4. Fourier Analysis

Section 4.1 Definition of the Fourier Transform

Section 4.2 Dirac Delta Function

Section 4.3 Properties

Section 4.4 Uncertainty Principle

Section 4.5 Convolution and Correlation

Section 4.6 2D Fourier Transforms

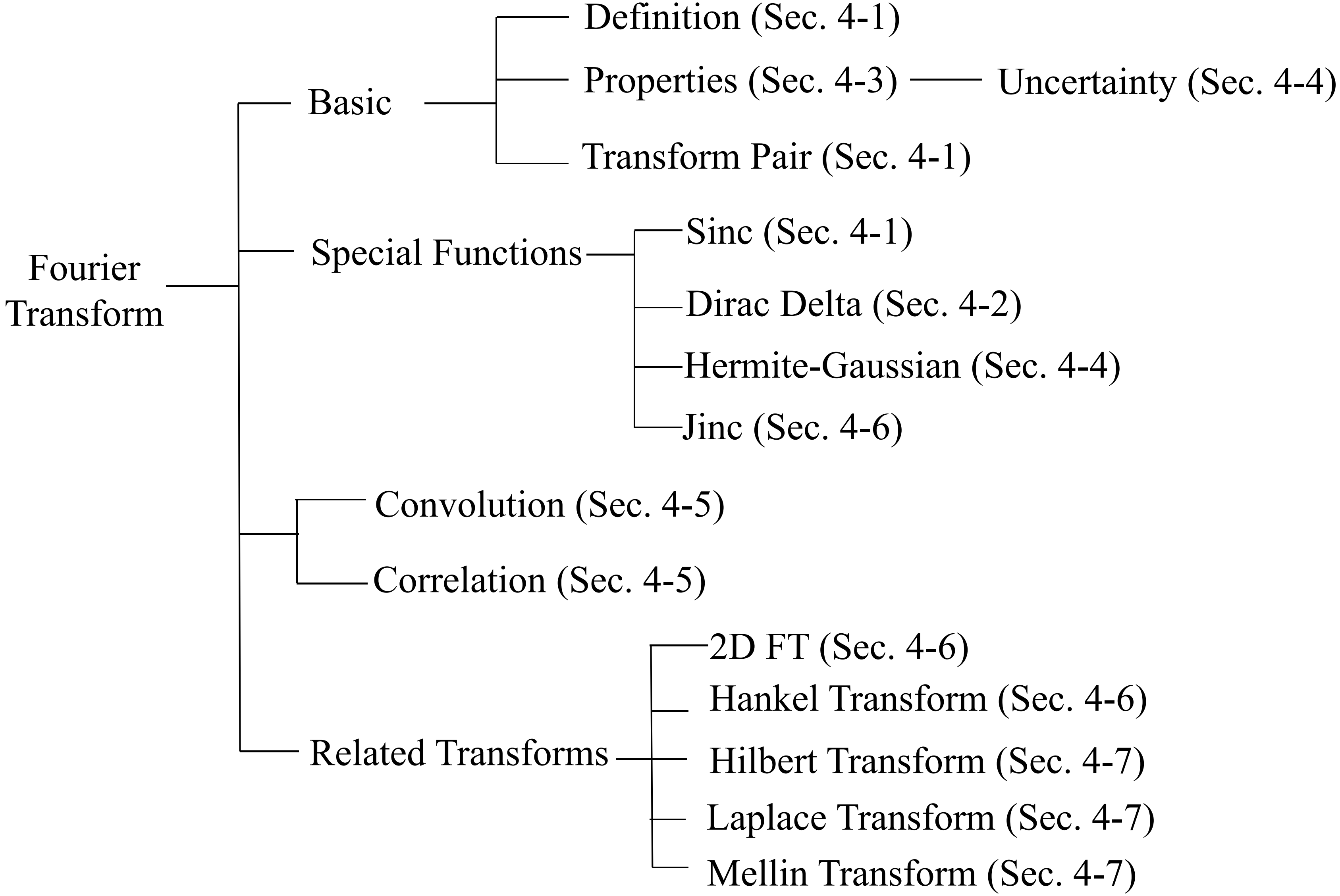
Section 4.7 The Operations Related to Fourier Transforms (只教不考)

[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

[2] D. G. Zill and Michael R. Cullen, Differential Equations-with Boundary-Value Problem (metric version), 9th edition, Cengage Learning, 2017.

[3] D. G. Zill, W. S. Wright, and J. J. Ding, Engineering Mathematics, Metric Edition, Cengage Learning, Taipei, Taiwan, 2019, Chapter 15.

Fourier Transform



- Fourier Transform
  - Basic
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4.1 Definition of the Fourier Transform

Fourier transform

$$\mathfrak{I}\left[g(x)\right]=\int_{-\infty}^{\infty}g(x)e^{-j2\pi fx}dx=G(f)$$

$\mathfrak{I}$ 代表 Fourier transform

inverse Fourier transform

$$\mathfrak{I}^{-1}\left[G(f)\right]=\int_{-\infty}^{\infty}G(f)e^{j2\pi fx}df=g(x)$$

[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

[2] D. G. Zill, W. S. Wright, and J. J. Ding, Engineering Mathematics, Metric Edition, Cengage Learning, Taipei, Taiwan, 2019, Sections 15-2.

Review: Fourier Series of the Complex Form

complete and orthogonal

$$\phi_n(x) = \exp\left(j\frac{2\pi}{T}nx\right) \qquad n = \ldots, -1, 0, 1, 2, 3, \ldots.$$

$$\phi_n(x+T) = \exp\left(j\frac{2\pi}{T}n(x+T)\right) = \exp\left(j\frac{2\pi}{T}nx\right)\exp(j2\pi n) = \phi_n(x)$$

form a complete and orthogonal set within $x \in \left[-\frac{T}{2}, \frac{T}{2}\right]$

$$\int_{-T/2}^{T/2} \phi_n(x)\phi_m^*(x)\,dx = \begin{cases} 0 & if\ m \neq n \\ T & if\ m = n \end{cases}$$

$$= \int_{-T/2}^{T/2} \exp\left(j\frac{2\pi}{T}(n-m)x\right)dx$$

$$= \frac{T}{j2\pi(n-m)} e^{j\frac{2\pi}{T}(n-m)x}\Big|_{-T/2}^{T/2} \qquad (n \neq m)$$

$$= \frac{T}{j2\pi(n-m)}\left(e^{j\pi(n-m)} - e^{-j\pi(n-m)}\right)$$

$$= \frac{T}{\pi(n-m)} \sin(\pi(n-m)) = 0$$

On page 280

p is replaced by $\frac{T}{2}$

$$\cos\left(\frac{n\pi}{p}x\right) + j\cdot\sin\left(\frac{n\pi}{p}x\right)$$

$$= \exp\left(j\frac{n\pi}{p}x\right)$$

$$= \exp\left(j\frac{2n\pi}{T}x\right)$$

frequency $= \frac{n}{2p} = \frac{n}{T}$

Review: Fourier Series of the Complex Form

(Compared to pages 277, 280)

