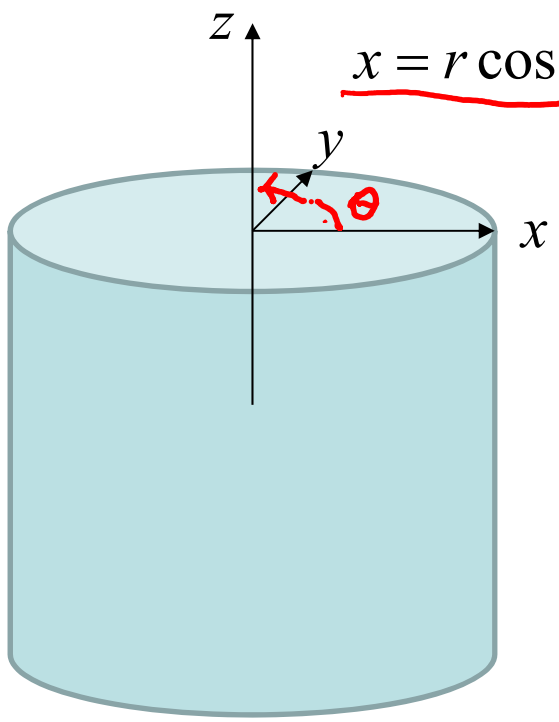
2.8 CYLINDRICAL COORDINATES



$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.2

2.8.1 Review for Special Functions

- **Bessel's equation of order v**

$$\boxed{x^2y'' + xy' + (x^2 - v^2)y = 0}$$ Solution: $c_1 J_v(x) + c_2 Y_v(x)$

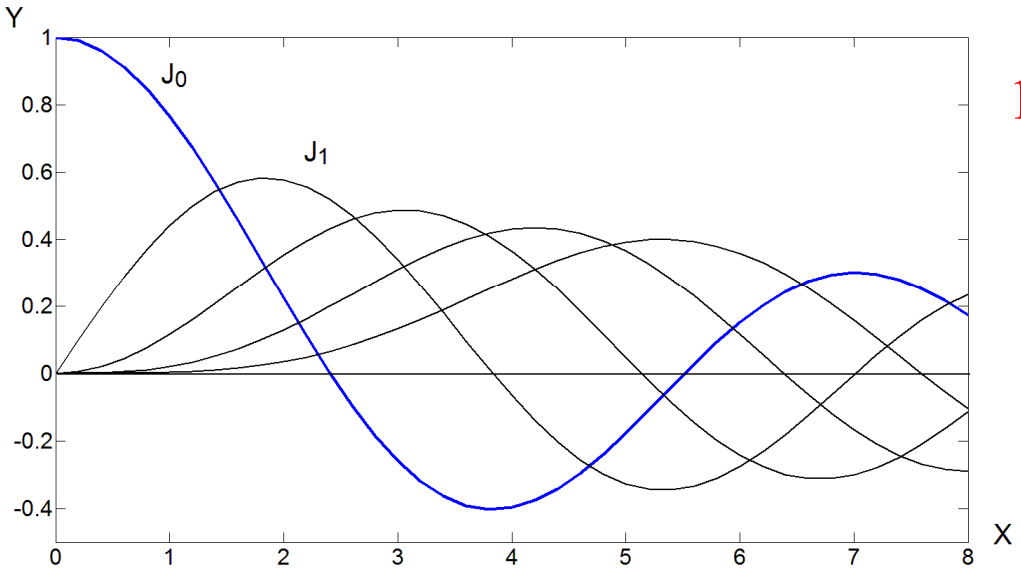
$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}$$: 1st kind Bessel function

$$Y_v(x) = \frac{\cos v\pi J_v(x) - J_{-v}(x)}{\sin v\pi}$$: 2nd kind Bessel function

where $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ (gamma function)

$$\Gamma(n+1) = n!$$

$$\Gamma(x+1) = x\Gamma(x)$$

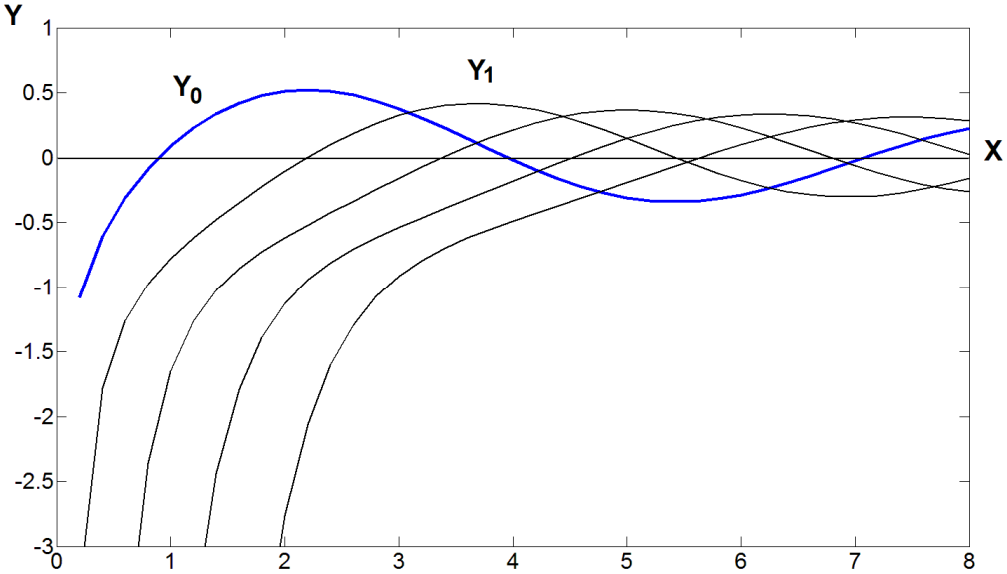


1st kind Bessel function

$$J_0(0) = 1$$

$$J_v(0) = 0 \quad for\ v \neq 0$$

$$\lim_{x \to \infty} J_v(x) = 0$$



2nd kind Bessel function

$$Y_v(0) \to -\infty$$

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

Zeros of J_0, J_1, Y_0, and Y_1

	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
x_1	2.4048	0.0000	0.8936	2.1971
x_2	5.5201	3.8317	3.9577	5.4297
x_3	8.6537	7.0156	7.0861	8.5960
x_4	11.7915	10.1735	10.2223	11.7492
x_5	14.9309	13.3237	13.3611	14.8974

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

J_0 is similar to sinc

$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$

but the zeros of sinc has a uniform distance

$\text{sinc}(x) = 0$ if x is a nonzero integer

• Generalization of Bessel's equation of order v

$$x^2y'' + xy' + (x^2 - v^2)y = 0 \qquad \text{解：} c_1J_v(x) + c_2Y_v(x)$$

$$x^2y'' + xy' + (\alpha^2x^2 - v^2)y = 0 \qquad \text{解：} c_1J_v(\alpha x) + c_2Y_v(\alpha x)$$

Proof: Set $t = \alpha x$

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \alpha\frac{dy}{dt}$$

Similarly, $$\frac{d^2y}{dx^2} = \frac{dt}{dx}\frac{d}{dt}\left(\frac{dy}{dx}\right) = \alpha\frac{d}{dt}\left(\alpha\frac{dy}{dt}\right) = \alpha^2\frac{d^2y}{dt^2}$$

$$\text{原式} = x^2y'' + xy' + (\alpha^2x^2 - v^2)y = \frac{t^2}{\alpha^2}\alpha^2\frac{d^2y}{dt^2} + \frac{t}{\alpha}\alpha\frac{dy}{dt} + (\alpha^2\frac{t^2}{\alpha^2} - v^2)y$$

$$= \boxed{t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - v^2)y = 0} \longrightarrow \text{對 } t \text{ 而言是 Bessel equation}$$

$$y = c_1J_v(t) + c_2Y_v(t) = c_1J_v(\alpha x) + c_2Y_v(\alpha x)$$

• Modified Bessel's equation of order v

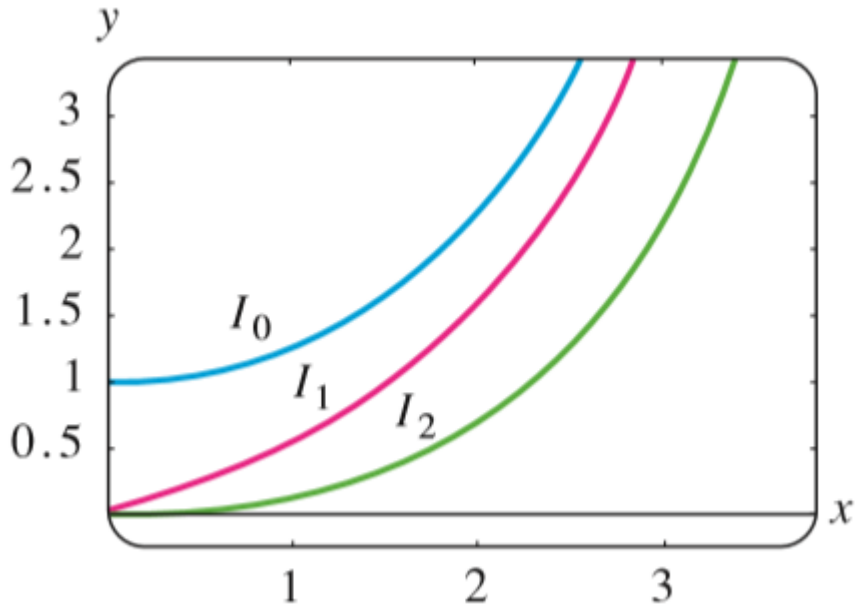
$$x^2y'' + xy' + (-x^2 - v^2)y = 0$$ 解：$c_1 I_v(x) + c_2 K_v(x)$

$$x^2y'' + xy' + (-\alpha^2 x^2 - v^2)y = 0$$ 解：$c_1 I_v(\alpha x) + c_2 K_v(\alpha x)$

其中 $I_v(x) = i^{-v} J_v(ix)$ 稱作是 modified Bessel function of the first kind of order v

$$K_v(x) = \frac{\pi}{2}\frac{I_{-v}(x) - I_v(x)}{\sin v\pi}$$ 稱作是 modified Bessel function of the second kind of order v

當 v 為整數時，也是取 limit

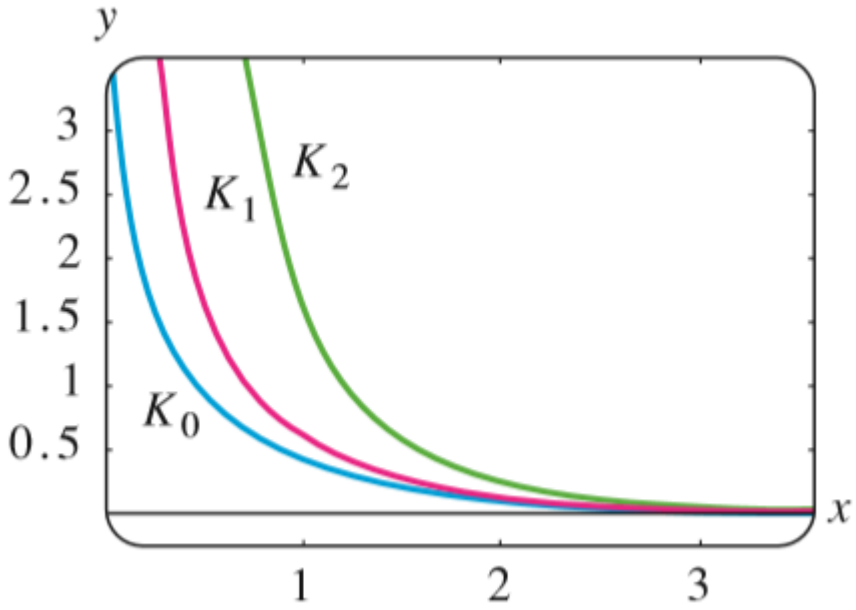


Modified Bessel functions of the first kind for $n = 0, 1, 2$

$$I_0(0) = 1$$

$$I_v(0) = 0 \quad for\ v \neq 0$$

$$I_v(x) \neq 0 \quad for\ x \neq 0$$



Modified Bessel functions of the second kind for $n = 0, 1, 2$

$$K_v(0) \to \infty$$

$$K_v(x) \neq 0 \quad for\ x \neq 0$$

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

• Legendre's equation of order *n*

$$\boxed{(1-x^2)y''-2xy'+n(n+1)y=0}$$

One of the solution: Legendre polynomials $P_n(x)$

When n is a non-negative integer
one of the solution has finite terms

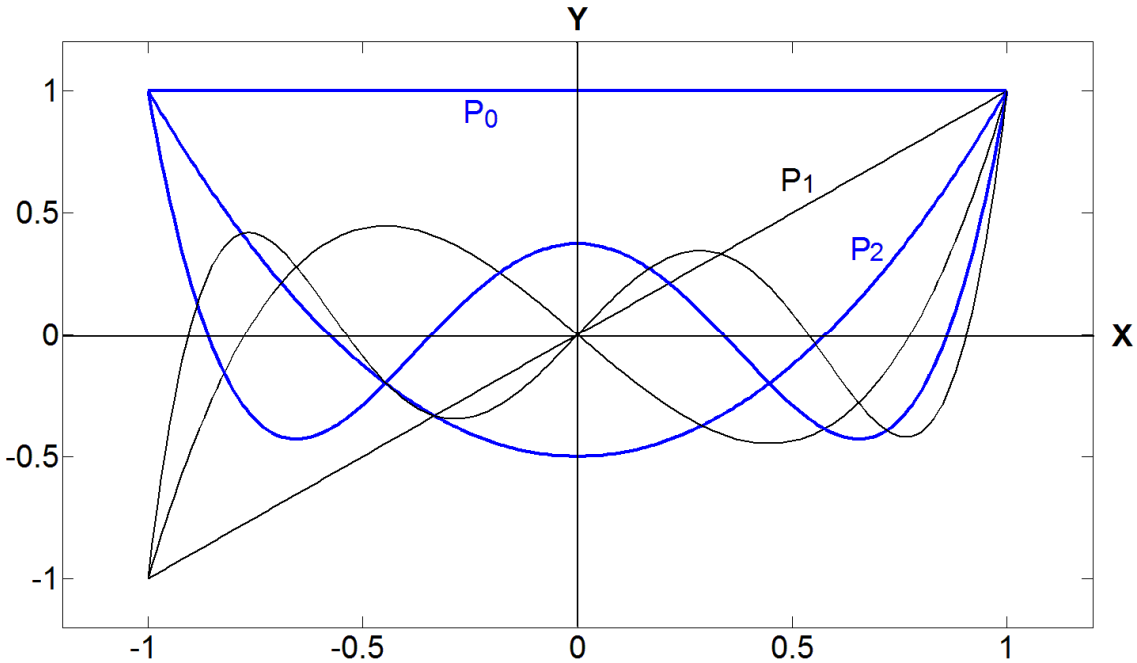
$$P_0(x)=1 \qquad P_1(x)=x$$

$$P_2(x)=\frac{1}{2}\left(3x^2-1\right) \qquad P_3(x)=\frac{1}{2}\left(5x^3-3x\right)$$

$$P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right) \qquad P_5(x)=\frac{1}{8}\left(63x^5-70x^3+15x\right)$$

$$\int_{-1}^{1}P_n(x)P_m(x)\,dx=\frac{2}{2n+1}\delta_{n,m} = \begin{cases} \frac{2}{2n+1} & \text{if } n=m \\ 0 & \text{otherwise} \end{cases}$$

Legendre polynomials



Interval:

$x \in [-1, 1]$

BY NC SA

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 6.4.

