx)dx \quad \text{if } g(x) \text{ is even}$$

$$\int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}dx = -j2\int_0^{\infty} g(x)\sin(2\pi fx)dx$$

if $g(x)$ is odd

When $g(x)$ is odd ($g(x) = -g(-x)$)

Fourier transform ⟶ Fourier sine transform

$$\Im_s\left[g(x)\right] = \int_0^\infty g(x)\sin(2\pi fx)dx = G_s(f)$$

$$\Im_s^{-1}\left[G_s(f)\right] = 4\int_0^\infty G_s(f)\sin(2\pi fx)df = g(x)$$

Fourier, Fourier Cosine / Sine Transforms 的微分性質

(1) Fourier transform 的微分性質

$$\Im[g'(x)] = \int_{-\infty}^{\infty} g'(x) e^{-j2\pi fx} dx = g(x) e^{-j2\pi fx}\Big|_{-\infty}^{\infty} + j2\pi f \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx$$

$$= j2\pi f\, \Im[g(x)]$$

微分性質做了一些假設： $g(x) = 0$ when $x \to \infty$ and $x \to -\infty$

以此類推 $\Im\left[g^{(n)}(x)\right] = (j2\pi f)^{(n)} G(f)$

比較：對 Laplace transform

$$L\{f'(x)\} = sL\{f(x)\} - f(0) \qquad \int_{0}^{\infty} f(x) e^{-sx} dx$$

對 Fourier transform

$s \to j2\pi f$, without initial conditions

(2) Fourier cosine transform 的微分性質

$$\Im_c\left[g'(x)\right]=\int_0^{\infty} g'(x)\cos(2\pi fx)\,dx$$

$$=g(x)\cos(2\pi fx)\Big|_0^{\infty}+2\pi f\int_0^{\infty} f(x)\sin(2\pi fx)\,dx$$

$$=2\pi f\,\Im_s\left[g(x)\right]-g(0)$$

(3) Fourier sine transform 的微分性質

$$\Im_s\left[g'(x)\right]=\int_0^{\infty} g'(x)\sin(2\pi fx)\,dx$$

$$=g(x)\sin(2\pi fx)\Big|_0^{\infty}-2\pi f\int_0^{\infty} g(x)\cos(2\pi fx)\,dx$$

$$=-2\pi f\,\Im_c\left[g(x)\right]$$

注意： (1) Fourier sine, cosine transforms 互換

(2) α 正負號不同

(3) Fourier cosine transform 要考慮 initial condition

$$\Im_c\left[g'(x)\right] = 2\pi f\, \Im_s\left[g(x)\right] - g(0)$$

$$\Im_s\left[g'(x)\right] = -2\pi f\, \Im_c\left[g(x)\right]$$

$$\Im_c\left[g''(x)\right] = 2\pi f\, \Im_s\left[g'(x)\right] - g'(0) = -4\pi^2 f^2\, \Im_c\left[g(x)\right] - g'(0)$$

$$\Im_s\left[g''(x)\right] = -2\pi f\, \Im_c\left[g'(x)\right] = -4\pi^2 f^2\, \Im_s\left[g(x)\right] + 2\pi f\, g(0)$$

2.11 Solving PDEs by Fourier Transforms

(只教不考)

Differentiation ⟶ Multiplication

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 14.4

Method

(Condition 1) interval 為 $-\infty < v < \infty$ 時:

用 Fourier transform

(Condition 2) interval 為 $0 < v < \infty$，

有 "$u(v, \ldots..) = 0$ or a constant when $v = 0$" 的 boundary condition 時:

用 Fourier sine transform

(Condition 3) interval 為 $0 < v < \infty$，

有 "$\frac{\partial}{\partial v} u(v, \cdots\cdots) = 0$ or a constant when $v = 0$" 的 boundary condition 時:

用 Fourier cosine transform

使用 Fourier transform, Fourier cosine transform, Fourier sine transform 來解 partial differential equation (PDE) 的 BVP 或IVP 的**解法流程**

(Step 1) 以 page 262 的規則，來決定要針對 **哪一個** independent variable ，做**什麼** transform (Fourier, Fourier cosine, 或Fourier sine transform)