$x \in [-\frac{T}{2}, \frac{T}{2}]$

If $g(x) = g(x+T)$, then

$$g(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j\frac{2\pi}{T}nx\right)$$

where

$$c_n = \frac{\int_{-T/2}^{T/2} g(x) conj\left(\exp\left(j\frac{2\pi}{T}nx\right)\right)dx}{\int_{-T/2}^{T/2} \exp\left(j\frac{2\pi}{T}nx\right) conj\left(\exp\left(j\frac{2\pi}{T}nx\right)\right)dx}$$

$$c_n = \frac{\int_{-T/2}^{T/2} g(x)\exp\left(-j\frac{2\pi}{T}nx\right)dx}{T}$$

4.1.1 Derivation and Physical Meaning

Fourier transform can be viewed as the Fourier series where

$$\boxed{T \to \infty}$$

Note that, if we set $g_n = c_n T$

$$g_n = \int_{-T/2}^{T/2} g(x)\exp\left(-j\frac{2\pi}{T}nx\right)dx \qquad g(x) = \sum_{n=-\infty}^{\infty} \frac{g_n}{T}\exp\left(j\frac{2\pi}{T}nx\right)$$

Then we set $\Delta_f = 1/T$

$$g_n = \int_{-T/2}^{T/2} g(x)\exp\left(-j2\pi n\Delta_f x\right)dx \qquad g(x) = \sum_{n=-\infty}^{\infty} g_n\Delta_f \exp\left(j2\pi n\Delta_f x\right)$$

$$G(f) = \int_{-T/2}^{T/2} g(x)\exp\left(-j2\pi fx\right)dx \qquad g(x) = \sum_{n=-\infty}^{\infty} G(f)\exp\left(j2\pi fx\right)\Delta_f$$

where $f = n\Delta_f$, $G(f) = g_n$

If $T \to \infty$, $\Delta_f \to 0$

$$G(f) = \int_{-\infty}^{\infty} g(x)\exp\left(-j2\pi fx\right)dx \qquad g(x) = \int_{-\infty}^{\infty} G(f)\exp\left(j2\pi fx\right)df$$

Physical Meaning of the Fourier Transform:

expanding a signal as a combination of $\exp(j2\pi fx)$

$\exp(j2\pi fx)$ period: $1/f$, frequency: f

$$e^{j2\pi f(x+\frac{1}{f})} = e^{j2\pi fx} e^{j2\pi}$$
$$= e^{j2\pi fx}$$

$G(f)$: the expansion coefficient for $\exp(j2\pi fx)$

$$g(x) = \int_{-\infty}^{\infty} G(f) e^{j2\pi fx} df$$

When will the Fourier transform exist?

Sufficient Conditions:

(1) $\int_{-\infty}^{\infty} \left| g(x) \right| dx < \infty$

(2) $g(x)$ is of bounded variations (It means that $g(x)$ can be represented by a curve of finite length in any finite interval of x).

4.1.2 Transform Pair

[Example 1] Find the Fourier transform of

$$g(x) = \exp\left(-3|x|\right)$$

(Solution):

$$\mathcal{F}\{g(x)\} = \int_{-\infty}^{\infty} e^{-3|x|} e^{-j2\pi fx} dx = \int_{-\infty}^{0} e^{3x} e^{-j2\pi fx} dx + \int_{0}^{\infty} e^{-3x} e^{-j2\pi fx} dx$$

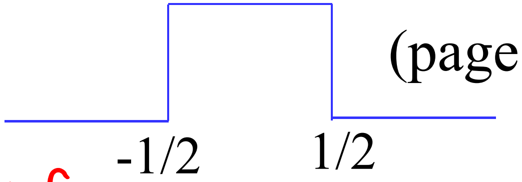
$$= \left.\frac{e^{3x} e^{-j2\pi fx}}{3 - j2\pi f}\right|_{-\infty}^{0} + \left.\frac{e^{-3x} e^{-j2\pi fx}}{-3 - j2\pi f}\right|_{0}^{\infty} = \frac{1}{3 - j2\pi f} - \frac{1}{-3 - j2\pi f}$$

$$= \frac{6}{9 + (2\pi f)^2}$$

[Example 2]

Find the Fourier transform of the rectangular function Π(*x*) where

$$\Pi(x)=\begin{cases}1 & \text{for } -1/2 < x < 1/2,\\ 0 & \text{otherwise.}\end{cases}$$



(page 339)

(Solution):

$$\frac{e^{-j\pi f}-e^{j\pi f}}{-j2\pi f}=\frac{-j2\sin(\pi f)}{-j2\pi f}$$

$$\mathscr{F}\{g(x)\}=\int_{-1/2}^{1/2}e^{-j2\pi fx}dx=\left.\frac{e^{-j2\pi fx}}{-j2\pi f}\right|_{-1/2}^{1/2}=\frac{\sin(\pi f)}{\pi f}$$

$$\mathscr{F}\{g(x)\}=\text{sinc}\, f$$ (page 338)

[Example 3]

Find the Fourier transform of the Dirac delta function $\delta(x)$

(Solution): From the sifting property of $\delta(x)$:

$$\int_{-\infty}^{\infty}\delta(x-x_0)\,y(x)\,dx = y(x_0) \qquad \text{(see page 344)}$$

$$\frac{1}{2b}\lim_{b\to 0}\int_{-b}^{b} e^{-j2\pi fx}dx = \frac{1}{2b}\int_{-b}^{b} 1\cdot dx$$

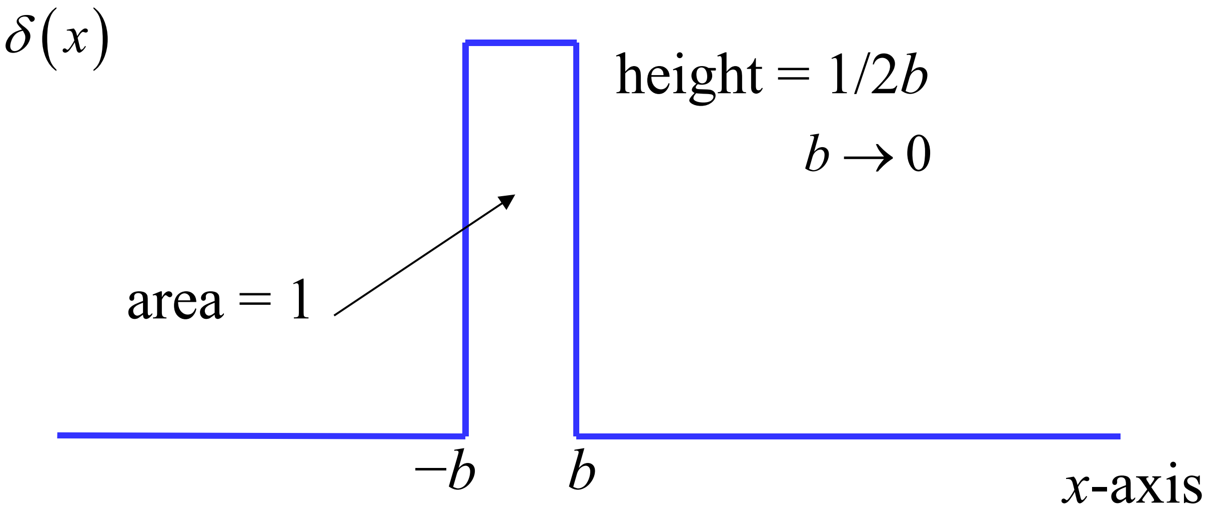
we have

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty}\delta(x)\,e^{-j2\pi fx}dx = e^{-j2\pi f\cdot 0} = 1$$

$$= \frac{2b}{2b} = 1$$

page 344 (2)