2.8.2 PDE for Cylindrical Coordinates

Laplacian in Cylindrical Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

page 183

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}} + \frac{\partial^2 u}{\partial z^2}.$$

For the case of **radial symmetry** $\underline{\dfrac{\partial u}{\partial \theta} = 0}$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}.$$

[Example 1] Steady Temperatures in a Circular Cylinder

$\nabla^2 u = 0$

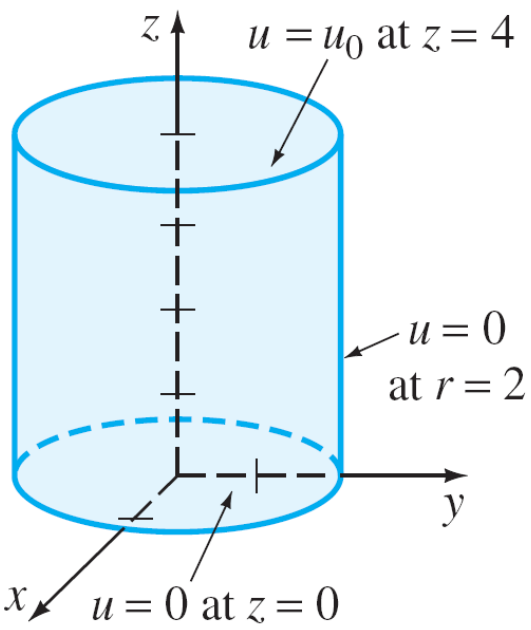


FIGURE 13.2.5 Circular cylinder in Example 2

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.2.

$$u(2,z)=0, \quad 0<z<4$$

$$u(r,0)=0, \quad u(r,4)=u_0, \quad 0 \le r < 2.$$

It is a problem of the 2D heat equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial t} = 0 \qquad u(r, \cancel{\theta}, z)$$

steady temperatures $\downarrow$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

radial symmetry $\downarrow$ (since $u = 0$ when $r = 2$)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad u(r, z)$$

$$u(2, z) = 0, \quad 0 < z < 4$$

$$u(r, 0) = 0, \quad u(r, 4) = u_0, \quad 0 < r < 2$$

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$$

(Step 1) $u(r,z)=R(r)Z(z)$

$$R''(r)Z(z)+\frac{1}{r}R'(r)Z(z)+R(r)Z''(z)=0$$

$$\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}+\frac{Z''(z)}{Z(z)}=0$$

(Step 2) $$\frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z}=-\lambda$$

$rR''+R'+\lambda rR=0$ $\qquad$ $Z''-\lambda Z=0$ $\qquad$ 2 ODEs

$R(2)Z(z)=0$

From $u(2,z)=0,\quad u(r,0)=0,$

$R(2)=0 \qquad Z(0)=0$

$$rR''+R'+\lambda rR=0 \qquad R(2)=0$$

$$Z''-\lambda Z=0 \qquad Z(0)=0$$

Cauchy – Euler

(Step 3)

Case 1: $\lambda=0$ $\qquad rR''+R'=0 \xrightarrow{\times r} r^2R''+rR'=0$

auxiliary: $m(m\text{-}1)+m=0, \quad m=0, 0$

$$R(r)=c_1+c_2\ln r \qquad 0\le r<2$$

Since $\ln 0\to-\infty$, $c_2=0$

Since $R(2)=0\to c_1+c_2\ln 2=0\to c_1=0$

$R(r)=0$ (trivial)

Case 2: $\lambda = -\alpha^2 < 0$ $\quad rR'' + R' - \alpha^2 rR = 0$

$$r^2 R'' + rR' - \alpha^2 r^2 R = 0$$

compared to modified Bessel function (page 205)

$$x^2 y'' + xy' - (x^2 + v^2) y = 0 \qquad \text{solution}: c_1 I_v(x) + c_2 K_v(x)$$

$v=0$

$$x^2 y'' + xy' - (\alpha^2 x^2 + v^2) y = 0 \qquad \text{solution}: c_1 I_v(\alpha x) + c_2 K_v(\alpha x)$$

$I_v(x)$: modified Bessel function of the 1st kind

$K_v(x)$: modified Bessel function of the 2nd kind

The solution of $rR'' + R' - \alpha^2 rR = 0$ is

page 206 $R(r) = c_1 I_0(\alpha r) + c_2 \cancel{K_0(\alpha r)}$

(page 206) Since $K_0(0) \to \infty$, $c_2 = 0$

Since $R(2) = 0$, $\quad c_1 I_0(2\alpha) = 0$ $\Longrightarrow$ $R(r) = 0$ (trivial)

(page 206) From $I_0(x) \neq 0$ for all $x \neq 0 \to c_1 = 0$

Case 3: $\lambda = \alpha^2 > 0$

$$rR'' + R' + \alpha^2 rR = 0$$

$$r^2 R'' + rR' + \alpha^2 r^2 R = 0$$

compared to the Bessel function on page 204

$v = 0$

$$x^2 y'' + xy' + (\alpha^2 x^2 - v^2) y = 0$$

solution : $c_1 J_v(\alpha x) + c_2 Y_v(\alpha x)$

$$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$$

(page 202)

Since $Y_0(0) \to -\infty$, $c_2 = 0$ $\quad R(r) = c_1 J_0(\alpha r)$

Since $R(2) = 0$ $\quad c_1 J_0(2\alpha) = 0$

Therefore,

$\boxed{R(r) = c_1 J_0(\alpha_n r)}$ where $\alpha_n = x_n / 2$

$2\alpha = x_n$

$\alpha = \frac{x_n}{2}$

x_n are the zeros of $J_0(x)$, i.e., $J_0(x_n) = 0$

(defined on page 203) $\quad \lambda_n = \alpha_n^2$, $\lambda_n = x_n^2 / 4$

(Step 4) Try to solve

$$Z'' - \lambda Z = 0 \qquad Z(0) = 0$$

Since $\lambda_n = \alpha_n^2$,

$$Z'' - \alpha_n^2 Z = 0$$

sinh(0)=0, cosh(0)=1

$$Z(z) = c_3 \cosh(\alpha_n z) + c_4 \sinh(\alpha_n z)$$

From $Z(0) = 0, \quad c_3 = 0$

$$\boxed{Z(z) = c_4 \sinh(\alpha_n z)} \qquad \text{where} \quad \alpha_n = x_n / 2 \qquad J_0(x_n) = 0$$

(Step 5) $u(r,z) = R(r)Z(z) = A_n J_0(\alpha_n r)\sinh(\alpha_n z)$

(Step 6) $u(r,z) = \sum_{n=1}^{\infty} A_n \sinh \alpha_n z J_0(\alpha_n r).$

where $\alpha_n = x_n / 2 \qquad J_0(x_n) = 0$

(Step 7) From $u(r,4)=u_0$,

$$u_0=\sum_{n=1}^{\infty}A_n\sinh(4\alpha_n)J_0(\alpha_n r).$$

From Eq. (9) in Section 11.5 of D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017

$\alpha_n=\frac{x_n}{2}$, $\alpha_m=\frac{x_m}{2}$

$$\int_0^2 rJ_0(\alpha_n r)J_0(\alpha_m r)dr=\begin{cases}0 & if\ m\neq n\\ 2J_1^2(2\alpha_n) & if\ m=n\end{cases}$$

$$\int_0^2 u_0 rJ_0(\alpha_m r)dr=\sum_{n=1}^{\infty}A_n\sinh(4\alpha_n)\int_0^2 J_0(\alpha_n r)rJ_0(\alpha_m r)dr$$

$$=A_m\sinh(4\alpha_m)2J_1^2(2\alpha_m)$$

$$A_n=\frac{u_0}{2\sinh(4\alpha_n)J_1^2(2\alpha_n)}\int_0^2 rJ_0(\alpha_n r)dr=\frac{u_0}{\alpha_n\sinh(4\alpha_n)J_1(2\alpha_n)}$$

(Solution):

$$\boxed{u(r,z) = u_0 \sum_{n=1}^{\infty} \frac{1}{\alpha_n \sinh(4\alpha_n) J_1(2\alpha_n)} \sinh(\alpha_n z) J_0(\alpha_n r)}$$

where $\alpha_n = x_n / 2 \qquad J_0(x_n) = 0$

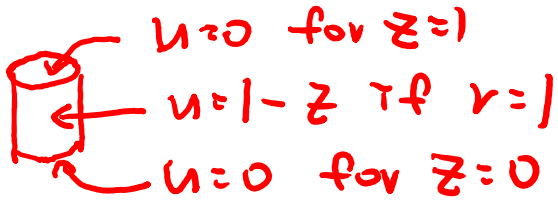
[Example 2] Steady Temperatures in a Circular Cylinder

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}=0$$

u is independent of θ

$$u(1,z)=1-z,\quad 0<z<1$$

$$u(r,0)=0,\quad u(r,1)=0\quad 0<r<1$$



$R(r)Z(0)=0$

(Solution): $Z(0)=0$

(Step 1) $u(r,z)=R(r)Z(z)$

$$R''(r)Z(z)+\frac{1}{r}R'(r)Z(z)+R(r)Z''(z)=0$$

(Step 2) $$\frac{R''+\frac{1}{r}R'}{R}=-\frac{Z''}{Z}=-\lambda$$

$$rR''+R'+\lambda rR=0 \qquad Z''-\lambda Z=0$$

From $u(r,0)=0,\ u(r,1)=0$

$$Z(0)=0,\quad Z(1)=0$$

(Step 3): Solve

Solve z first (two zero constraints)

$$Z'' - \lambda Z = 0 \qquad Z(0) = 0, \quad Z(1) = 0$$

There is no non-trivial solution for $\lambda = 0$ and $\lambda > 0$

$Z(z) = c_2 \sin \alpha z \qquad z(1) = c_2 \sin \alpha = 0$

$\alpha = n\pi$

When $\lambda < 0$, set $\lambda = -\alpha^2$,

$$Z'' + \alpha^2 Z = 0$$

$$Z(z) = c_1 \cos \alpha z + c_2 \sin \alpha z$$

From $Z(0) = 0, \quad Z(1) = 0$

$$\boxed{Z(z) = c_2 \sin \alpha_n z} \quad \text{where} \quad \alpha_n = n\pi \qquad \lambda_n = -\alpha_n^2 = -n^2\pi^2$$

(Step 4): Solve

$$rR'' + R' + \lambda r R = 0$$

$$rR'' + R' - n^2\pi^2 r R = 0$$

$$rR''+R'-n^2\pi^2 rR=0 \qquad r^2R''+rR'-n^2\pi^2r^2R=0$$

$v=0,\ \alpha=n\pi$

Since $x^2y''+xy'-(\alpha^2x^2+v^2)y=0$ solution : $c_1I_v(\alpha x)+c_2K_v(\alpha x)$

the solution of $r^2R''+rR'-n^2\pi^2r^2R=0$ is

$$R(r)=c_3I_0(n\pi r)+c_4K_0(n\pi r)$$

(page 205)

$I_v(x)$: modified Bessel function of the 1st kind

page 206 $K_v(x)$: modified Bessel function of the 2nd kind

Since $K_0(0)\to\infty$

$$\boxed{R(r)=c_3I_0(n\pi r)}$$

(Step 5): $u(r,z)=R(r)Z(z)$ $\qquad u_n(r,z)=A_nI_0(n\pi r)\sin(n\pi z)$

(Step 6): $$\boxed{u(r,z)=\sum_{n=1}^{\infty}A_nI_0(n\pi r)\sin(n\pi z)}$$

$$\boxed{u(r,z)=\sum_{n=1}^{\infty}A_nI_0(n\pi r)\sin(n\pi z)}$$

(Step 7): From $u(1,z)=1-z$

$$\sum_{n=1}^{\infty}A_nI_0(n\pi)\sin(n\pi z)=1-z$$

From the Fourier sine series

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi}{p}x \qquad b_n=\frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx$$

we have

$1-z$ $\quad A_nI_0(n\pi)$ $\quad p=1$

$$A_nI_0(n\pi)=2\int_0^1(1-z)\sin(n\pi z)dz=\frac{2}{n\pi}(z-1)\cos(n\pi z)\Big|_0^1-\int_0^1\frac{2}{n\pi}\cos(n\pi z)dz$$

$\int u'v=uv-\int uv'$

$u=1-z$

$v=\frac{-\cos(n\pi z)}{n\pi}$

$$A_n=\frac{2}{n\pi I_0(n\pi)}$$

$$\boxed{u(r,z)=2\sum_{n=1}^{\infty}\frac{I_0(n\pi r)}{n\pi I_0(n\pi)}\sin(n\pi z)}$$

2.9 SPHERICAL COORDINATES

Spherical Coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

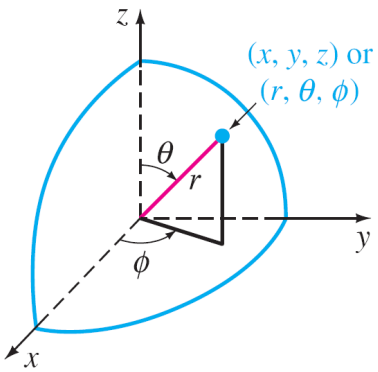


FIGURE 13.3.1 Spherical coordinates of a point (x, y, z) are (r, θ, ϕ).

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.3.

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.3

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta.$$

Laplacian in Spherical Coordinates

The 3D Laplacian is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\Downarrow$$

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}.}$$

$\Downarrow$ Suppose that u is independent of ϕ.

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}.}$$

[Example 1] Steady Temperatures in a Sphere

$\nabla^2 u = 0$

Find the steady-state temperature $u(r, \theta)$ within the sphere shown in Figure 13.3.2.