(Step 2) 對 PDE 做 Step 1 所決定的 transform, 則原本的 PDE 變成針對另外一個 independent variable 的 ordinary differential equation (ODE)

(Step 3) 將 Step 2 所得出的 ODE 的解算出來

(Step 4) Step 3 所得出來的解會有一些 constants，可以對 initial conditions (或 boundary conditions) 做 transform 將 constants 解出

(※ 和 Step 1 所做的 transform 一樣，只是 transform 的對象變成是 initial 或 boundary conditions，見 pages 265, 268 的例子)

(Step 5) 最後，別忘了做 inverse transform (畫龍點睛）

[Example 1]

heat equation: $k\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ $\quad -\infty < x < \infty \quad$ $t > 0$

subject to $u(x,0) = g(x)$ where $g(x) = \begin{cases} u_0, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

Step 1 決定針對 x 做 Fourier transform

$$\Im_{x\to f}\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t) e^{-i2\pi fx} dx = U(f,t)$$

$FT(u''(x)) = (j2\pi f)^2 U(f)$

Step 2 $\Im_{x\to f}\left\{k\dfrac{\partial^2 u}{\partial x^2}\right\} = \Im_{x\to f}\left\{\dfrac{\partial u}{\partial t}\right\}$

$$-k4\pi^2 f^2 U(f,t) = \frac{\partial U(f,t)}{\partial t}$$

原本對 x, t 兩個變數做偏微分

經過 Fourier transform 之後，
只剩下對 t 做偏微分

$$\frac{dU(f,t)}{dt}+4k\pi^2 f^2 U(f,t)=0$$

對於 t 而言，是 1st order ODE

Step 3 $U(f,t)=c\,e^{-4k\pi^2 f^2 t}$ 這邊的 c 值，對 t 而言是 constant ，

但是可能會 **dependent on *f*** (特別注意)

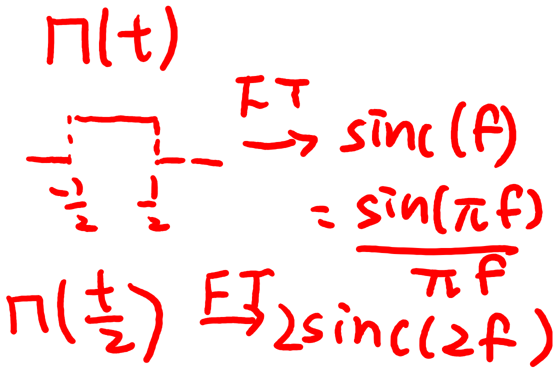
Step 4 根據 $u(x,0)=g(x)$ 將 c 解出

和 Step 1 一樣，也是針對 x 做 Fourier transform

只是對象改成 initial condition

$$\Im_{x\to f}\{u(x,0)\}=\int_{-\infty}^{\infty} g(x)e^{-i2\pi fx}dx=\int_{-1}^{1} u_0 e^{-i2\pi fx}dx$$

$$=u_0\frac{e^{-i2\pi f}-e^{i2\pi f}}{-i2\pi f}=u_0\frac{\sin(2\pi f)}{\pi f}$$



因為 $\Im_{x\to f}\{u(x,0)\}=U(f,0)$

$$U(f,0)=u_0\frac{\sin(2\pi f)}{\pi f}$$

$$U(f,t)=c\,e^{-4k\pi^2 f^2 t} \xleftrightarrow{\text{比較係數}} U(f,0)=u_0\frac{\sin(2\pi f)}{\pi f}$$

解出 $c=u_0\dfrac{\sin(2\pi f)}{\pi f}$

$$U(f,t)=u_0\frac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}$$

Step 5 未完待續，別忘了最後要做 inverse Fourier transform

$$u(x,t)=\Im^{-1}_{f\to x}\left[U(f,t)\right]=\int_{-\infty}^{\infty}u_0\frac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}e^{j2\pi fx}df$$