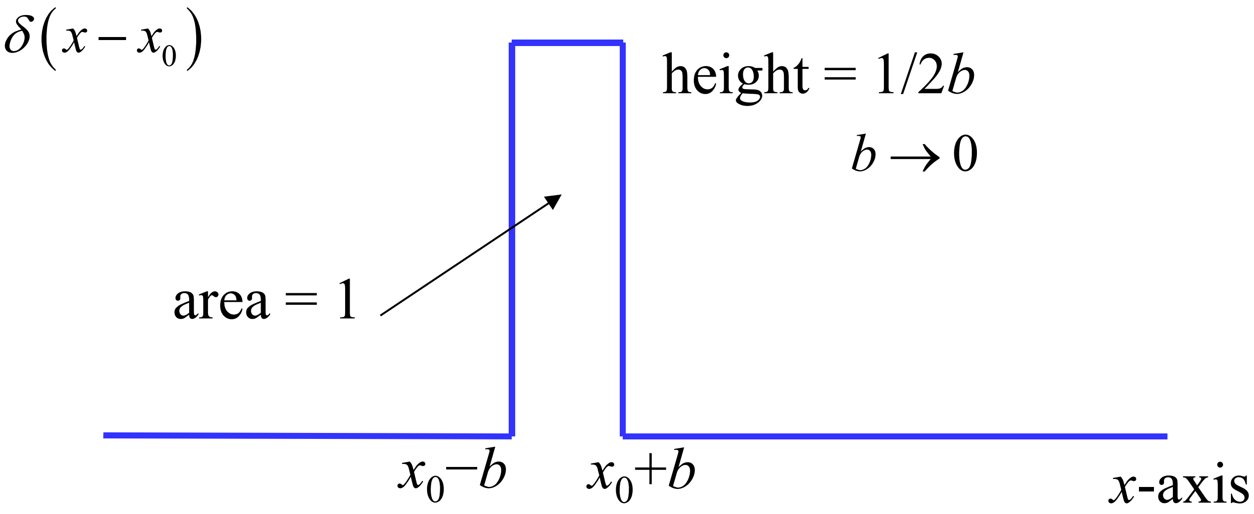
k=0



Note:

More generally,

$$\mathcal{F}\left\{\delta\left(x-x_0\right)\right\}=\int_{-\infty}^{\infty}\delta\left(x-x_0\right)e^{-j2\pi fx}dx=e^{-j2\pi x_0 f}$$



Linearity Property of the Fourier Transform

If

$$\Im\left[g_1(x)\right]=G_1(f) \qquad \Im\left[g_2(x)\right]=G_2(f)$$

then

$$\Im\left[\alpha g_1(x)+\beta g_2(x)\right]=\alpha G_1(f)+\beta G_2(f)$$

Duality Property of the Fourier Transform

If $g(x) = \sqcap(x)$, $G(f) = \text{sinc}(f)$

$\mathfrak{F}(\text{sinc}(x)) = \sqcap(-f) = \sqcap(f)$

$$\begin{tikzcd} \text{If} & \Im[g(x)] \arrow[dr] & = G(f) \arrow[dl] \\ \text{then} & \Im[G(x)] & = g(-f) \end{tikzcd}$$

(Proof): Since

$$\Im^{-1}[G(f)] = \int_{-\infty}^{\infty} G(f) e^{j2\pi fx} df = g(x)$$

$$\int_{-\infty}^{\infty} G(x) e^{j2\pi xf} dx = g(f)$$

f is replaced by x
x is replaced by f

$$\int_{-\infty}^{\infty} G(x) e^{-j2\pi fx} dx = g(-f) \implies \Im[G(x)] = g(-f)$$

Q: How do we compute the Fourier transforms of sinc(x) and 1?

[Example 4] Find the Fourier transform of $\mathrm{sinc}(x)$ where

(Solution): Since

$$\Im\left[\Pi(x)\right] = \mathrm{sinc}(f)$$

from the duality property, we have

$$\Im\left[\mathrm{sinc}(x)\right] = \Pi(-f) = \Pi(f)$$

[Example 5] Find the Fourier transform of exp($j2\pi k x$)

(Solution): Since

$$\mathcal{F}\left\{\delta\left(x-k\right)\right\}=e^{-j2\pi k f}, \qquad \mathcal{F}\left\{\delta\left(x+k\right)\right\}=e^{j2\pi k f}$$

from the duality property, we have

$$\mathcal{F}\left\{e^{j2\pi kx}\right\}=\delta\left(-f+k\right)=\delta\left(f-k\right)$$

(Here we apply the fact that $\delta(x) = \delta(-x)$).

Specially,

$$\mathfrak{I}\{1\}=\delta\left(f\right)$$

(Note): Although 1 does not satisfy the sufficient condition on page 326, its Fourier transform exists.

[Example 6] Find the Fourier transform of $\cos(2\pi k x)$

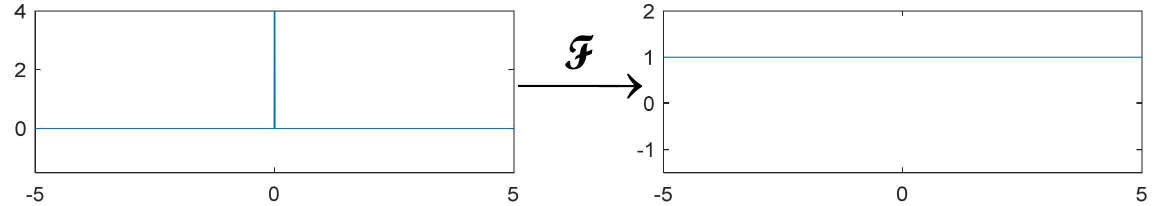
(Solution):

☆ ☆

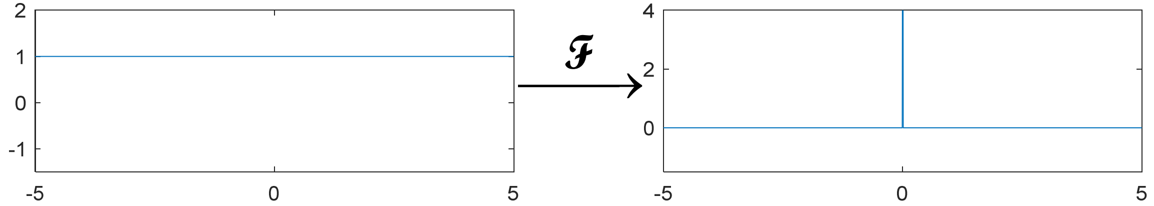
Some basic Fourier transform pairs

$g(x)$	$G(f)=\mathcal{F}\{g(x)\}$
✓ (1) $\delta(x)$	1
✓ (2) 1	$\delta(f)$ $2\pi\delta(w)$
✓ (3) $\delta(x-k)$	$\exp(-j2\pi kf)$
✓ (4) $\exp(j2\pi kx)$	$\delta(f-k)$
(5) $\cos(2\pi kx)=\frac{e^{j2\pi kx}+e^{-j2\pi kx}}{2}$	$\frac{1}{2}\delta(f-k)+\frac{1}{2}\delta(f+k)$
(6) $\sin(2\pi kx)=\frac{-je^{j2\pi kx}+je^{-j2\pi kx}}{2}$	$\frac{-j}{2}\delta(f-k)+\frac{j}{2}\delta(f+k)$
✓ (7) $\prod(x)$	$\text{sinc}(f)$
✓ (8) $\text{sinc}(x)$	$\prod(f)$
(9) $\exp(-kx)U(x)\ (k>0)$	$U(x)=\begin{cases}1 & x>0\\ 0 & x\le 0\end{cases}$ $\frac{1}{k+j2\pi f}$
(10) $\exp(-k\lvert x\rvert)\ (k>0)$	$\frac{2k}{k^2+4\pi^2f^2}$