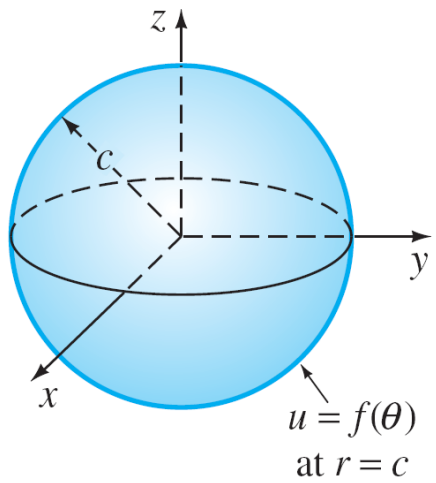


FIGURE 13.3.2 Dirichlet problem for a sphere in Example 1

From D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 13.3.

SOLUTION

Steady Temperatures

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Analogous to page 184-188

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \qquad u(r,\theta)$$

With $\quad u(c,\theta) = f(\theta), \quad 0 < \theta < \pi$

(Step 1): $\quad u = R(r)\Theta(\theta),$

$$R''(r)\Theta(\theta) + \frac{2}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) + \frac{\cot\theta}{r^2}R(r)\Theta'(\theta) = 0$$

divided by $u = R\Theta$

$$\frac{R''(r)}{R(r)} + \frac{2}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} + \frac{\cot\theta}{r^2}\frac{\Theta'(\theta)}{\Theta(\theta)} = 0$$

$$r^2\frac{R''(r)}{R(r)} + 2r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} + \cot\theta\frac{\Theta'(\theta)}{\Theta(\theta)} = 0$$

(Step 2): $$\frac{r^2R''+2rR'}{R}=-\frac{\Theta''+\cot\theta\Theta'}{\Theta}=\lambda.$$

2 ODEs

$$r^2R''+2rR'-\lambda R=0 \qquad \sin\theta\Theta''+\cos\theta\Theta'+\lambda\sin\theta\Theta=0.$$

(Step 3): Try to solve

$$\sin\theta\Theta''(\theta)+\cos\theta\Theta'(\theta)+\lambda\sin\theta\Theta(\theta)=0. \qquad 0\le\theta\le\pi$$

Set $x=\cos(\theta)$ $\quad -1\le x\le 1$

$$\frac{d}{d\theta}\Theta=\frac{dx}{d\theta}\frac{d}{dx}\Theta=-\sin\theta\frac{d}{dx}\Theta$$

$$\frac{d^2}{d\theta^2}\Theta=\frac{d}{d\theta}\frac{d}{d\theta}\Theta=-\frac{d}{d\theta}\left(\sin\theta\frac{d}{dx}\Theta\right)=-\cos\theta\frac{d}{dx}\Theta-\sin\theta\frac{d}{d\theta}\frac{d}{dx}\Theta$$

$$=-\cos\theta\frac{d}{dx}\Theta-\sin\theta\frac{dx}{d\theta}\frac{d^2}{dx^2}\Theta=-\cos\theta\frac{d}{dx}\Theta+\sin^2\theta\frac{d^2}{dx^2}\Theta$$

$$\sin^3\theta\frac{d^2\Theta}{dx^2}-2\sin\theta\cos\theta\frac{d\Theta}{dx}+\lambda\sin\theta\Theta=0$$

$$\sin^3\theta\frac{d^2\Theta}{dx^2}-2\sin\theta\cos\theta\frac{d\Theta}{dx}+\lambda\sin\theta\Theta=0$$

$x = \cos(\theta)$

$$\sin^2\theta\frac{d^2\Theta}{dx^2}-2\cos\theta\frac{d\Theta}{dx}+\lambda\Theta=0$$

$$\left(1-x^2\right)\frac{d^2\Theta}{dx^2}-2x\frac{d\Theta}{dx}+\lambda\Theta=0$$

$-1 \le x \le 1$

page 207

When $\lambda = n(n+1)$, it becomes Legendre's equation of order n

This is one of the applications of the Legendre equation

The solution is

$$\Theta(x)=cP_n(x)$$

$P_n(x)$: the Legendre polynomial of order n

$$\Theta(\theta)=cP_n(\cos\theta)$$

$n = 0, 1, 2, \ldots.$

Note: The other linearly independent solution may not have finite derivatives at $x = \pm 1$.

(Step 4): Try to solve

$$r^2 R'' + 2rR' - \lambda R = 0$$

Since $\lambda = n(n+1)$,

Cauchy - Euler

$$r^2 R'' + 2rR' - n(n+1)R = 0$$

Auxiliary: $m(m-1) + 2m - n(n+1) = 0$

$m^2 + m - n(n+1) = 0$

$(m-n)(m+n+1) = 0$

$$m = n, \quad -(n+1)$$

$$R(r) = c_1 r^n + c_2 \cancel{r^{-(n+1)}}$$

Since $0^{-(n+1)} \to \infty$ $\quad c_2 = 0$

$$\boxed{R(r) = c_1 r^n}$$

(Step 5): $u_n(r,\theta) = A_n r^n P_n(\cos\theta)$ $\quad n = 0, 1, 2, \ldots.$

(Step 6): $$\boxed{u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)}$$

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta)$$

(Step 7): From $u(c,\theta)=f(\theta)$

$$\sum_{n=0}^{\infty}A_n c^n P_n(\cos\theta)=f(\theta)$$

We have known that page 207

$$\int_{-1}^{1}P_n(x)P_m(x)\,dx=\frac{2}{2n+1}\delta_{n,m}$$

$$\delta_{n,n}=1$$

$$\delta_{n,m}=0 \quad \text{if } n\neq m$$

Set $x=\cos(\theta)$ $\quad \dfrac{dx}{d\theta}=-\sin\theta$

$$\int_{\pi}^{0}P_n(\cos\theta)P_m(\cos\theta)(-\sin\theta)d\theta=\frac{2}{2n+1}\delta_{n,m}$$

$$\int_{0}^{\pi}P_n(\cos\theta)P_m(\cos\theta)\sin\theta d\theta=\frac{2}{2n+1}\delta_{n,m}$$

(orthogonal with the weight function $\sin\theta$)

$$f(\theta)=\sum_{m=0}^{\infty}A_m c^m P_m(\cos\theta) \qquad \int_0^{\pi}P_n(\cos\theta)P_m(\cos\theta)\sin\theta d\theta=\frac{2}{2n+1}\delta_{n,m}$$

$$\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta=\sum_{m=0}^{\infty}A_m c^m\int_0^{\pi}P_m(\cos\theta)P_n(\cos\theta)\sin\theta d\theta$$

$$\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta=A_n c^n\frac{2}{2n+1}$$

$$A_n=\frac{2n+1}{2c^n}\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta$$

$$\boxed{u(r,\theta)=\sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta d\theta\right)\left(\frac{r}{c}\right)^n P_n(\cos\theta)}$$

Another Method for Solving PDEs: Method of Characteristics (只教不考)

Method of Characteristics

linear

The method is suitable for the 1st Order PDE.

Suppose that there is a 1st Order PDE as follows:

$$a_1\left(x_1,x_2,\cdots,x_k,u\right)\frac{\partial u\left(x_1,x_2,\cdots,x_k\right)}{\partial x_1}+a_2\left(x_1,x_2,\cdots,x_k,u\right)\frac{\partial u\left(x_1,x_2,\cdots,x_k\right)}{\partial x_2}$$

$$+\cdots\cdots+a_k\left(x_1,x_2,\cdots,x_k,u\right)\frac{\partial u\left(x_1,x_2,\cdots,x_k\right)}{\partial x_k}=g\left(x_1,x_2,\cdots,x_k,u\right)$$

Method of Characteristics

$$a_1\left(x_1,x_2,\cdots,x_k,u\right)\frac{\partial u\left(x_1,x_2,\cdots,x_k\right)}{\partial x_1}+a_2\left(x_1,x_2,\cdots,x_k,u\right)\frac{\partial u\left(x_1,x_2,\cdots,x_k\right)}{\partial x_2}$$

$$+\cdots\cdots+a_k\left(x_1,x_2,\cdots,x_k,u\right)\frac{\partial u\left(x_1,x_2,\cdots,x_k\right)}{\partial x_k}=g\left(x_1,x_2,\cdots,x_k,u\right)$$

We set a variable s and suppose that

$$\frac{\partial x_1}{\partial s}=a_1\left(x_1,x_2,\cdots,x_k,u\right),\quad \frac{\partial x_2}{\partial s}=a_2\left(x_1,x_2,\cdots,x_k,u\right),$$

$$\cdots\cdots,\quad \frac{\partial x_k}{\partial s}=a_k\left(x_1,x_2,\cdots,x_k,u\right).$$

Then, the original equation can be expressed as

$$\frac{\partial x_1}{\partial s}\frac{\partial u}{\partial x_1}+\frac{\partial x_2}{\partial s}\frac{\partial u}{\partial x_2}+\cdots\cdots+\frac{\partial x_k}{\partial s}\frac{\partial u}{\partial x_k}=g\left(x_1,x_2,\cdots,x_k,u\right)$$

$$\Downarrow$$

$$\frac{du}{ds}=g\left(x_1,x_2,\cdots,x_k,u\right)\Longrightarrow$$ Solve the 1st order ODE for u

Then, we apply the process as follows to solve u:

(Step 1) Solve the relation between x_1, x_2, …, x_k and s from each of the following ODEs:

$$\begin{aligned}
\frac{\partial x_1}{\partial s} &= a_1\left(x_1, x_2, \cdots, x_k, u\right) \\
\frac{\partial x_2}{\partial s} &= a_2\left(x_1, x_2, \cdots, x_k, u\right), \\
&\vdots \\
\frac{\partial x_k}{\partial s} &= a_k\left(x_1, x_2, \cdots, x_k, u\right).
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
x_1 &= a_1\left(x_1, x_2, \cdots, x_k, u\right)s + c_1 \\
x_2 &= a_2\left(x_1, x_2, \cdots, x_k, u\right)s + c_2 \\
&\vdots \\
x_k &= a_k\left(x_1, x_2, \cdots, x_k, u\right)s + c_k
\end{aligned}$$

(Step 2) Try to find the general equation b such that

$$b\left(x_1, x_2, \cdots, x_k\right) = c_b \qquad \text{where } c_b \text{ is a constant}$$

(Step 3) Solve the following ODE

$$\frac{du}{ds} = g\left(x_1(s), x_2(s), \cdots, x_k(s), u\right)$$ Solve the 1st order ODE for u

The solution has the form of $u = u_1\left(s, c_g\right)$

where c_g is some unknown constant.

(Specially, for case where $g = 0$, $u = c_g$)

(Step 4) Replace c_g by

$$c_g = f\left(c_b\right) = f\left(b\left(x_1, x_2, \cdots, x_k, u\right)\right)$$

where f is any function, then the solution is

$$u = u_1\left(s\left(x_1\right), f\left(c_b\right)\right)$$ Here, we express s as a function of x_1.

(Specially, for case where $g = 0$, $u = f(c_b)$)

[Example 1] Solve

$$\frac{\partial u(x,y)}{\partial x} = 2\frac{\partial u(x,y)}{\partial y}$$

(Solution): $$\frac{\partial u(x,y)}{\partial x} - 2\frac{\partial u(x,y)}{\partial y} = 0$$

(1) $$\frac{\partial x}{\partial s} = 1, \quad x = s + c_x$$

$$\frac{\partial y}{\partial s} = -2, \quad y = -2s + c_y$$

(2) $$2x + y = c_b$$

(3) $$\frac{du}{ds} = 0, \quad u = c_g = f(c_b)$$

$$u = f(2x + y)$$ where f is any function.

page 234

$u = XY$

$X'Y = 2XY'$

$\frac{X'}{2X} = \frac{Y'}{Y} = \lambda$

$X' - \lambda 2X = 0$

$Y' - \lambda Y = 0$

$X(x) = C_1 e^{2\lambda x}$

$Y(y) = C_2 e^{\lambda y}$

$u = \sum_\lambda C_\lambda e^{\lambda(2x+y)}$

[Example 2] Solve

$$\frac{\partial u(x,y,z)}{\partial x}=\frac{\partial u(x,y,z)}{\partial y}-\frac{\partial u(x,y,z)}{\partial z}+u(x,y,z)$$

$u = XYZ$

$X'YZ - XY'Z + XYZ' = XYZ$

(Solution):

$$\frac{\partial u(x,y,z)}{\partial x}-\frac{\partial u(x,y,z)}{\partial y}+\frac{\partial u(x,y,z)}{\partial z}=u(x,y,z)$$

$\frac{X'}{X} - \frac{Y'}{Y} + \frac{Z'}{Z} = 1$

$\frac{Z'}{Z} = \lambda_1, \quad \frac{Y'}{Y} = \lambda_2$

$\frac{X'}{X} = 1 + \lambda_2 - \lambda_1$

(1) $\frac{\partial x}{\partial s}=1, \quad \frac{\partial y}{\partial s}=-1, \quad \frac{\partial z}{\partial s}=1$

$x = s + c_x, \quad y = -s + c_y, \quad z = s + c_z$

$Z = c_1 e^{\lambda_1 z}, \quad Y = c_2 e^{\lambda_2 y}$

$X = c_3 e^{(1+\lambda_2-\lambda_1)x}$

(2) Note that

$x + y = c_x + c_y$ and $y + z = c_y + c_z$

$b_1(x,y,z) = x + y$

$b_2(x,y,z) = y + z$

can all generate a constant. Although x-z can also generate a constant, it is dependent on x+y and y+z. Therefore, a general way to obtain a constant is

$$f(x+y,\, y+z) = c$$

$u = XYZ$

$= c_{\lambda_1\lambda_2} e^{\lambda_1(z-x)} e^{\lambda_2(y+x)} e^{x}$

where f is any function with two independent variables.