不易化簡，課本僅依據 $\dfrac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}$ 對 f 而言是 even function 將 $u(x, t)$ 化簡為

$$u(x,t)=u_0\int_{-\infty}^{\infty}\frac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}(\cos(2\pi fx)+j\sin(2\pi fx))df$$

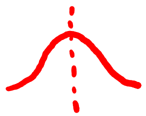
$$=u_0\int_{-\infty}^{\infty}\frac{\sin(2\pi f)\cos(2\pi fx)}{\pi f}e^{-4k\pi^2 f^2 t}df$$

[Example 2] Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < 1 \qquad y > 0$$

$$u(0, y) = 0 \qquad u(1, y) = e^{-y} \qquad y > 0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \qquad 0 < x < 1$$



u is possibly even for y

Fourier cosine series

Step 1 決定針對 y 做 Fourier cosine transform

$$\mathfrak{I}_{c, y \to f}\{u(x, y)\} = \int_0^{\infty} u(x, y)\cos(2\pi f y)dy = U(x, f)$$

Step 2 $$\mathfrak{I}_{c, y \to f}\left\{\frac{\partial^2 u}{\partial x^2}\right\} + \mathfrak{I}_{c, y \to f}\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \mathfrak{I}_{c, y \to f}\{0\}$$

page 260

from $\mathfrak{I}_c[g''(y)] = -4\pi^2 f^2 \mathfrak{I}_c[g(y)] - g'(0)$

$$\frac{d^2 U(x, f)}{dx^2} - 4\pi^2 f^2 U(x, f) = 0$$ 對於 x 的 2nd order ODE

Step 3 $\dfrac{d^2U(x,f)}{dx^2} - 4\pi^2 f^2 U(x,f) = 0$

⇨ $U(x,f) = c_1 \cosh(2\pi fx) + c_2 \sinh(2\pi fx)$

Step 4 由 $u(0,y)=0$ $u(\pi,y)=e^{-y}$ 來解 c_1, c_2 (may be dependent on f)

和 Step 1 一樣，也是針對 y 做 Fourier cosine transform

只是對象改成 boundary conditions

(1) $U(0,f) = \Im_{c,y\to f}\{u(0,y)\} = \int_0^\infty 0\cdot\cos(2\pi fy)dy = 0$

(2) $U(1,f) = \Im_{c,y\to f}\{u(1,y)\} = \int_0^\infty e^{-y}\cdot\cos(2\pi fy)dy = \dfrac{1}{1+(2\pi f)^2}$

(可以用Laplace transform 的「取巧法」)

$$U(x,f)=c_1\cosh(2\pi fx)+c_2\sinh(2\pi fx)$$

分別代入 $U(0,f)=0$ $\qquad$ $U(1,f)=\dfrac{1}{1+(2\pi f)^2}$

$$c_1=0 \qquad c_1\cosh(2\pi f)+c_2\sinh(2\pi f)=\frac{1}{1+(2\pi f)^2}$$

$$\Longrightarrow \quad c_1=0 \qquad c_2=\frac{1}{(1+4\pi^2 f^2)\sinh(2\pi f)}$$

$$U(x,f)=\frac{\sinh(2\pi fx)}{(1+4\pi^2 f^2)\sinh(2\pi f)}$$

Step 5 inverse cosine transform

$$u(x,y)=\Im_{c,f\to y}^{-1}\left[U(x,f)\right]=4\int_0^{\infty}\frac{\sinh(2\pi fx)}{(1+4\pi^2 f^2)\sinh(2\pi f)}\cos(2\pi fy)df$$

(算到這裡即可，難以繼續化簡)

本節需要注意的地方

(1) 微分公式當中，Fourier cosine transform 和 Fourier sine transform 會有互換的情形。(See page 259)

(2) 在解 boundary value problem 時，要了解

何時用 Fourier transform，

何時用 Fourier cosine transform,

何時用 Fourier sine transform (see page 262)

(3) 在解 partial differential equation 時，往往只針對一個 independent variable 做 Fourier transform, 另一個 independent variable 不受影響，如 Examples 1 and 2 的例子

計算過程中，自己要清楚是對哪一個 independent variable 做Fourier transform

※ 本講義中習慣用下標做記號