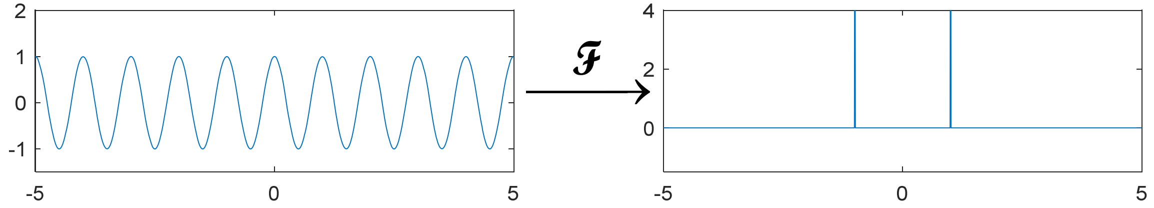
Some basic Fourier transform pairs



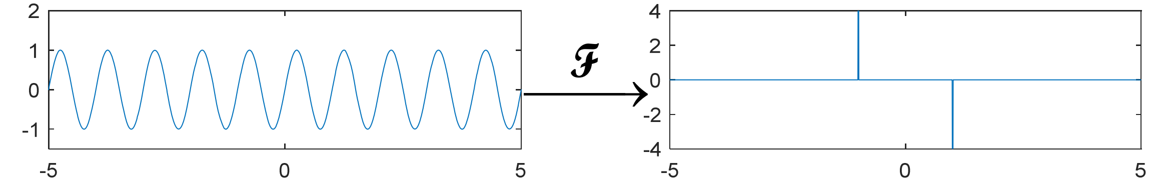
$$\delta(x) \xrightarrow{\mathcal{F}} 1$$



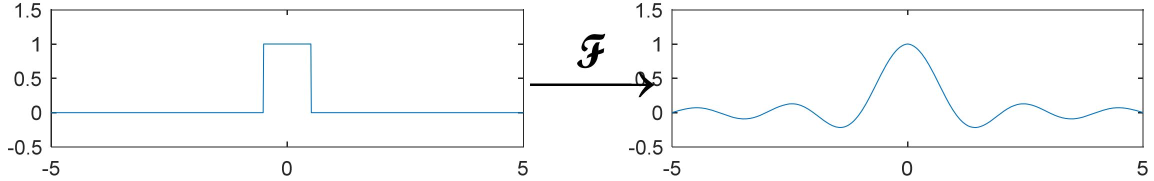
$$1 \xrightarrow{\mathcal{F}} \delta(f)$$



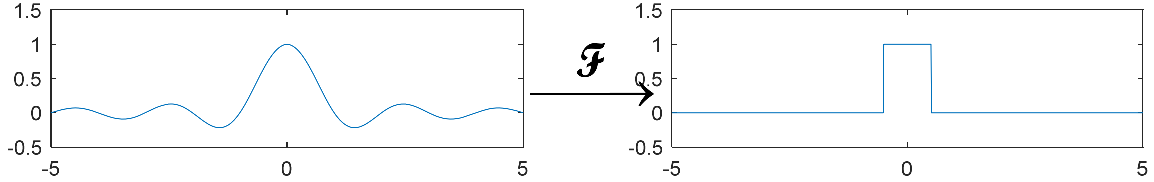
$$\cos(2\pi x) \xrightarrow{\mathcal{F}} \delta\left(f-1\right)/2 + \delta\left(f+1\right)/2$$



$$\sin(2\pi x) \xrightarrow{\mathcal{F}} -j\delta\left(f-1\right)/2 + j\delta\left(f+1\right)/2$$



$$\Pi(x) \xrightarrow{\mathcal{F}} \mathrm{sinc}(f)$$



$$\mathrm{sinc}(t) \xrightarrow{\mathcal{F}} \Pi(f)$$

附錄八 Summary of Popular Special Functions

(1) Sinc Function $\operatorname{sinc} x = \dfrac{\sin(\pi x)}{\pi x}$

$$\operatorname{sinc}(0) = \frac{\pi\cos(\pi\cdot 0)}{\pi} = 1$$

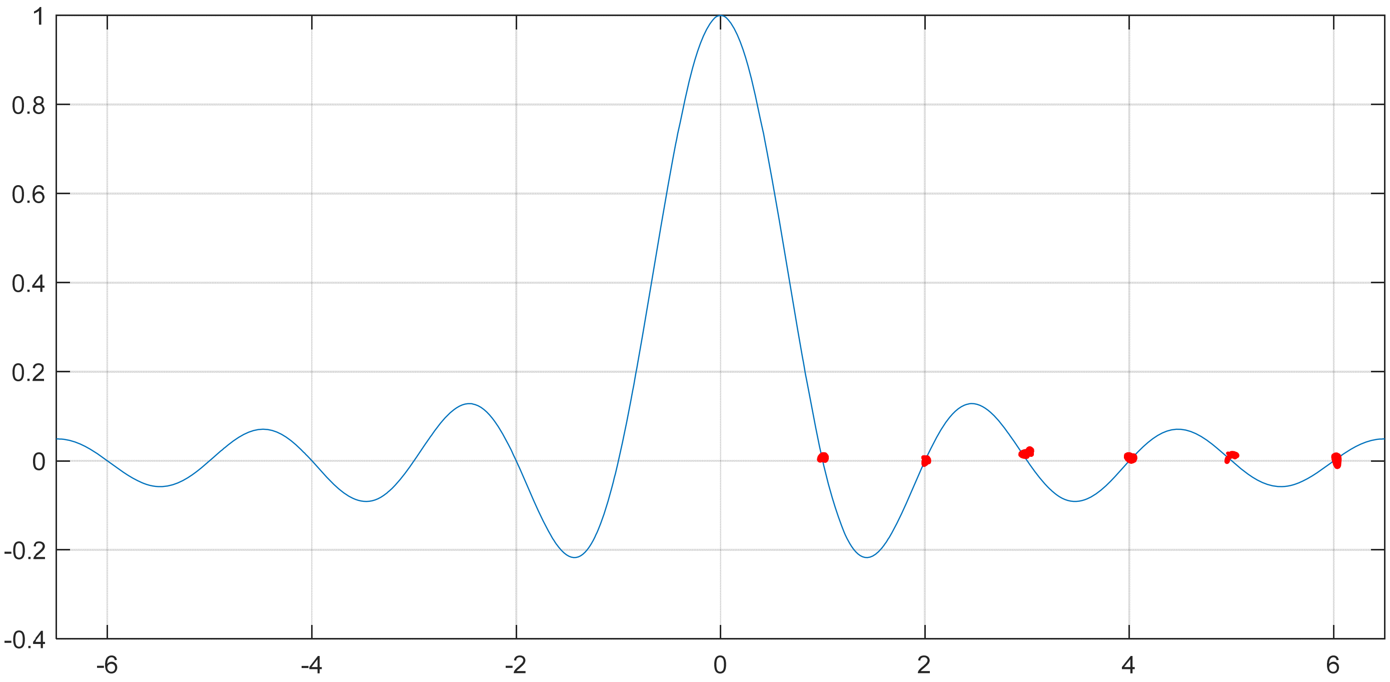
$\operatorname{sinc} 0 = 1,$ (L'hospital's rule)