(3) $$\frac{\partial x}{\partial s}\frac{\partial u(x,y,z)}{\partial x}+\frac{\partial y}{\partial s}\frac{\partial u(x,y,z)}{\partial y}+\frac{\partial z}{\partial s}\frac{\partial u(x,y,z)}{\partial z}=u(x,y,z)$$

$$\frac{du}{ds}=u,$$ Since $x=s+c_x$

$$u=c_1e^{s}=c_1e^{x-c_x}=ce^{x}$$ Here we set $c=c_1e^{-c_x}$

$$u=f(x+y,\,y+z)e^{x}$$

$y+z-x-y$

$$u(x,y,z)$$
$$=\sum_{\lambda_1}\sum_{\lambda_2}C_{\lambda_1\lambda_2}e^{\lambda_1(z-x)}e^{\lambda_2(y+x)}e^{x}$$
$$=\sum_{\lambda_1}\sum_{\lambda_2}C_{\lambda_1\lambda_2}e^{\lambda_1(y+z)}e^{(\lambda_2-\lambda_1)(y+x)}e^{x}$$

附錄五 Review for Laplace Transforms

2-sided Laplace $\int_{-\infty}^{\infty} e^{-st} f(t)\, dt$

Laplace transform (1-sided form) $F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$

Differentiation Property for the Laplace transform

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$\mathcal{L}(f''(t)) = s^2 \mathcal{L}(f(t)) - s f(0) - f'(0)$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots\cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$s^a f^{(b)}(0), \quad a + b = n - 1$$

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$\exp(at)$	$\frac{1}{s-a}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$

Seven Important Properties of the Laplace Transform

input	Laplace transform
(1) Differentiation $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots\cdots$ $-sf^{(n-2)}(0) - f^{(n-1)}(0)$
(2) Multiplication by t $t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
(3) Integration $\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$

(續)

input	Laplace transform
(4) Multiplication by exp $e^{at} f(t)$	$F(s-a)$
(5.1) Translation $u(t)$: unit step $f(t-a)u(t-a)$	$e^{-as}F(s)$
(5.2) Translation $g(t)u(t-a)$	$e^{-as}\mathcal{L}\{g(t+a)\}$
(6) Convolution $y(t)=\int_0^t f(\tau)g(t-\tau)d\tau$	$Y(s)=F(s)G(s)$
(7) Periodic Input $f(t)=f(t+T)$	$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$

2.10 Solving PDEs by Laplace Transforms

(只教不考)

In this section, we see that a linear PDE with constant coefficients is transformed into an ODE using the 1-sided Laplace transform.

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 14.2

Transform of a Function of Two Variables

$$U(x,s) = \mathcal{L}\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t)dt$$

To avoid confusing, we denote it as

$$U(x,s) = \mathcal{L}_{t\to s}\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t)dt$$

Transform of Partial Derivatives

$$\mathcal{L}_{t\to s}\left\{\frac{\partial u}{\partial t}\right\} = sU(x,s) - u(x,0),$$

$$\mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2U(x,s) - su(x,0) - u_t(x,0).$$

where

$$u_t(x,t) = \frac{\partial}{\partial t}u(x,t)$$

Because we are transforming with respect to t,

(not to x)

Be careful

$$\mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2U}{dx^2}.$$

for 1-sided Laplace
$0 < t < \infty$

[Example 1] Laplace Transform of a PDE

Find the Laplace transform of the wave equation $a^2 \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, $\boxed{t > 0}$.
(respect to t)

SOLUTION

$$\mathcal{L}_{t\to s}\left\{a^2 \frac{\partial^2 u}{\partial x^2}\right\} = \mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$

where $u_t = \dfrac{\partial}{\partial t} u$

$$a^2 \frac{d^2}{dx^2}\mathcal{L}_{t\to s}\{u(x,t)\} = s^2 \mathcal{L}_{t\to s}\{u(x,t)\} - su(x,0) - u_t(x,0)$$

$$a^2 \frac{d^2 U(x,s)}{dx^2} - s^2 U(x,s) = -su(x,0) - u_t(x,0).$$

ODE for x

The process for solving the BVP or IVP of a partial differential equation (PDE) by the Laplace transform

The range of the independent variable should be $[0, \infty)$

(Step 1) Apply the Laplace transform for one independent variable to change the PDE into an ordinary differential equation (ODE) with another independent variable.

(Step 2) Solve the ODE obtain in Step 1.

(Step 3) Solution in Step 2 contains some constants. Find these constants by transforming the initial conditions.

(Step 4) Inverse transform.

[Example 2] Using the Laplace Transform to Solve a BVP

$X'' + \lambda X = 0$

$T'' + \lambda T = 0$

$u = XT$

$X''T = XT'' = 0$

$\frac{X''}{X} = \frac{T''}{T} = -\lambda$

Solve $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1, \quad t > 0$$

Subject to $$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0$$

$$u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin \pi x, \quad 0 < x < 1.$$

$X(0) = X(1) = 0$

$X(x) = C \sin \pi n x$

$X'' = -C\pi^2 n^2 \sin \pi n x$

$\lambda = \pi^2 n^2$

$T = C_1 \sin \pi n t + C_2 \cos \pi n t$

$u(x,t) = \sum_n C_n \sin n\pi x \sin n\pi t$

SOLUTION

(Step 1) From Example 1 and the given initial conditions,

$$\frac{d^2 U(x,s)}{dx^2} - s^2 U(x,s) = -\sin \pi x,$$

page 245

where $U(x,s) = \mathcal{L}_{t \to s}\{u(x,t)\}$.

$$\mathcal{L}_{t \to s}\{u(0,t)\} = U(0,s) = 0 \qquad \mathcal{L}_{t \to s}\{u(1,t)\} = U(1,s) = 0.$$

s is a constant for x

(Step 2) $$\frac{d^2U(x,s)}{dx^2} - s^2U(x,s) = -\sin \pi x,$$

$m^2 - s^2 = 0$

$m = \pm s$

$c_3 e^{sx} + c_4 e^{-sx}$

Complementary Function

$$U_c(x,s) = c_1 \cosh sx + c_2 \sinh sx.$$

Particular Solution

$$U_p(x,s) = a \sin \pi x + b \cos \pi x.$$

$$a = \frac{1}{s^2 + \pi^2}, \quad b = 0$$

$(-a\pi^2 - as^2)\sin \pi x + (-b\pi^2 - bs^2)\cos \pi x$

$= -\sin \pi x$

$$U(x,s) = U_c(x,s) + U_p(x,s)$$

$$= c_1 \cosh sx + c_2 \sinh sx + \frac{1}{s^2 + \pi^2} \sin \pi x.$$

$$U(x,s) = c_1 \cosh sx + c_2 \sinh sx + \frac{1}{s^2+\pi^2}\sin \pi x.$$

for $x=0$, $U(0,s)=c_1$

(Step 3) From

$$\mathcal{L}_{t\to s}\{u(0,t)\} = U(0,s) = 0 \qquad \mathcal{L}_{t\to s}\{u(1,t)\} = U(1,s) = 0.$$

$$c_1 = 0. \qquad c_1 \cancel{\cosh} s + c_2 \sinh s = 0$$

We have $c_1 = 0$ and $c_2 = 0$.

$$U(x,s) = \frac{1}{s^2+\pi^2}\sin \pi x$$

(Step 4) $$u(x,t) = \mathcal{L}^{-1}_{s\to t}\left\{\frac{1}{s^2+\pi^2}\sin \pi x\right\} = \frac{1}{\pi}\sin \pi x \mathcal{L}^{-1}_{s\to t}\left\{\frac{\pi}{s^2+\pi^2}\right\}.$$

page 240 (4)
$k=\pi$

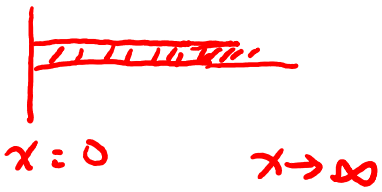
$$u(x,t) = \frac{1}{\pi}\sin \pi x \sin \pi t.$$

[Example 3] The Wave Equation with Gravity

Solve $$a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \quad x > 0, \quad t > 0$$

Subject to $$u(0,t) = 0, \quad \lim_{x\to\infty} \frac{\partial u}{\partial x} = 0, \quad t > 0$$

$$u(x,0) = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad x > 0$$



t=0

Since $\left.\frac{\partial u}{\partial x}\right|_{x=0}$ is unknown we perform the Laplace transform along t-axis

Solution:

(Step 1) $$\mathcal{L}_{t\to s}\left\{a^2 \frac{\partial^2 u}{\partial x^2}\right\} - \mathcal{L}_{t\to s}\{g\} = \mathcal{L}_{t\to s}\left\{\frac{\partial^2 u}{\partial t^2}\right\}$$

$1 \to \frac{1}{s}$

$$a^2 \frac{\partial^2 U(x,s)}{\partial x^2} - \frac{g}{s} = s^2 U(x,s) - su(x,0) - u_t(x,0) = s^2 U(x,s)$$

$$\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s^2}{a^2} U(x,s) = \frac{g}{a^2 s}$$

(Step 2) $$\frac{\partial^2 U(x,s)}{\partial x^2} - \frac{s^2}{a^2} U(x,s) = \frac{g}{a^2 s}$$

ODE for x

$-s^2/a^2$ is a constant for x

$m^2 - \frac{s^2}{a^2} = 0$

$m = \pm \frac{s}{a}$

Complementary Function

$$U_c(x,s) = c_1 \cosh\left(\frac{s}{a}x\right) + c_2 \sinh\left(\frac{s}{a}x\right)$$

Particular Solution

$$U_p(x, \cancel{y}\, s) = c_3$$

$$c_3 = -\frac{g}{s^3}$$

$$U(x,s) = c_1 \cosh\left(\frac{s}{a}x\right) + c_2 \sinh\left(\frac{s}{a}x\right) - \frac{g}{s^3}$$

$$U(x,s) = c_1 \cosh\left(\frac{s}{a}x\right) + c_2 \sinh\left(\frac{s}{a}x\right) - \frac{g}{s^3}$$

$U(0,s) = c_1 - g/s^3$

(Step 3) From initial conditions

$$\mathcal{L}_{t\to s}\{u(0,t)\} = U(0,s) = 0 \qquad \mathcal{L}_{t\to s}\{\lim_{x\to\infty}\frac{\partial u(x,t)}{\partial x}\} = \lim_{x\to\infty}\frac{\partial}{\partial x}U(x,s) = 0.$$

$$c_1 - \frac{g}{s^3} = 0$$

$$c_1 = \frac{g}{s^3}$$

$$\lim_{x\to\infty} c_1 \frac{s}{a}\sinh\left(\frac{s}{a}x\right) + c_2 \frac{s}{a}\cosh\left(\frac{s}{a}x\right) = 0$$

$$\lim_{x\to\infty} c_1 \frac{s}{2a}\exp\left(\frac{s}{a}x\right) + c_2 \frac{s}{2a}\exp\left(\frac{s}{a}x\right) = 0$$

$$c_2 = -c_1 = -\frac{g}{s^3}$$

Note: from page 110

$$\lim_{x\to\infty}\sinh(x) = \lim_{x\to\infty}\cosh(x) = \exp(x)/2$$

$\frac{e^x \pm e^{-x}}{2}$

$$U(x,s) = \frac{g}{s^3}\left(\cosh\left(\frac{s}{a}x\right) - \sinh\left(\frac{s}{a}x\right)\right) - \frac{g}{s^3} = \frac{g}{s^3}\exp\left(-\frac{s}{a}x\right) - \frac{g}{s^3}$$

$$U(x,s) = \frac{g}{s^3}\exp\left(-\frac{s}{a}x\right) - \frac{g}{s^3}$$

page 240 (2), n=2

$t^2 \to \frac{2}{s^3}$

(Step 4)

$$u(x,t) = \mathcal{L}^{-1}_{s\to t}\left\{\frac{g}{s^3}\exp\left(-\frac{s}{a}x\right) - \frac{g}{s^3}\right\}$$

page 242 (5.1)

a is replaced by $\frac{s}{a}$

$$u(x,t) = \frac{1}{2}g\left(t-\frac{x}{a}\right)^2 \mathcal{U}\left(t-\frac{x}{a}\right) - \frac{1}{2}gt^2$$

where $\mathcal{U}(t)$ is the unit step function

We have applied the translation property:

$$\mathcal{L}\{f(t-a)u(t-a)\} = \exp(-as)F(s)$$

附錄六 Review for Fourier Transforms

Fourier transform $s \to j2\pi f$

$$\Im[g(x)] = \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx = G(f)$$

$\Im$ 代表 Fourier transform

$e^{-j2\pi fx}$ has f periods for $x \in (0,1)$

inverse Fourier transform

$$\Im^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f) e^{j2\pi fx} df = g(x)$$

文獻上其他 Fourier transform 的定義

$$\Im[g(x)] = \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx = G(\omega)$$

$\omega = 2\pi f$

$$\Im^{-1}[G(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega x} d\omega = g(x)$$

或者

$$\Im[g(x)] = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-j\omega x} dx = G(\omega)$$

$$\Im^{-1}[G(\omega)] = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G(\omega) e^{j\omega x} d\omega = g(x)$$

或者

$$\Im[g(x)] = \int_{-\infty}^{\infty} g(x) e^{j\alpha x} dx = G(\alpha)$$

$\alpha : -\omega = -2\pi f$

$$\Im^{-1}[G(\alpha)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\alpha) e^{-j\alpha x} d\alpha = g(x)$$

When $g(x)$ is even ($g(x) = g(-x)$)

Fourier transform $\longrightarrow$ Fourier cosine transform

$$\boxed{\begin{aligned}\Im_c\left[g(x)\right] &= \int_0^\infty g(x)\cos(2\pi fx)dx = G_c(f)\\ \Im_c^{-1}\left[G_c(f)\right] &= 4\int_0^\infty G_c(f)\cos(2\pi fx)df = g(x)\end{aligned}}$$

Remember that if $k(x)$ is even

$$\int_{-\infty}^{\infty} k(x)dx = 2\int_0^\infty k(x)dx$$

If $k(x)$ is odd

$$\int_{-\infty}^{\infty} k(x)dx = 0$$

$$e^{-j2\pi fx} = \cos(2\pi fx) - j\sin(2\pi fx)$$

When $g(x)$ is odd ($g(x) = -g(-x)$)

Fourier transform $\longrightarrow$ Fourier sine transform

$$\boxed{\begin{aligned} \Im_s\left[g(x)\right] &= \int_0^\infty g(x)\sin(2\pi fx)dx = G_s(f) \\ \Im_s^{-1}\left[G_s(f)\right] &= 4\int_0^\infty G_s(f)\sin(2\pi fx)df = g(x) \end{aligned}}$$