$\operatorname{sinc} n = 0$ if n is a nonzero integer,

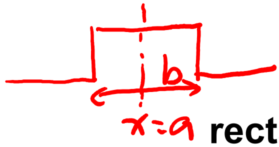
$$\operatorname{sinc}(n) = \frac{\sin(n\pi)}{n\pi} = 0 \quad \text{if } n \neq 0$$

$\operatorname{sinc} x = \operatorname{sinc}(-x)$

Applications: sampling theorem; ideal filters



$\Pi\left(\frac{x-a}{b}\right)$



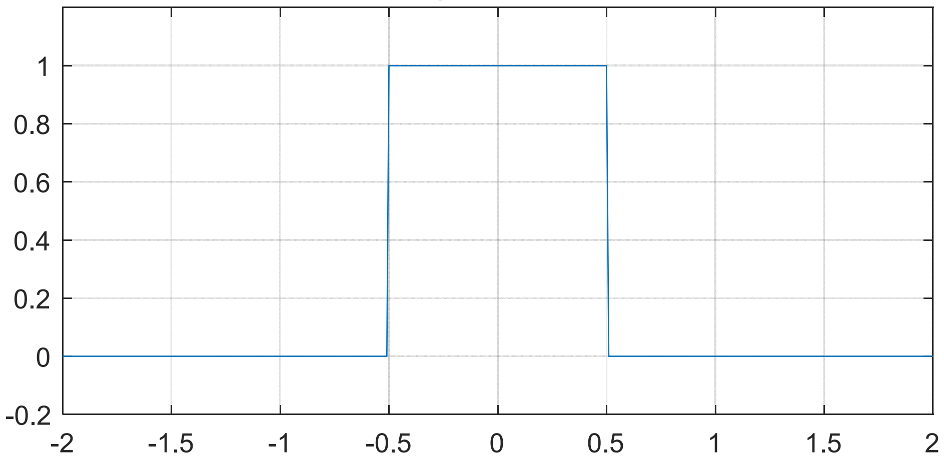
a : center
b : width

(2) Rectangular Function

$$\Pi(x)=\begin{cases}1 & for\ \ |x|<1/2\\ 0 & otherwise\end{cases}$$

In general,

$$\Pi\left(\frac{x-a}{b}\right)=\begin{cases}1 & if\ \ a-\frac{b}{2}<x<a+\frac{b}{2}\\ 0 & otherwise\end{cases}$$

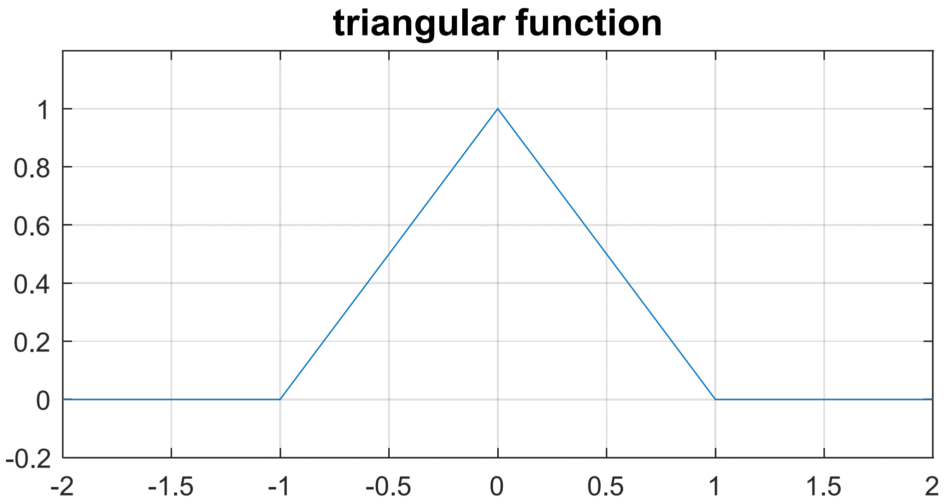


(3) Triangular Function

$$\text{tri}(x)=\begin{cases}1-|x| & for\ \ |x|<1\\ 0 & otherwise\end{cases}$$

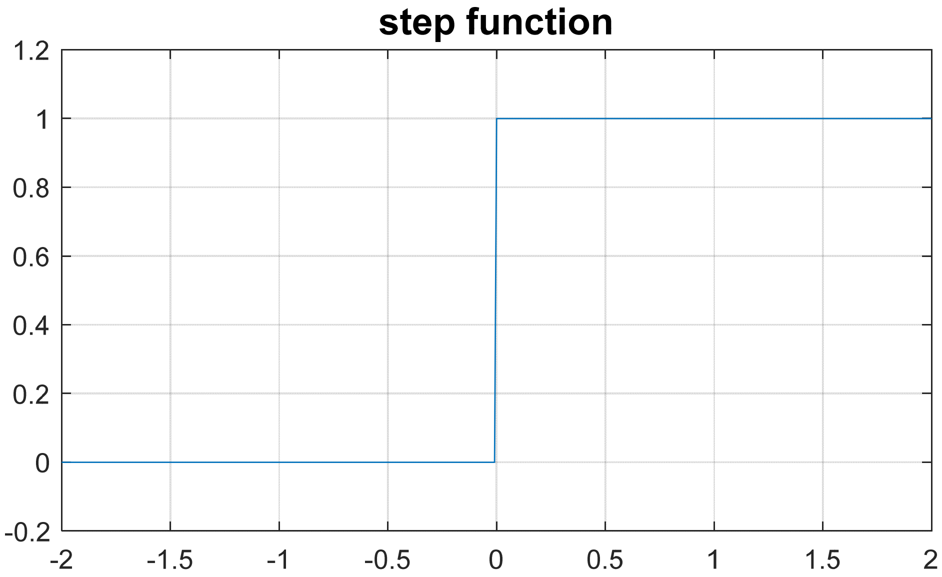
$\text{tri}(x)=\Pi(x)\underset{\uparrow}{*}\Pi(x)$

convolution



(4) Step Function

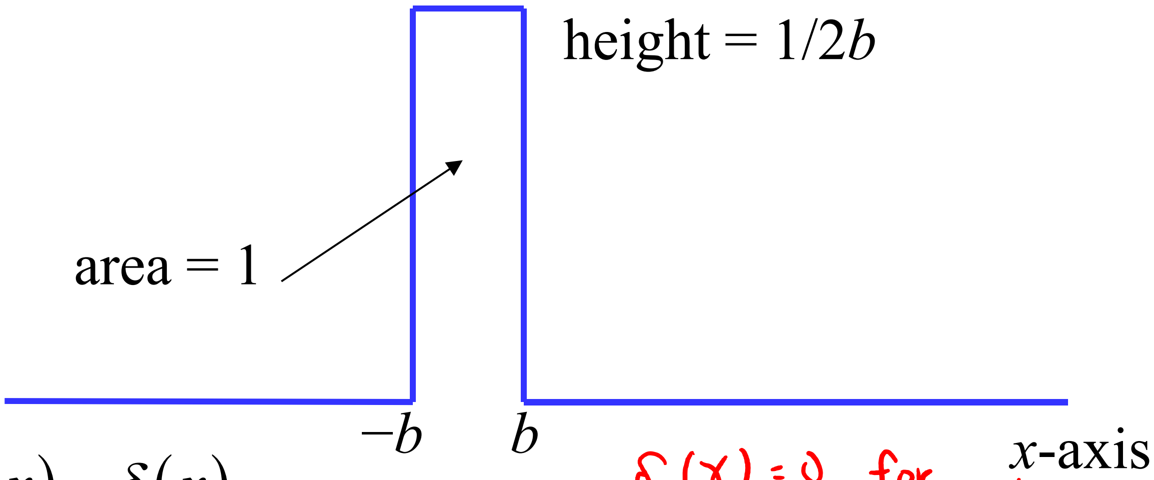
$$U(x)=\begin{cases}1 & for\ x>0\\ 0 & for\ x<0\end{cases}$$



4.2 Dirac Delta Functions

The Dirac delta function does not have a fixed definition. It is in fact the limitation of a distribution.

(1) $r_b(x)$



$$\lim_{b\to 0} r_b(x) = \delta(x)$$

$\delta(x) = 0$ for $x \neq 0$

$\delta(0) \to \infty$

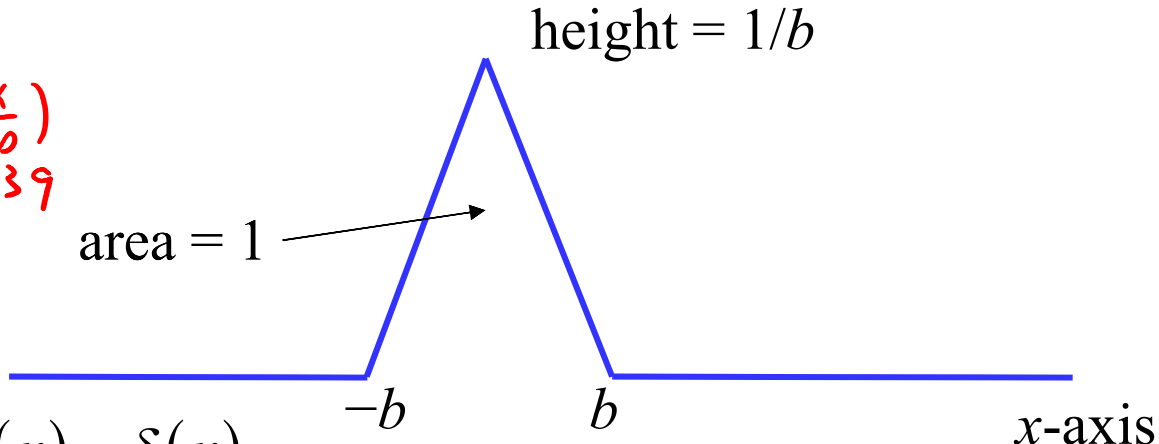
$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$

$\delta(x) = \delta(-x)$ even

(2) $tri_b(x)$

$tri_b(x) = \frac{1}{b} tri\left(\frac{x}{b}\right)$

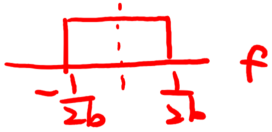
page 339



$$\lim_{b\to 0} tri_b(x) = \delta(x)$$

$sinc(x) \xrightarrow{FT} \Pi(f)$

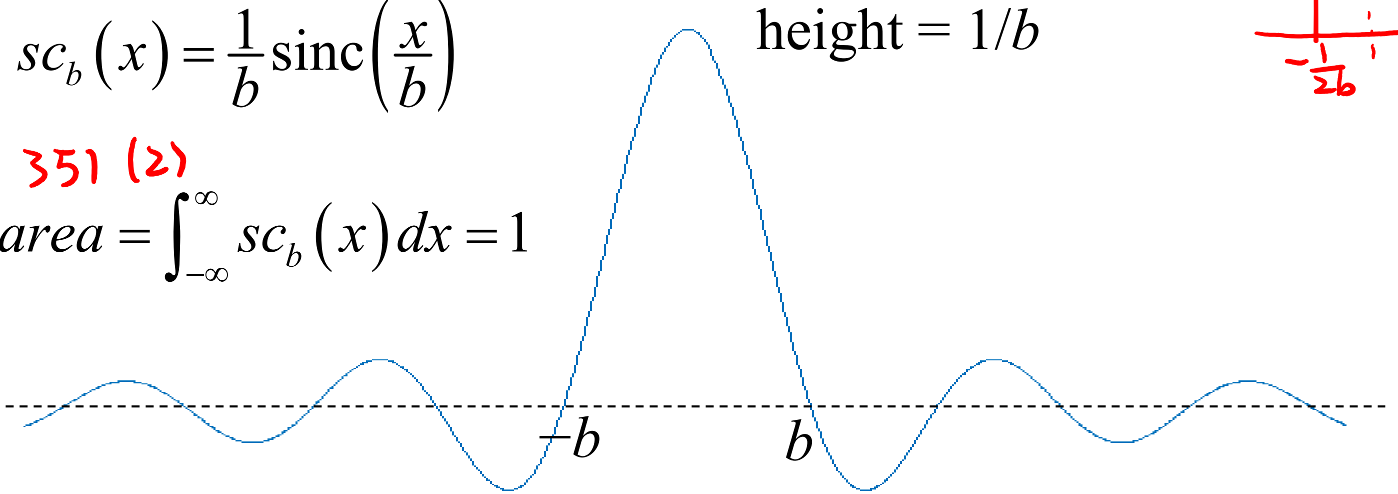
page 351 (5), $a = \frac{1}{b}$ $\frac{1}{b} sinc\left(\frac{x}{b}\right) \to \Pi(bf) = \Pi\left(\frac{f}{1/b}\right)$



(3) $sc_b(x) = \frac{1}{b}\text{sinc}\left(\frac{x}{b}\right)$

page 351 (2)

$$area = \int_{-\infty}^{\infty} sc_b(x)\,dx = 1$$



$$\lim_{b\to 0} sc_b(x) = \delta(x)$$

Definition of the Dirac delta function:

$$\left\{\begin{array}{lll} (1) & \int_{-\infty}^{\infty}\delta(x)\,dx=1 & \\ (2) & \delta(x)=0 & \text{if } x\neq 0 \\ (3) & \delta(x)=\delta(-x) & \end{array}\right.$$

Properties of the Dirac delta function:

(1) $\delta(x)=\frac{d}{dx}U(x)$ $U(x)$: unit step function

$$\delta(x-k)=\frac{d}{dx}U(x-k)$$

(2) Sifting property

$$\int_{-\infty}^{\infty}\delta(x-k)g(x)dx=g(k)$$

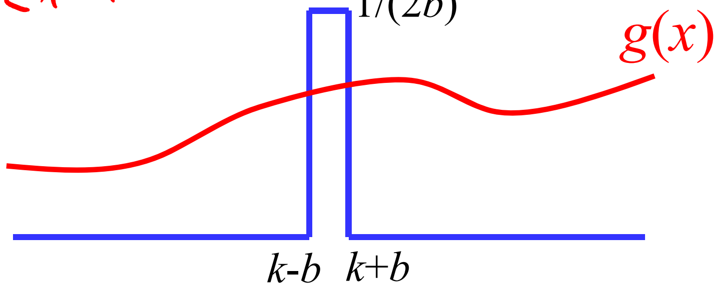
(Proof):

$$\int_{-\infty}^{\infty}\delta(x-k)g(x)dx$$

$$=\lim_{b\to 0}\frac{1}{2b}\int_{k-b}^{k+b}g(x)dx$$

$$=\lim_{b\to 0}\frac{1}{2b}g(k)\int_{k-b}^{k+b}dx=g(k)$$

$g(x)\hat{=}g(k)$ for $k-b<x<k+b$



(3) Sifting property (without integral)

$$\delta(x-k)g(\underline{x})=\delta(x-k)g(\underline{k})$$

(4) Scaling property

$\delta(x) = |a|\,\delta(ax)$

$|a|\,\delta(ax) = 0$ if $x \neq 0$

$|a|\,\delta(ax) = |a|\,\delta(-ax)$

∴ page 343 (2)(3) are satisfied

$\int_{-\infty}^{\infty} |a|\,\delta(ax)\,dx = 1$, ∴ page 343 (1) is also satisfied

$$\delta(ax)=\frac{1}{|a|}\delta(x)$$

$a \neq 0$

(to balance the integral)