Fourier, Fourier Cosine / Sine Transforms 的微分性質

(1) Fourier transform 的微分性質

$$\Im[g'(x)] = \int_{-\infty}^{\infty} g'(x) e^{-j2\pi fx} dx = g(x) e^{-j2\pi fx}\Big|_{-\infty}^{\infty} + j2\pi f \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx$$

$$= j2\pi f\, \Im[g(x)]$$

微分性質做了一些假設： $g(x) = 0$ when $x \to \infty$ and $x \to -\infty$

page 239, $s \to j2\pi f$

以此類推 $\boxed{\Im\left[g^{(n)}(x)\right] = (j2\pi f)^{(n)} G(f)}$

比較：對 Laplace transform

$$L\{f'(x)\} = sL\{f(x)\} - f(0) \qquad \int_0^{\infty} f(x) e^{-sx} dx$$

對 Fourier transform

$s \to j2\pi f$, without initial conditions

(2) Fourier cosine transform 的微分性質

$\int u'v = uv - \int uv'$

$$\Im_c[g'(x)] = \int_0^\infty g'(x)\cos(2\pi fx)\,dx$$

$$= g(x)\cos(2\pi fx)\Big|_0^\infty + 2\pi f\int_0^\infty f(x)\sin(2\pi fx)\,dx$$

$$= 2\pi f\,\Im_s[g(x)] - g(0)$$

(3) Fourier sine transform 的微分性質

$$\Im_s[g'(x)] = \int_0^\infty g'(x)\sin(2\pi fx)\,dx$$

$$= g(x)\sin(2\pi fx)\Big|_0^\infty - 2\pi f\int_0^\infty g(x)\cos(2\pi fx)\,dx$$

$$= -2\pi f\,\Im_c[g(x)]$$

注意： (1) Fourier sine, cosine transforms 互換

(2) α 正負號不同

(3) Fourier cosine transform 要考慮 initial condition

$$\Im_c\left[g'(x)\right] = 2\pi f\,\Im_s\left[g(x)\right] - g(0)$$

$$\Im_c\left(g''(x)\right) = 2\pi f\,\Im_s\left(g'(x)\right) - g'(0)$$

$$\Im_s\left[g'(x)\right] = -2\pi f\,\Im_c\left[g(x)\right]$$

$$\boxed{\Im_c\left[g''(x)\right] = 2\pi f\,\Im_s\left[g'(x)\right] - g'(0) = -4\pi^2 f^2\,\Im_c\left[g(x)\right] - g'(0)}$$

$$\boxed{\Im_s\left[g''(x)\right] = -2\pi f\,\Im_c\left[g'(x)\right] = -4\pi^2 f^2\,\Im_s\left[g(x)\right] + 2\pi f\,g(0)}$$

2.11 Solving PDEs by Fourier Transforms

(只教不考)

Differentiation $\longrightarrow$ Multiplication

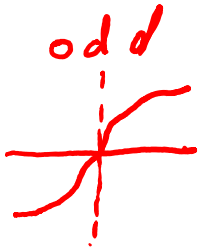
D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Section 14.4

Method

(Condition 1) interval 為 $-\infty < v < \infty$ 時:

用 Fourier transform

odd



(Condition 2) interval 為 $0 < v < \infty$，

有 "$u(v, \ldots..) = 0$ or a constant when $v = 0$" 的 boundary condition 時:

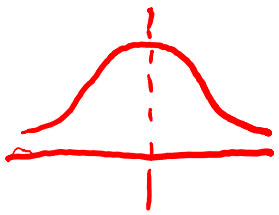
用 Fourier sine transform odd

(Condition 3) interval 為 $0 < v < \infty$，

有 "$\frac{\partial}{\partial v} u(v, \cdots\cdots) = 0$ or a constant when $v = 0$" 的 boundary condition 時:

用 Fourier cosine transform even

even



使用 Fourier transform, Fourier cosine transform, Fourier sine transform 來解 partial differential equation (PDE) 的 BVP 或IVP 的解法流程

(Step 1) 以 page 262 的規則，來決定要針對 哪一個 independent variable ，做什麼 transform (Fourier, Fourier cosine, 或Fourier sine transform)

(Step 2) 對 PDE 做 Step 1 所決定的 transform, 則原本的 PDE 變成針對另外一個 independent variable 的 ordinary differential equation (ODE)

(Step 3) 將 Step 2 所得出的 ODE 的解算出來

(Step 4) Step 3 所得出來的解會有一些 constants，可以對 initial conditions (或 boundary conditions) 做 transform 將 constants 解出

(※ 和 Step 1 所做的 transform 一樣，只是 transform 的對象變成是 initial 或 boundary conditions，見 pages 265, 268 的例子)

(Step 5) 最後，別忘了做 inverse transform (畫龍點睛)

[Example 1]

heat equation: $k\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t}$ $\quad -\infty<x<\infty \quad t>0$

subject to $u(x,0)=g(x)$ where $g(x)=\begin{cases} u_0, & |x|<1 \\ 0, & |x|>1 \end{cases}$

$g(x)=u_0\,\mathrm{rect}\left(\frac{x}{2}\right)$

Step 1 決定針對 x 做 Fourier transform

$$\Im_{x\to f}\{u(x,t)\}=\int_{-\infty}^{\infty}u(x,t)e^{-i2\pi fx}dx=U(f,t)$$

$G(f)=2u_0\,\mathrm{sinc}(2f)$

$=2u_0\dfrac{\sin(2\pi f)}{2\pi f}$

$\mathrm{sinc}(x)=\dfrac{\sin \pi x}{\pi x}$

Step 2 $\Im_{x\to f}\left\{k\dfrac{\partial^2 u}{\partial x^2}\right\}=\Im_{x\to f}\left\{\dfrac{\partial u}{\partial t}\right\}$

原本對 x, t 兩個變數做偏微分

$$-k4\pi^2 f^2 U(f,t)=\frac{\partial U(f,t)}{\partial t}$$

經過 Fourier transform 之後，只剩下對 t 做偏微分

ODE for t

$$\frac{dU(f,t)}{dt}+4k\pi^2 f^2 U(f,t)=0$$

→ constant for t

對於 t 而言，是 1st order ODE

Step 3 $U(f,t)=c\,e^{-4k\pi^2 f^2 t}$ 這邊的 c 值，對 t 而言是 constant，

但是可能會 **dependent on *f*** (特別注意)

Step 4 根據 $u(x, 0) = g(x)$ 將 c 解出

和 Step 1 一樣，也是針對 x 做 Fourier transform

只是對象改成 initial condition

$$\Im_{x\to f}\{u(x,0)\}=\int_{-\infty}^{\infty} g(x)e^{-i2\pi fx}dx=\int_{-1}^{1} u_0 e^{-i2\pi fx}dx$$

$$=u_0\frac{e^{-i2\pi f}-e^{i2\pi f}}{-i2\pi f}=u_0\frac{\sin(2\pi f)}{\pi f}$$

因為 $\Im_{x\to f}\{u(x,0)\}=U(f,0)$

$$U(f,0)=u_0\frac{\sin(2\pi f)}{\pi f} = C$$

$$U(f,t)=c\,e^{-4k\pi^2 f^2 t} \xleftrightarrow{\text{比較係數}} U(f,0)=u_0\frac{\sin(2\pi f)}{\pi f}$$

解出 $c=u_0\dfrac{\sin(2\pi f)}{\pi f}$

$$U(f,t)=u_0\frac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}$$

Step 5 未完待續，別忘了最後要做 inverse Fourier transform

$$u(x,t)=\Im^{-1}_{f\to x}\left[U(f,t)\right]=\int_{-\infty}^{\infty}u_0\frac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}e^{j2\pi fx}df$$

不易化簡，課本僅依據 $\dfrac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}$ 對 f 而言是 even function 將 $u(x, t)$ 化簡為

$$u(x,t)=u_0\int_{-\infty}^{\infty}\frac{\sin(2\pi f)}{\pi f}e^{-4k\pi^2 f^2 t}(\cos(2\pi fx)+j\sin(2\pi fx))df$$

$$=u_0\int_{-\infty}^{\infty}\frac{\sin(2\pi f)\cos(2\pi fx)}{\pi f}e^{-4k\pi^2 f^2 t}df$$

hard to further simplify

[Example 2] Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<1 \qquad y>0$$

$$u(0,y)=0 \qquad u(1,y)=e^{-y} \qquad y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0}=0 \qquad 0<x<1$$

Step 1 決定針對 y 做 Fourier cosine transform

$$\Im_{c,y\to f}\{u(x,y)\}=\int_0^\infty u(x,y)\cos(2\pi fy)dy=U(x,f)$$

Step 2 $$\Im_{c,y\to f}\left\{\frac{\partial^2 u}{\partial x^2}\right\}+\Im_{c,y\to f}\left\{\frac{\partial^2 u}{\partial y^2}\right\}=\Im_{c,y\to f}\{0\}$$

from $\Im_c[g''(y)]=-4\pi^2 f^2\Im_c[g(y)]-g'(0)$

$$\frac{d^2U(x,f)}{dx^2}-4\pi^2 f^2 U(x,f)=0$$ 對於 x 的 2nd order ODE

Step 3 $$\frac{d^2U(x,f)}{dx^2}-4\pi^2 f^2 U(x,f)=0$$

$$\Rightarrow \quad U(x,f)=c_1\cosh(2\pi fx)+c_2\sinh(2\pi fx)$$

Step 4 由 $u(0,y)=0 \quad u(\pi,y)=e^{-y}$ 來解 c_1, c_2 (may be dependent on f)

和 Step 1 一樣，<u>也是針對 y 做 Fourier cosine transform</u>

只是對象改成 boundary conditions

(1) $U(0,f)=\Im_{c,y\to f}\{u(0,y)\}=\int_0^\infty 0\cdot\cos(2\pi fy)dy=0$

(2) $U(1,f)=\Im_{c,y\to f}\{u(1,y)\}=\int_0^\infty e^{-y}\cdot\cos(2\pi fy)dy=\frac{1}{1+(2\pi f)^2}$

(可以用Laplace transform 的「取巧法」)

$$U(x,f)=c_1\cosh(2\pi fx)+c_2\sinh(2\pi fx)$$

分別代入 $U(0,f)=0$ $\qquad U(1,f)=\dfrac{1}{1+(2\pi f)^2}$

$$c_1=0 \qquad c_1\cosh(2\pi f)+c_2\sinh(2\pi f)=\frac{1}{1+(2\pi f)^2}$$

$$\Rightarrow \quad c_1=0 \qquad c_2=\frac{1}{(1+4\pi^2 f^2)\sinh(2\pi f)}$$

$$U(x,f)=\frac{\sinh(2\pi fx)}{(1+4\pi^2 f^2)\sinh(2\pi f)}$$

Step 5 inverse cosine transform

$$u(x,y)=\Im^{-1}_{c,f\to y}\left[U(x,f)\right]=4\int_0^\infty \frac{\sinh(2\pi fx)}{(1+4\pi^2 f^2)\sinh(2\pi f)}\cos(2\pi fy)df$$

(算到這裡即可，難以繼續化簡)

本節需要注意的地方

(1) 微分公式當中，Fourier cosine transform 和 Fourier sine transform 會有互換的情形。(See page 259)

(2) 在解 boundary value problem 時，要了解

何時用 Fourier transform，

何時用 Fourier cosine transform,

何時用 Fourier sine transform (see page 262)

(3) 在解 partial differential equation 時，往往只針對一個 independent variable 做 Fourier transform, 另一個 independent variable 不受影響，如 Examples 1 and 2 的例子

計算過程中，自己要清楚是對哪一個 independent variable 做Fourier transform

※ 本講義中習慣用下標做記號

3. Function Approximation

Section 3.1 Review for Orthogonal Basis Expansion

Section 3.2 Polynomial Approximation Using Legendre Polynomials

Section 3.3 Generalization for Function Set Approximation

Section 3.4 Other Orthogonal Polynomials (只教不考)

[1] D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017.

[2] R. Beals, Special Functions and Orthogonal Polynomials, Cambridge Studies in Advanced Mathematics, vol. 153, Cambridge University Press, 2016.

Function Approximation

- Function Approximation
  - Theorems for Orthogonal Basis (Sec. 3.1)
  - by Polynomials
    - Legendre Polynomials (Sec. 3.2)
    - Other Polynomials (Sec. 3.4，只教不考)
  - by Other Function Set (Sec. 3.3)

Sec. 3.1 Review for Orthogonal Basis Expansion

(1) inner product on an interval $[a, b]$

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx \qquad (f_1, f_2 \text{ 為 real 時})$$

or $$(f_1, f_2) = \int_a^b f_1(x) f_2^*(x) dx \qquad \text{(more standard definition)}$$

(2) orthogonal on an interval $[a, b]$

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) dx = 0 \qquad (f_1, f_2 \text{ 為 real 時})$$

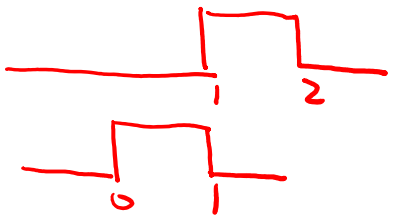
$$(f_1, f_2) = \int_a^b f_1(x) f_2^*(x) dx = 0 \qquad \text{(more standard definition)}$$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Sections 11-1~11-3.

注意：任何 even function 和任何 odd function 在 [-b, b] 之間必為 orthogonal, (a = -b)

$$\int_{-b}^{b} f_1(x) f_2(x)\, dx = 0$$

even odd



(3) orthogonal set

For a set of functions $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ……., if

$$\int_{a}^{b} \phi_m(x) \phi_n^*(x)\, dx = 0 \quad \text{for any } m \neq n$$

then $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, …….. is an orthogonal set on the interval [a, b]

(4) square norm

$$\|f(x)\|^2 = (f(x), f(x)) = \int_a^b f(x) f^*(x)\, dx = \int_a^b |f(x)|^2\, dx$$

(5) norm

$$\|f(x)\| = \sqrt{(f(x), f(x))} = \sqrt{\int_a^b f(x) f^*(x)\, dx}$$

(6) orthonormal set

對一個 orthogonal set, 若更進一步的滿足

$$\int_a^b \phi_n(x)\phi_n^*(x)\, dx = 1 \quad \text{for all } n$$

則被稱為 orthonormal set

$$(v(x), v(x)) = \left(\frac{\psi(x)}{\|\psi(x)\|}, \frac{\psi(x)}{\|\psi(x)\|}\right) = \frac{1}{\|\psi(x)\|^2}(\psi(x), \psi(x)) = \frac{\|\psi(x)\|^2}{\|\psi(x)\|^2} = 1$$

(7) normalize

$$\psi(x) \xrightarrow{\text{normalize}} v(x) = \frac{\psi(x)}{\|\psi(x)\|}$$

將 norm 變為 1

[Example 1] Suppose that $[a, b] = [-1, 1]$. Then

$$\left\{ \frac{1}{\sqrt{2}} e^{j\pi nx} \mid n = \cdots\cdots, -2, -1, 0, 1, 2, 3, \cdots\cdots \right\}$$

forms an orthonormal set.