$x' = ax$ $\quad dx' = a\,dx$

$dx = dx'/a$

for $a>0$ $\quad \int_{-\infty}^{\infty}\delta(ax)\,dx=\int_{-\infty}^{\infty}\delta(x')\frac{dx'}{a}=\frac{1}{a}\int_{-\infty}^{\infty}\delta(x)\,dx = \frac{1}{a} = \frac{1}{|a|}$

for $a<0$ $\quad \int_{-\infty}^{\infty}\delta(ax)\,dx=\int_{\infty}^{-\infty}\delta(x')\frac{dx'}{a}=\frac{-1}{a}\int_{-\infty}^{\infty}\delta(x)\,dx = \frac{1}{|a|}$

(5) Convolution property

$$g(x)*\delta(x)=g(x)$$

FT ↓ ↑

$G(f)\cdot 1 = G(f)$

Specially, $\quad \delta(x)*\delta(x)=\delta(x)$

(6) Integral for exponential functions

$$\int_{-\infty}^{\infty} e^{j2\pi fx} df = \delta(x)$$

It is directly from the fact that

$$\Im\{\delta(x)\} = 1, \qquad \Im^{-1}\{1\} = \delta(x)$$

(7) Generalization of the integral for exponential functions

$$\int_{-\infty}^{\infty} e^{j2\pi f\, g(x)} df = \delta(g(x))$$

← How do we define it?

$\delta(x^2+1) = 0$ for all x

since $x^2+1 \neq 0$ for all x

(8) $\delta(g(x))$

If $g(x) = 0$ only at $x = x_0$, then

$$\delta\big(g(x)\big) = \frac{\delta(x-x_0)}{|g'(x_0)|}$$

(Proof): $g(x) = g(x_0) + g'(x_0)(x-x_0) + \cdots\cdots$ remained terms

$$g\left(x\right) \cong g'(x_0)(x-x_0) \qquad \text{when} \quad x \cong x_0$$

$$\delta\big(g(x)\big) = \delta\big(g'(x_0)(x-x_0)\big) = \frac{\delta(x-x_0)}{|g'(x_0)|}$$

page 345(4)

$a = g'(x_0)$

In general, if $g(x) = 0$ only at $x = x_1, x_2, \ldots, x_N$, then

$$\delta\big(g(x)\big) = \sum_{n=1}^{N} \frac{\delta(x-x_n)}{|g'(x_n)|}$$

Ex: $\delta(x^2-1)$

$= \frac{\delta(x+1)}{2} + \frac{\delta(x-1)}{2}$

if $g(x) = x^2-1$

$g(\pm 1) = 0$

$g'(x) = 2x$

$|g'(\pm 1)| = 2$

(9) Derivative of $\delta(x)$ $\quad \delta'(x) = \frac{d}{dx}\delta(x)$

$$\underline{\delta'(x) * g(x)} = \underline{g'(x)}$$

$u = \delta(\tau)$, $v = g(\tau - \tau)$

(Proof):

$\int u'v = uv - \int uv'$

$$\delta'(x) * g(x) = \int_{-\infty}^{\infty} \delta'(\tau) g(x-\tau) d\tau$$

$$= \delta(\tau) g(x-\tau)\Big|_{\tau\to-\infty}^{\tau\to\infty} - \int_{-\infty}^{\infty} \delta(\tau) \frac{d}{d\tau} g(x-\tau) d\tau$$

page 344, $k = 0$

$$= \int_{-\infty}^{\infty} \delta(\tau) g'(x-\tau) d\tau$$ (from the sifting property)

$$= g'(x)$$

$\frac{d}{d\tau} g(x-\tau) = \left(\frac{d}{d\tau}(x-\tau)\right) \cdot g'(x-\tau)$

(10) Properties related to derivative of $\delta(x)$

(i) $\delta'(x) = -\delta'(-x)$

(ii) $\int_{-\infty}^{\infty} \delta'(x-x_0)\, g(x)\, dx = -g'(x_0)$

(iii) $\delta'(x-x_0)\, g(x) = \delta'(x-x_0)\, g(x_0) - \delta(x-x_0)\, g'(x_0)$

(Proof): Since

$$\delta(x-x_0)\, g(x) = \delta(x-x_0)\, g(x_0)$$

$$\frac{d}{dx}\delta(x-x_0)\, g(x) = \frac{d}{dx}\delta(x-x_0)\, g(x_0)$$

$$\delta'(x-x_0)\, g(x) + \delta(x-x_0)\, g'(x) = \delta'(x-x_0)\, g(x_0)$$

$$\delta'(x-x_0)\, g(x) = \delta'(x-x_0)\, g(x_0) - \delta(x-x_0)\, g'(x_0)$$

(11) Higher order derivative of $\delta(x)$

$$\underline{\delta^{(n)}(x)*g(x)=g^{(n)}(x)}$$

$$\int_{-\infty}^{\infty}\delta^{(n)}(x-x_0)g(x)dx=(-1)^n g^{(n)}(x_0)$$

$$\int_{-\infty}^{\infty}\delta^{(n)}(x-x_0)dx=0 \quad \text{when } n>0$$

$$\delta^{(n)}(x)=0 \qquad \text{when } x\neq 0$$

$$\delta^{(n)}(x)=(-1)^n\delta^{(n)}(-x)$$

4.3 Properties of the Fourier Transform

4.3.1 List of Properties

☆☆

$$G(f)=\mathscr{F}[g(x)]=\int_{-\infty}^{\infty}g(x)\exp(-j2\pi f\,x)\,dx$$

(1) Recovery (inverse Fourier transform)	$g(x)=\int_{-\infty}^{\infty}G(f)\exp(j2\pi f\,x)\,df$
(2) Integration (DC property)	$G(0)=\int_{-\infty}^{\infty}g(x)\,dx$ $\quad g(0)=\int_{-\infty}^{\infty}G(f)\,df$
(3) Modulation	$\mathscr{F}\left[g(x)e^{j2\pi f_0 x}\right]=G(f-f_0)$
(4) Time Shifting	$\mathscr{F}[g(x-x_0)]=G(f)e^{-j2\pi f\,x_0}$
(5) Scaling	$\mathscr{F}[g(ax)]=\frac{1}{\lvert a\rvert}G\left(\frac{f}{a}\right)$
(6) Time Reverse	$a=-1 \downarrow$ $\mathscr{F}[g(-x)]=G(-f)$