(Proof): $\left(\frac{1}{\sqrt{2}} e^{j\pi nx}, \frac{1}{\sqrt{2}} e^{j\pi mx} \right) = \frac{1}{2} \int_{-1}^{1} e^{j\pi nx} e^{-j\pi mx} dx$

$\cos(\pi(n-m)) + j\sin(\pi(n-m))$

If $m \neq n$,

$$\left(\frac{1}{\sqrt{2}} e^{j\pi nx}, \frac{1}{\sqrt{2}} e^{j\pi mx} \right) = \left. \frac{e^{j\pi(n-m)x}}{j2\pi(n-m)} \right|_{-1}^{1} = \frac{e^{j\pi(n-m)} - e^{-j\pi(n-m)}}{j2\pi(n-m)} = \frac{\sin(\pi(n-m))}{\pi(n-m)} = 0$$

If $m = n$,

$$\left(\frac{1}{\sqrt{2}} e^{j\pi nx}, \frac{1}{\sqrt{2}} e^{j\pi mx} \right) = \frac{1}{2} \int_{-1}^{1} 1 dx = 1$$

(8) complete

In continuous function cases, ex: $1, x, x^2, x^3, \ldots$ are complete (but not orthogonal)

若在 interval $[a, b]$ 之間，任何一個 function $f(x)$ 都可以表示成 $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \ldots\ldots\ldots$ 的 linear combination

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots\cdots\cdots = \sum_{n=0}^{\infty} c_n\phi_n(x)$$

則 $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \ldots\ldots\ldots$ 被稱作 complete

how to find c_n

$f(x) = \sum_{m=0}^{\infty} c_m \phi_m(x)$, $(f(x), \phi_n(x)) = \sum_{m=0}^{\infty} c_m(\phi_m, \phi_n) = c_n(\phi_n, \phi_n)$

if $(\phi_m, \phi_n) = 0$ when $m \neq n$

(9) orthogonal series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x) \quad \text{where} \quad \boxed{c_n = \frac{\int_a^b f(x)\phi_n^*(x)\,dx}{\int_a^b \phi_n(x)\phi_n^*(x)\,dx}}$$

Specially, if $\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \ldots\ldots\ldots$ are orthonormal

$$\boxed{c_n = \int_a^b f(x)\phi_n^*(x)\,dx}$$

(10) weight

inner product with <u>weight function</u>

$$\left(f_1(x), f_2(x)\right) = \int_a^b w(x) f_1(x) f_2^*(x)\, dx$$

$w(x)$ is called the <u>weight function</u>

With the weight function

(10-1) orthogonal 的定義改成

$$\left(f_m, f_n\right) = \int_a^b w(x) f_m(x) f_n^*(x)\, dx = 0 \quad \text{for } m \neq n$$

(10-2) square norm 的定義改成 $\left\| f(x) \right\|^2 = \int_a^b w(x) f(x) f^*(x)\, dx$

(10-3) norm 的定義改成 $\left\| f(x) \right\| = \sqrt{\int_a^b w(x) f(x) f^*(x)\, dx}$

(10-4) orthonormal 的定義改成

$$\int_a^b w(x) f_m(x) f_n^*(x) dx = 0 \quad \text{for } m \neq n$$

$$\int_a^b w(x) f_n(x) f_n^*(x) dx = 1$$

(10-5) normalize 的算法改成

$$v(x) = \frac{\psi(x)}{\|\psi(x)\|} = \frac{\psi(x)}{\sqrt{\int_a^b w(x)\psi(x)\psi^*(x)dx}}$$

(10-6) orthogonal series expansion of $f(x)$ 的算法改成

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \qquad c_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x) dx}{\int_a^b w(x) \phi_n(x) \phi_n^*(x) dx}$$

orthonormal case: $c_n = \int_a^b w(x) f(x) \phi_n^*(x) dx$

trigonometric functions ← orthogonal , complete

$$\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \ldots\ldots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \sin\frac{3\pi}{p}x, \ldots\ldots\right\}$$

form a complete and orthogonal set on the interval of $[-p\ , p]$

$\cos\frac{n\pi}{p}x, \quad \sin\frac{n\pi}{p}x$ 週期：$\frac{2p}{n}$ 頻率：$\frac{n}{2p}$

$$\cos\frac{n\pi}{p}x = \cos\left(\frac{n\pi}{p}x + 2\pi\right) = \cos\left(\frac{n\pi}{p}\left(x + \frac{2p}{n}\right)\right)$$

Fourier Series (expanded by trigonometric functions)

from page 277

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$\frac{a_0}{2} = \frac{\int_{-p}^{p} f(x)\cdot 1\,dx}{\int_{-p}^{p} 1\cdot 1\,dx} = \frac{\int_{-p}^{p} f(x)\,dx}{2p}$$

$$a_n = \frac{\int_{-p}^{p} f(x)\cos\left(\frac{n\pi}{p}x\right)}{\int_{-p}^{p}\cos^2\left(\frac{n\pi}{p}x\right)dx}$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \qquad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx$$

$$\int_{-p}^{p}\cos^2\frac{2n\pi}{p}x\,dx = \frac{1}{2}\int_{-p}^{p} 1\cdot dx + \int_{-p}^{p}\cos\frac{2n\pi x}{p}\,dx = 2p/2 = p$$

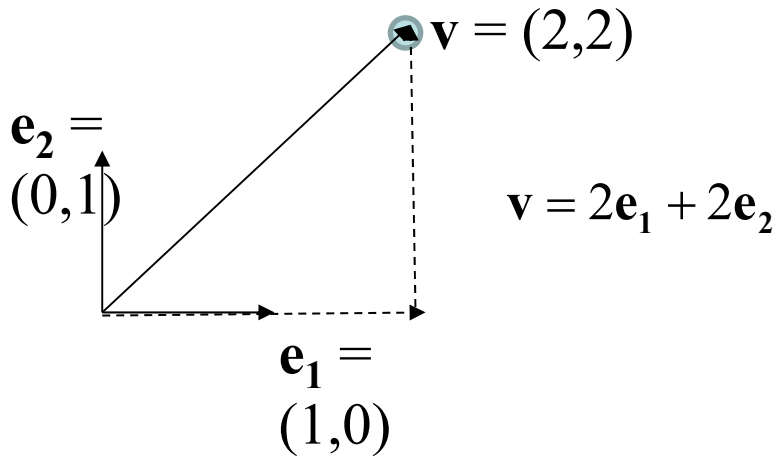
Q: Why should we use the orthogonal basis?

(1) The expansion coefficients are easier to determined,

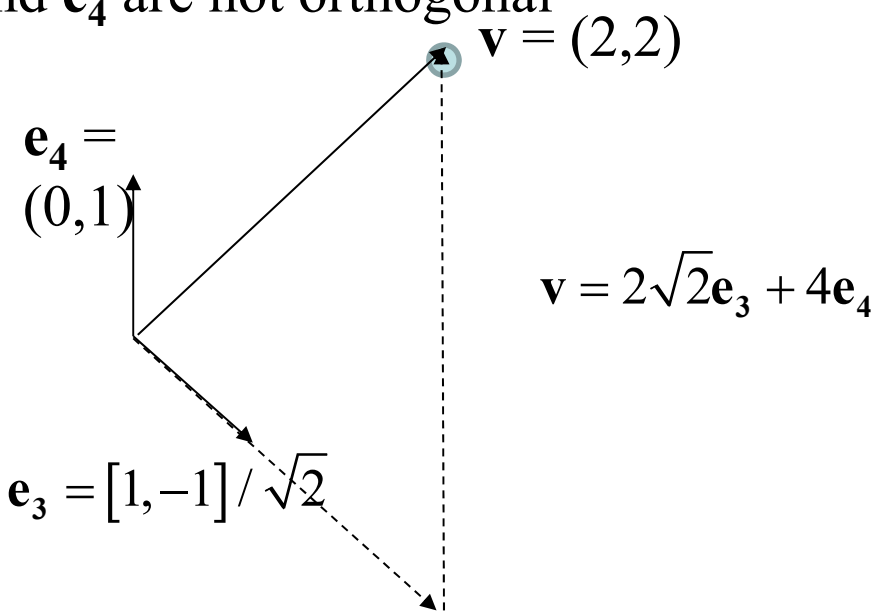
(2) No Interference

(3) More basis functions $\rightarrow$ Less error

$\mathbf{e_1}$ and $\mathbf{e_2}$ are orthogonal



$\mathbf{e_3}$ and $\mathbf{e_4}$ are not orthogonal



Section 3-2 Polynomial Approximation Using Legendre Polynomials ← complete and orthogonal

- **Legendre's equation of order *n***

$$\boxed{(1-x^2)y''-2xy'+n(n+1)y=0}$$

One of the solution: Legendre polynomials

$n=0$ $\quad P_0(x)=1$

$n=1$ $\quad P_1(x)=x$

$n=2$ $\quad P_2(x)=\frac{1}{2}\left(3x^2-1\right)$

$n=3$ $\quad P_3(x)=\frac{1}{2}\left(5x^3-3x\right)$

$n=4$ $\quad P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right)$

$n=5$ $\quad P_5(x)=\frac{1}{8}\left(63x^5-70x^3+15x\right)$

D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017, Sections 6-4, 11-5.

3.2.1 Legendre Polynomial

Legendre's Equation $(1-x^2)y''-2xy'+n(n+1)y=0$

$$y(x)=\sum_{k=0}^{\infty}c_k x^k$$ 代入，得出

Two linearly independent solutions are

$$y_1(x)=c_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6+\cdots\cdots\right]$$

$$y_2(x)=c_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5-\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7+\cdots\cdots\right]$$

(a) When n is not an integer, both the two solutions have infinite number of terms.

(b) When n is an even integer, $y_1(x)$ has finite number of terms.

In $y_1(x)$, the coefficient of x^k is zero when $k > n$.

(c) When n is an odd integer, $y_2(x)$ has finite number of terms.

In $y_2(x)$, the coefficient of x^k is zero when $k > n$.

$y_1(x)$ when n is an even integer and $y_2(x)$ when n is an odd integer are called the Legendre polynomials (denoted by $P_n(x)$).

通常選

$$c_0 = (-1)^{n/2}\frac{1\cdot 3\cdot\cdots\cdots(n-1)}{2\cdot 4\cdot\cdots\cdots n} \qquad c_1 = (-1)^{(n-1)/2}\frac{1\cdot 3\cdot\cdots\cdots n}{2\cdot 4\cdot\cdots\cdots(n-1)}$$

(讓 $P_n(1)$ 一律等於 1)

$\mathrm{span}(1, x, x^2, \cdots x^N) = \mathrm{span}(P_0(x), P_1(x) \cdots, P_N(x))$

$\sum_{n=0}^{N} C_n x^n$

由 $y_1(x)$

$$P_0(x) = 1$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

由 $y_2(x)$

$$P_1(x) = x$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

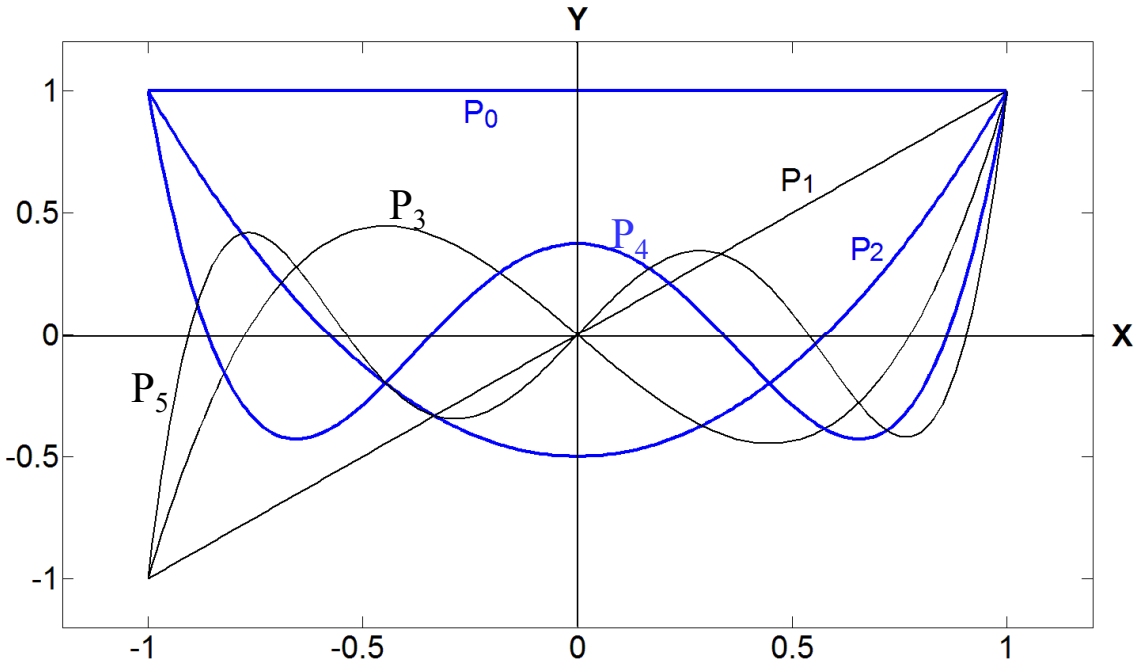
$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

In general,

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2 - 1\right)^n}$$

Rodrigues' formula

Legendre polynomials



Interval:

$x \in [-1, 1]$

BY NC SA

(1) $P_n(-x) = (-1)^n P_n(x)$ even / odd symmetry

(2) $P_n(1) = 1$ $P_n(-1) = (-1)^n$

(3) $P_n(0) = 0$ when n is odd

(4) $P_n'(0) = 0$ when n is even

(5) $(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$ recursive relation

(6) $\int_{-1}^{1} P_n(x) P_n(x)\, dx = \frac{2}{2n+1}$

$w(x) = 1$

$\sqrt{\frac{2n+1}{2}} P_n(x)$ $n = 0, 1, 2, \ldots\ldots$

form a complete and orthonormal set.

(7) $\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0$ If $m \neq n$ orthogonality property

Orthogonality property 才是 Legendre polynomials 最重要的性質

3.2.2 Expansion by Legendre Polynomials

若任何在 $x \in [-1, 1]$ 區間為 continuous 的函式 $f(x)$

皆可表示為

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

由於 $\int_{-1}^{1} f(x) P_m(x)\,dx = \underbrace{\sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_m(x) P_n(x)\,dx = a_m \int_{-1}^{1} P_m(x) P_m(x)\,dx}_{\text{根據 orthogonality property}}$

page 277

所以

page 288

$$a_n = \frac{\int_{-1}^{1} f(x) P_n(x)\,dx}{\int_{-1}^{1} P_n(x) P_n(x)\,dx} = \underline{\left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\,dx}$$

Q: How do we use the Nth order polynomial to approximate a function $f(x)$ for $x \in [-1, 1]$?

Answer:

note

$$f(x) = \sum_{n=0}^{N} a_n P_n(x)$$

where $$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x)\, dx$$

The detail of proof can be seen from Section 3.3.

[Example 2] Suppose that

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \quad \text{when } -1 \le x \le 1.$$

Try to approximate $f(x)$ by a 2nd order polynomial

$$f(x) \cong f_2(x) = a_0 + a_1 x + a_2 x^2$$

span$(1, x, x^2)$
$=$ span$(P_0(x), P_1(x), P_2(x))$

to minimize $\left\| f(x) - f_2(x) \right\|^2 = \int_{-1}^{1} \left(f(x) - f_2(x) \right)^2 dx$

(Solution):

Note that 1, x, x^2 are not a orthonormal set with $x \in [-1, 1]$.
Therefore, instead, we may adopt the <u>Legendre polynomials</u>.

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

Then $$f(x) \cong f_2(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)$$

where $$c_0 = \frac{1}{2}\int_{-1}^{1} f(x) P_0(x)\,dx = \frac{1}{2}\int_{-1}^{1} \cos\left(\frac{\pi}{2}x\right)dx = \frac{2}{\pi}$$ $\frac{1}{\pi}\sin\frac{\pi}{2}x\Big|_{-1}^{1}$

$$c_1 = \frac{3}{2}\int_{-1}^{1} f(x) P_1(x)\,dx = \frac{3}{2}\int_{-1}^{1} \underbrace{x}_{\text{odd}}\,\underbrace{\cos\left(\frac{\pi}{2}x\right)}_{\text{even}}dx = 0$$ page 274

$$c_2 = \frac{5}{2}\int_{-1}^{1} f(x) P_2(x)\,dx = \frac{5}{2}\int_{-1}^{1} \frac{1}{2}\left(3x^2 - 1\right)\cos\left(\frac{\pi}{2}x\right)dx$$

$$= \frac{1}{2}\frac{5}{2}\left(\frac{2}{\pi}\sin\left(\frac{\pi}{2}x\right)\left(3x^2-1\right)\Big|_{-1}^{1} - \int_{-1}^{1}\frac{12}{\pi}x\sin\left(\frac{\pi}{2}x\right)dx\right)$$

$$= \frac{5}{4}\left(\frac{8}{\pi} - \frac{96}{\pi^3}\right)$$

$\int uv' = uv - \int u'v$

$u = \frac{1}{2}(3x^2-1)$

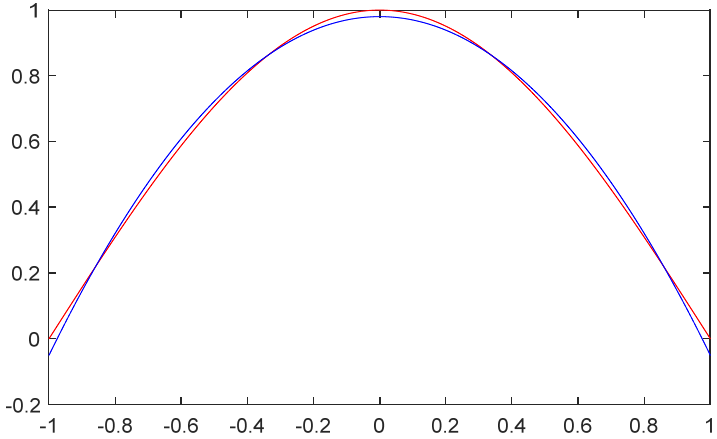
$v' = \cos\frac{\pi}{2}x$

$v = \frac{2}{\pi}\sin\frac{\pi}{2}x$

$$f_2(x) = \frac{2}{\pi}(1) + 0\cdot x + \frac{5}{4}\left(\frac{8}{\pi} - \frac{96}{\pi^3}\right)\frac{1}{2}\left(3x^2 - 1\right)$$

$$f_2(x) = \left(\frac{15}{\pi} - \frac{180}{\pi^3}\right)x^2 + \left(-\frac{3}{\pi} + \frac{60}{\pi^3}\right)$$

$$f(x) = \cos\left(\frac{\pi}{2}x\right) \qquad f_2(x) = \left(\frac{15}{\pi} - \frac{180}{\pi^3}\right)x^2 + \left(-\frac{3}{\pi} + \frac{60}{\pi^3}\right)$$



red line: $f(x)$

blue line: $f_2(x)$

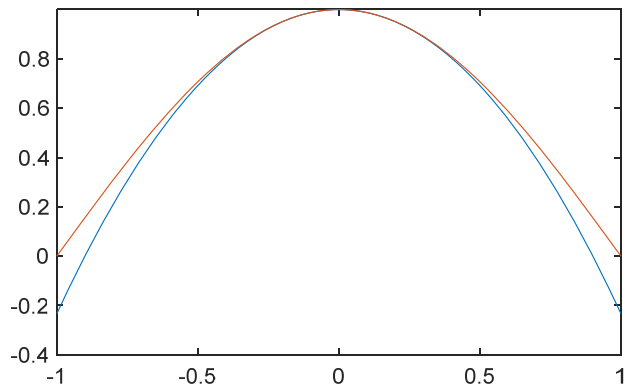
Approximation error (expressed by $\|f(x) - f_2(x)\|^2$) is

$$\int_{-1}^{1} \left(f(x) - f_2(x)\right)^2 dx = 0.00059606$$

Comparison: When using the Taylor series,

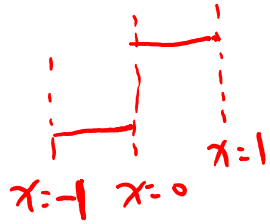
$$f(x) \cong T(x) = 1 - \frac{\pi^2}{8}x^2$$

$$\int_{-1}^{1} \left(f(x) - T(x)\right)^2 dx = 0.0125$$



[Example 3] (Zill page 456)

$$f(x)=\begin{cases}0, & -1<x<0\\ 1, & 0\le x<1.\end{cases}$$



SOLUTION

Using $$c_n=\frac{\int_a^b f(x)P_n^*(x)\,dx}{\int_a^b P_n(x)P_n^*(x)\,dx}$$

$$c_0=\frac{1}{2}\int_{-1}^{1}f(x)P_0(x)dx=\frac{1}{2}\int_0^1 1\cdot 1dx=\frac{1}{2}$$

$$c_1=\frac{3}{2}\int_{-1}^{1}f(x)P_1(x)dx=\frac{3}{2}\int_0^1 1\cdot xdx=\frac{3}{4}$$

$$c_2=\frac{5}{2}\int_{-1}^{1}f(x)P_2(x)dx=\frac{5}{2}\int_0^1 1\cdot\frac{1}{2}(3x^2-1)dx=0$$

$$c_3=-\frac{7}{16},\quad c_4=0,\quad c_5=\frac{11}{32},\quad \cdots\cdots\cdots$$

$$f(x)=\frac{1}{2}P_0(x)+\frac{3}{4}P_1(x)-\frac{7}{16}P_3(x)+\frac{11}{32}P_5(x)+\cdots.$$

If we want to approximate $f(x)$ by the 1st order polynomial,

$$f(x) \cong \underline{c_0 P_0(x) + c_1 P_1(x)} = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) = \frac{1}{2} + \frac{3}{4} x$$

If we want to approximate $f(x)$ by the 2nd order polynomial,

$$f(x) \cong \underline{c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)} = \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + 0 P_2(x) = \frac{1}{2} + \frac{3}{4} x$$

If we want to approximate $f(x)$ by the 3rd order polynomial,

不同

$$f(x) \cong \underline{c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x)}$$

$$= \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + 0 P_2(x) - \frac{7}{16} P_3(x) = \frac{1}{2} + \frac{45}{32} x - \frac{35}{32} x^3$$

3.2.3 Generalization for the Interval

Problem : $w(x) = 1$

How do we expand $f(x)$ where the range of x is $[a, b]$ and $a \neq -1$, $b \neq 1$?

(1) $$g(x) = f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right)$$ Note: The range of x is changed into [-1, 1]

$g(-1) = f\left(\frac{a-b}{2} + \frac{a+b}{2}\right) = f(a)$, $g(1) = f(b)$

(2) Expand $g(x)$ by Legendre polynomials

$$g(x) \cong \sum_{n=0}^{N} c_n P_n(x) \qquad \text{where} \quad c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} g(x) P_n(x)\,dx$$

(3) $$f(x) = g\left(\frac{2}{b-a}\left(x - \frac{a+b}{2}\right)\right) \cong \sum_{n=0}^{N} c_n P_n\left(\frac{2}{b-a}\left(x - \frac{a+b}{2}\right)\right)$$

$f(a) = g(-1)$

$f(b) = g(1)$

Section 3.3 Generalization for Function Set Approximation

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, …… $\phi_N(x)$ are a set of independent functions where $x \in [a, b]$ and the weight function is $w(x)$.

Problem :

try to find c_n

How do we expand $f(x)$ by $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$ …. $\phi_N(x)$

$$f_N(x) = \sum_{n=0}^{N} c_n \phi_n(x)$$

norm of $f(x) - f_N(x)$

where $$error = \| f(x) - f_N(x) \| = \sqrt{\int_a^b w(x)\left(f(x) - f_N(x)\right)^2 dx}$$

is minimized.

- Orthogonal
  - Complete (Case 1)
  - Incomplete (Case 2)
- Non-Orthogonal
  - Complete, Incomplete (Case 3)
    (Using Gram-Schmidt, page 305)
  - Over-Complete
    (Using Compressive Sensing，page 310)

$1, x, x^2, \ldots$
$x+1, x+2, x+3, \ldots$

(Case 1): The function set is complete and orthogonal / orthonormal

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

$$c_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x) dx}{\int_a^b w(x) |\phi_n(x)|^2 dx}$$

(orthogonal case)

$error = 0$

$$c_n = \int_a^b w(x) f(x) \phi_n^*(x) dx$$

(orthonormal case)

(Case 2): The function set is incomplete and orthogonal

$$f(x) \cong f_N(x) = \sum_{n=0}^{N} d_n \phi_n(x) \qquad \boxed{d_n = \frac{\int_a^b w(x) f(x) \phi_n^*(x) dx}{\int_a^b w(x) |\phi_n(x)|^2 dx}}$$

$$\text{error} = \| f(x) - f_N(x) \| = \sqrt{\sum_{n=N+1}^{\infty} |d_n|^2 \int_a^b w(x) |\phi_n(x)|^2 dx}$$

$$= \sqrt{\int_a^b w(x) f^2(x) dx - \sum_{n=0}^{N} |d_n|^2 \int_a^b w(x) |\phi_n(x)|^2 dx}$$

Specially, if the function set is orthonormal

$$f(x) \cong f_N(x) = \sum_{n=0}^{N} d_n \phi_n(x) \qquad \boxed{d_n = \int_a^b w(x) f(x) \phi_n^*(x) dx}$$

$$\text{error} = \sqrt{\sum_{n=N+1}^{\infty} |d_n|^2} = \sqrt{\int_a^b w(x) f^2(x) dx - \sum_{n=0}^{N} |d_n|^2}$$

(Proof): The formula of d_n is directly from page 279.

If $\psi_n(x) = \dfrac{\phi_n(x)}{\|\phi_n(x)\|}$

then $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, ……., $\psi_N(x)$ form an orthonormal set and

$$f_N(x) = \sum_{n=0}^{N} d_n \|\phi_n(x)\| \psi_n(x) = \sum_{n=0}^{N} d_n \phi_n(x)$$

Suppose that $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, ……., $\psi_N(x)$ is a subset of the complete and orthonormal function set $\psi_0(x)$, $\psi_1(x)$, $\psi_2(x)$, ……..., $\psi_N(x)$, $\psi_{N+1}(x)$, ……. Then, $f(x)$ can be expressed as

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x) \quad \text{where } c_n = \int_a^b w(x) f(x) \psi_n^*(x)\, dx$$

Therefore,

$$f(x) - f_N(x) = \sum_{n=0}^{N} \left(c_n - d_n \|\phi_n(x)\|\right) \psi_n(x) + \sum_{n=N+1}^{\infty} c_n \psi_n(x)$$

$$f(x)-f_N(x)=\sum_{n=0}^{N}\left(c_n-d_n\left\|\phi_n(x)\right\|\right)\psi_n(x)+\sum_{n=N+1}^{\infty}c_n\psi_n(x)$$

Then, from Parseval's theorem on page 302,

$$\left\|f(x)-f_N(x)\right\|^2=\sum_{n=0}^{N}\left|c_n-d_n\left\|\phi_n(x)\right\|\right|^2+\sum_{n=N+1}^{\infty}\left|c_n\right|^2$$

To minimize $\left\|f(x)-f_N(x)\right\|^2$, we should set

$$d_n=\frac{c_n}{\left\|\phi_n(x)\right\|}=\frac{\int_a^b w(x)f(x)\phi_n^*(x)\,dx}{\int_a^b w(x)\left|\phi_n(x)\right|^2dx}\qquad \text{for } n=0,1,\ldots,N$$

In this case,

$$\text{error}=\left\|f(x)-f_N(x)\right\|=\sqrt{\sum_{n=N+1}^{\infty}\left|c_n\right|^2}=\sqrt{\sum_{n=0}^{\infty}\left|c_n\right|^2-\sum_{n=0}^{N}\left|c_n\right|^2}$$

$$=\sqrt{\sum_{n=0}^{\infty}\left|c_n\right|^2-\sum_{n=0}^{N}\left|d_n\right|^2\int_a^b w(x)\left|\phi_n(x)\right|^2dx}$$

Parseval's theorem on page 302

$$=\sqrt{\int_a^b w(x)f^2(x)\,dx-\sum_{n=0}^{N}\left|d_n\right|^2\int_a^b w(x)\left|\phi_n(x)\right|^2dx}$$

3.3.1 Parseval's Theorem

[Theorem 3.3.1] Parseval's Theorem (Energy Preservation Theorem)

Suppose that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, …….. are a complete and orthonormal function set for $x \in [a, b]$ and

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \qquad \text{where} \qquad c_n = \left(f(x), \phi_n(x)\right) = \int_a^b w(x) f(x) \phi_n^*(x)\, dx$$

Then

$$\|f(x)\|^2 = \left(f(x), f(x)\right) = \sum_{n=0}^{\infty} |c_n|^2 \qquad \int_a^b w(x) |f(x)|^2\, dx = \sum_{n=0}^{\infty} |c_n|^2$$

(Proof):

$$\int_a^b w(x)\left|f(x)\right|^2 dx = \int_a^b w(x) f(x) f^*(x) dx$$

$$= \int_a^b w(x) \sum_{n=0}^{\infty} c_n \phi_n(x) \sum_{m=0}^{\infty} c_m^* \phi_m^*(x) dx$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n c_m^* \int_a^b w(x) \phi_n(x) \phi_m^*(x) dx$$

Since $\int_a^b w(x) \phi_n(x) \phi_m^*(x) dx = 0$ if $n \neq m$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n c_m^* \int_a^b w(x) \phi_n(x) \phi_m^*(x) dx = \sum_{n=0}^{\infty} c_n c_n^* \int_a^b w(x) \phi_n(x) \phi_n^*(x) dx$$

$$= \sum_{n=0}^{\infty} \left|c_n\right|^2 \int_a^b w(x) \left|\phi_n(x)\right|^2 dx$$

$$\int_a^b w(x)\left|f(x)\right|^2 dx = \sum_{n=0}^{\infty} \left|c_n\right|^2 \int_a^b w(x) \phi_n(x) \phi_m^*(x) dx$$

The result on the previous page leads to Parseval's theorem:

$$\int_a^b w(x)\left|f(x)\right|^2 dx = \sum_{n=0}^{\infty}\left|c_n\right|^2 \int_a^b w(x)\left|\phi_n(x)\right|^2 dx$$

Specially, if the function set $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\ldots\ldots\ldots\}$ is orthonormal

$$\int_a^b w(x)\left|f(x)\right|^2 dx = \sum_{n=0}^{\infty}\left|c_n\right|^2$$

3.3.2 Non-Orthogonal Function Set Expansion

(Case 3)

How do we generate an orthonormal function set $\{\phi_0(x), \phi_1(x), \phi_2(x), \phi_3(x), \ldots\ldots\ldots\}$ if the input $\{p_0(x), p_1(x), p_2(x), p_3(x), \ldots\ldots\ldots\}$ is independent but not an orthogonal function set ?

Gram-Schmidt process

$$\phi_0(x) = \frac{p_0(x)}{\|p_0(x)\|} \qquad n = 0$$

$$n = n+1$$

$$g_n(x) = p_n(x) - \sum_{m=0}^{n-1} \left(p_n(x), \phi_m(x)\right)\phi_m(x)$$

$$\phi_n(x) = \frac{g_n(x)}{\|g_n(x)\|}$$

$$(P_k(n), \phi_m(n)) = \sum_{n=a}^{b} w(n) P_k(n) \phi_m^*(n)$$

$$\|g_k(n)\| = \sqrt{\sum_{n=a}^{b} w(n) |g_k(n)|^2}$$

Proof for orthogonality of the function set generated from the Gram-Schmidt process

Suppose that we have known that $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ……., $\phi_{n\text{-}1}(x)$ are orthonormal.

If $k < n$,

$$\left(\phi_n(x), \phi_k(x)\right) = \left(\frac{g_n(x)}{\|g_n(x)\|}, \phi_k(x)\right)$$

$$= \frac{1}{\|g_n(x)\|}\left(p_n(x) - \sum_{m=0}^{n-1}\left(p_n(x), \phi_m(x)\right)\phi_m(x), \quad \phi_k(x)\right)$$

$$= \frac{1}{\|g_n(x)\|}\left[\left(p_n(x), \phi_k(x)\right) - \sum_{m=0}^{n-1}\left(p_n(x), \phi_m(x)\right)\left(\phi_m(x), \phi_k(x)\right)\right]$$

$$= \frac{1}{\|g_n(x)\|}\left[\left(p_n(x), \phi_k(x)\right) - \left(p_n(x), \phi_k(x)\right)\right] = 0$$

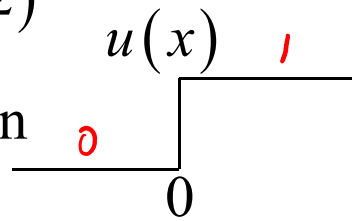
$(\phi_m, \phi_k) = 0$ if $m \neq k$
$= 1$ if $m = k$

[Example 4]

Derive $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, …….. such that

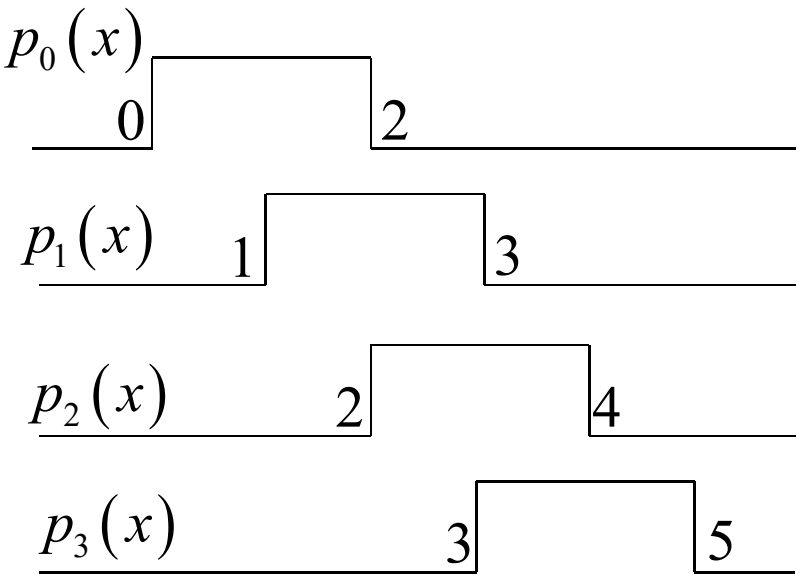
$$\phi_n(x) = \sum_{k=0}^{n} \tau_{n,k} p_k(x), \qquad p_k(x) = u(x-k) - u(x-k-2)$$

$u(x)$: unit step function



and

$$\int_0^\infty \phi_n(x)\phi_m(x)\,dx = 0 \quad \text{when } m \neq n \qquad \int_0^\infty \phi_n(x)\phi_n(x)\,dx = 1$$



⇒ orthonormalization

Answer:

Set $p_k(x) = u(x-k) - u(x-k-2)$

$$\phi_0(x) = \frac{p_0(x)}{\|p_0(x)\|} = \frac{u(x) - u(x-2)}{\sqrt{2}}$$

$$g_1(x) = p_1(x) - (p_1(x), \phi_0(x))\phi_0(x)$$

$$= u(x-1) - u(x-3) - \frac{1}{\sqrt{2}}\left(\frac{u(x) - u(x-2)}{\sqrt{2}}\right)$$

$$\phi_1(x) = \frac{g_1(x)}{\|g_1(x)\|} = \frac{u(x-1) - u(x-3) - \frac{1}{2}(u(x) - u(x-2))}{\sqrt{6}/2}$$

$$g_2(x) = p_2(x) - (p_2(x), \phi_0(x))\phi_0(x) - (p_2(x), \phi_1(x))\phi_1(x)$$

$$= u(x-2) - u(x-4) - \frac{2}{\sqrt{6}} \frac{u(x-1) - u(x-3) - \frac{1}{2}(u(x) - u(x-2))}{\sqrt{6}/2}$$

$$= u(x-2) - u(x-4) - \frac{2}{3}u(x-1) + \frac{2}{3}u(x-3) + \frac{1}{3}(u(x) - u(x-2))$$

$$\phi_2(x) = \frac{g_2(x)}{\|g_2(x)\|}$$

$$= \frac{u(x-2) - u(x-4) - \frac{2}{3}(u(x-1) - u(x-3)) + \frac{1}{3}(u(x) - u(x-2))}{2/\sqrt{3}}$$

……

……

附錄七 Compressive Sensing

The problems that compressive sensing deals with:

Suppose that $b_0(t)$, $b_1(t)$, $b_2(t)$, $b_3(t)$ …….. form an over-complete and non-orthogonal spanning set.

$b_0(t), b_1(t), b_2(t), b_3(t) \cdots\cdots$
may be dependent

(Problem 1) We want to minimize $\|c\|_0$ （$\| \ \|_0$ 是 zero-order norm， $\|c\|_0$ 意指c_m 的值不為 0 的個數） such that

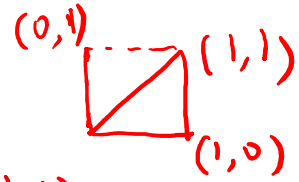
$$x(t) = \sum_m c_m b_m(t)$$

$e_1 = [1, 0]$
$e_2 = [0, 1]$
$e_\theta = [\cos\theta, \sin\theta]$

(Problem 2) We want to minimize $\|c\|_0$ such that

$\|(1,1) - e_{\frac{\pi}{4}} \cdot \sqrt{2}\| = 0 \quad \theta \in [0, \pi)$

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$



(Problem 3) When $\|c\|_0$ is limited to M, we want to minimize

$\|(1,1) - 1 \cdot e_1\| = 1$

$\|(1,1) - 1 \cdot e_2\| = 1$

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$

Examples for the spanning set used for compressive sensing
(Since they are not linearly independent, they are not basis).

3-atom form

$$b_m(t) = \exp(j2\pi f_m t)\exp(-\frac{\pi(t-t_m)^2}{\sigma_m^2})$$

4-atom form

$$b_m(t) = \exp(j2\pi(f_m t + \eta_m t^2))\exp(-\frac{\pi(t-t_m)^2}{\sigma_m^2})$$

Problems: over-complete, non-orthogonal, too many functions in the sets

Section 3.4 Other Orthogonal Polynomials

(只教不考)

In addition to Legendre Polynomials, there are many other orthogonal polynomials. However, their weight functions and intervals are different.

[1] R. Beals, Special Functions and Orthogonal Polynomials, Cambridge Studies in Advanced Mathematics, vol. 153, Cambridge University Press, 2016.

[2] M. R. Spiegel, Mathematical Handbook, Schaum, 1990.

Summary of Weights and Supports of Orthogonal Polynomials

Name	Weight Function $w(x)$	Support
Legendre	1	$x \in [-1, 1]$
Jacobi	$(1+x)^n(1-x)^m$	$x \in [-1, 1]$
Associated Legendre (Ultraspherical, Gegenbauer)	$(1-x^2)^m$	$x \in [-1, 1]$
Chebyshev (1st Kind)	$(1-x^2)^{-1/2}$	$x \in [-1, 1]$
Chebyshev (2st Kind)	$(1-x^2)^{1/2}$	$x \in [-1, 1]$
Laguerre	e^{-x}	$x \in [0, \infty)$
Associated Laguerre	$x^m e^{-x}$	$x \in [0, \infty)$
Hermite	e^{-x^2}	$x \in (-\infty, \infty)$

- Associated Legendre Functions

$$\boxed{P_{n,m}(x) = \frac{d^m}{dx^m} P_n(x)}$$

$P_n(x)$: the Legendre polynomial of order n, in fact, $P_n(x) = P_{n,0}(x)$

They are orthogonal for $x \in [-1, 1]$, $w(x) = \left(1 - x^2\right)^m$

$$\int_{-1}^{1} \left(1 - x^2\right)^m P_{n,m}(x) P_{k,m}(x) dx = 0 \qquad \text{if } n \neq k$$

$$\int_{-1}^{1} \left(1 - x^2\right)^m \left(P_{n,m}(x)\right)^2 dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$$

Ex: When $m = 1$,

$$P_{1,1}(x) = 1 \qquad P_{2,1}(x) = 2x$$

$$P_{3,1}(x) = \frac{15}{2} x^2 - \frac{3}{2} \qquad P_{4,1}(x) = \frac{35}{2} x^3 - \frac{15}{2} x$$

- Chebychev polynomials 電子學和 filter design 常用

$$P_n(\cos\theta) = \cos n\theta$$

$$P_n(x) = \sum_{k=0}^{n/2} \binom{n}{2k} x^{n-2k} \left(1-x^2\right)^k$$

They are orthogonal for $x \in [-1, 1]$ $\quad w(x) = \dfrac{1}{\sqrt{1-x^2}}$

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} P_n(x) P_k(x)\, dx = 0 \quad \text{if } n \neq k$$

$$\int_{-1}^{1} \frac{P_n^2(x)}{\sqrt{1-x^2}} dx = \begin{cases} \pi & \text{if } n = 0, \\ \pi/2 & \text{otherwise} \end{cases}$$

They are the solutions of $(1-x^2)P_n''(x) - xP_n'(x) + n^2 P_n(x) = 0$

$$P_0(x) = 1 \qquad P_1(x) = x$$

$$P_2(x) = 2x^2 - 1 \qquad P_3(x) = 4x^3 - 3x$$

- Laguerre polynomials

$$P_n(x) = e^x \frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

They are orthogonal for $x \in [0, \infty)$ $\quad w(x) = e^{-x}$

$$\int_0^\infty e^{-x} P_n(x) P_k(x)\, dx = 0 \qquad \int_0^\infty e^{-x} \left(P_n(x)\right)^2 dx = (n!)^2$$

if $n \neq k$

They are the solutions of $\quad xP_n''(x) + (1-x)P_n'(x) + nP_n(x) = 0$

$$P_0(x) = 1 \qquad P_1(x) = -x + 1$$

$$P_2(x) = x^2 - 4x + 2 \qquad P_3(x) = -x^3 + 9x^2 - 18x + 6$$

- Associated Laguerre polynomials

$$\boxed{P_{n.m}(x) = \frac{d^m}{dx^m}\left(P_n(x)\right)}$$

where $P_n(x)$ is the nth order Laguarre polynomial

They are orthogonal for $x \in [0, \infty)$ $\quad w(x) = x^m e^{-x}$

$$\int_0^\infty x^m e^{-x} P_{n,m}(x) P_{k,m}(x)\, dx = 0 \quad \text{if } n \neq k$$

$$\int_0^\infty x^m e^{-x} \left(P_{n,m}(x)\right)^2 dx = \frac{(n!)^2}{(n-m)!}$$

They are the solutions of $xP_n''(x) + (m+1-x)P_n'(x) + (n-m)P_n(x) = 0$

$$P_{1,1}(x) = -1 \qquad P_{2,1}(x) = 2x - 4$$

$$P_{3,1}(x) = -3x^2 + 18x - 18 \qquad P_{4,1}(x) = 4x^3 - 48x^2 + 144x - 96$$

- Hermite polynomials 電磁波、光學、頻譜分析常用

Eigenfunctions of the Fourier transform

$$P_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$$

They are orthogonal for $x \in (-\infty, \infty)$ $\quad w(x) = e^{-x^2}$

$$\int_{-\infty}^{\infty} e^{-x^2} P_n(x) P_k(x)\, dx = 0 \quad \text{if } n \neq k$$

$$\int_{-\infty}^{\infty} e^{-x^2} \left(P_n(x)\right)^2 dx = 2^n n! \sqrt{\pi}$$

They are the solutions of $P_n''(x) - xP_n'(x) + nP_n(x) = 0$

$P_0(x) = 1$ $\qquad P_1(x) = 2x$

$P_2(x) = 4x^2 - 2$ $\qquad P_3(x) = 8x^3 - 12x$

$P_4(x) = 16x^4 - 48x^2 + 12$ $\qquad P_5(x) = 32x^5 - 160x^3 + 120x$