(7) Real / Imaginary Input	If $g(x)$ is real, then $G(f) = G^*(-f)$; If $g(x)$ is pure imaginary, then $G(f) = -G^*(-f)$
(8) Even / Odd Input	If $g(x) = g(-x)$, then $G(f) = G(-f)$; If $g(x) = -g(-x)$, then $G(f) = -G(-f)$;
(9) Conjugation	$\mathcal{F}\left[g^*(x)\right] = G^*(-f)$ $\qquad$ $\mathcal{F}\left[g^*(-x)\right] = G^*(f)$
(10) Differentiation	$\mathcal{F}\left[g'(x)\right] = j2\pi f\, G(f)$
(11) Multiplication by *x*	$\mathcal{F}\left[xg(x)\right] = \frac{j}{2\pi} G'(f)$
(12) Division by *x*	$\mathcal{F}\left[\frac{g(x)}{x}\right] = -j2\pi \int_{-\infty}^{f} G(\mu)\, d\mu$
(13) Parseval's Theorem (Energy Preservation)	$\int_{-\infty}^{\infty} \lvert g(x)\rvert^2 dx = \int_{-\infty}^{\infty} \lvert G(f)\rvert^2 df$ $g(x)\, g^*(x)$
(14) Generalized Parseval's Theorem	$\int_{-\infty}^{\infty} g(x) h^*(x) dx = \int_{-\infty}^{\infty} G(f) H^*(f) df$

(15) Linearity	$\mathcal{F}\left[ag(x)+bh(x)\right]=aG(f)+bH(f)$
(16) Convolution	If $z(x)=g(x)*h(x)=\int_{-\infty}^{\infty}g(\tau)h(x-\tau)d\tau,$ then $Z(f)=G(f)H(f)$
(17) Multiplication	If $z(x)=g(x)h(x)$, then $Z(f)=G(f)*H(f)=\int_{-\infty}^{\infty}G(\mu)H(f-\mu)d\mu$
(18) Correlation	If $z(x)=\int_{-\infty}^{\infty}g(\tau)h^{*}(\tau-x)d\tau,$ then $Z(f)=G(f)H^{*}(f)$
(19) Two Times of Fourier Transforms	$\mathcal{F}\left\{\mathcal{F}\left[g(x)\right]\right\}=g(-x)$
(20) Four Times of Fourier Transforms	$\mathcal{F}\left[\mathcal{F}\left(\mathcal{F}\left\{\mathcal{F}\left[g(x)\right]\right\}\right)\right]=g(x)$

(Proof of (2) Integration Property)

$$G(f) = \int_{-\infty}^{\infty} \exp(-j2\pi f\, x)\, g(x)\, dx$$

$$G(0) = \int_{-\infty}^{\infty} \exp(-j2\pi 0\, x)\, g(x)\, dx = \int_{-\infty}^{\infty} g(x)\, dx$$

(Proof of (5) Scaling Property)

$$\mathcal{F}[g(ax)] = \int_{-\infty}^{\infty} \exp(-j2\pi f\, x)\, g(ax)\, dx$$

$$= \int_{-\infty}^{\infty} \exp\left(-j2\pi f\, \frac{x'}{a}\right) g(x') \frac{dx'}{|a|}$$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} \exp\left(-j2\pi \frac{f}{a} x'\right) g(x')\, dx' = \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

Property (6) is a special case of Property (5) where $a = -1$.

(Proof of (7) and (9))

$$G^*(-f)=\int_{-\infty}^{\infty}\overline{\exp(j2\pi f\,x)\,g(x)}\,dx$$

$$=\int_{-\infty}^{\infty}\exp(-j2\pi f\,x)\,g^*(x)\,dx=\mathcal{F}\left[g^*(x)\right]$$ ((9) is proven)

If $g(x)$ is real, then

$$G^*(-f)=\mathcal{F}\left[g^*(x)\right]=\mathcal{F}\left[g(x)\right]=G(f)$$

(Proof of (10) Differentiation Property)

$$g(x)=\mathcal{F}^{-1}\left[G(f)\right]=\int_{-\infty}^{\infty}\exp(j2\pi f\,x)\,G(f)\,df$$

$$\frac{d}{dx}g(x)=\int_{-\infty}^{\infty}j2\pi f\exp(j2\pi f\,x)\,G(f)\,df=\mathcal{F}^{-1}\left[j2\pi f\,G(f)\right]$$

[Example 1] Determine the Fourier transform of the following signal.

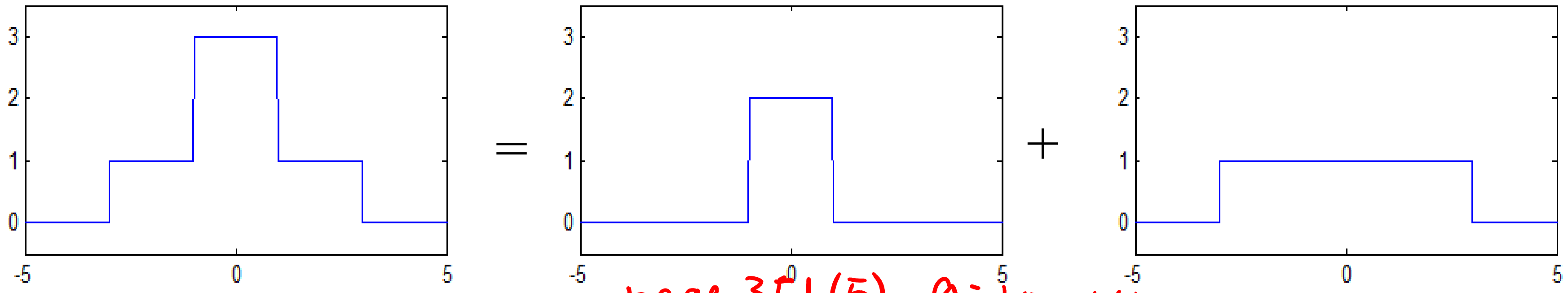
$$g(x) = 3 \quad \text{for } |x| < 1,$$

$$g(x) = 1 \quad \text{for } 1 < |x| < 3, \qquad g(x) = 0 \quad \text{for } |x| > 3$$

(Solution): Note that

$$g(x) = 2\Pi\left(\frac{x}{2}\right) + \Pi\left(\frac{x}{6}\right)$$

$g(x)$ $\qquad$ $2\Pi(x/2)$ $\qquad$ $\Pi(x/6)$



page 351 (5), a: 1/2, 1/6

Therefore, $$G(f) = 2 \cdot 2\operatorname{sinc}(2f) + 6\operatorname{sinc}(6f)$$

$$= 4\operatorname{sinc}(2f) + 6\operatorname{sinc}(6f)$$

[Example 2] Determine the Fourier transform of the following signal.

$$g(x) = x\exp(-|x|)$$

(10) k=1

(Solution): From page 336, we have

$$\mathcal{F}[\exp(-|x|)] = \frac{2}{1+4\pi^2 f^2}$$

Then, from the differentiation property $\mathcal{F}[xg(x)] = \frac{j}{2\pi}G'(f)$

page 352 (11)

$$\mathcal{F}[x\exp(-|x|)] = \frac{j}{2\pi}\frac{d}{df}\frac{2}{1+4\pi^2 f^2}$$

$$= -j\frac{8\pi f}{(1+4\pi^2 f^2)^2}$$

[Example 3] Determine the Fourier transform of the following signal.

$$g(x) = \exp\left(-3|x-1| + j6\pi x\right)$$

(Solution): Since

page 336 (10), $k=3$

$$\mathcal{F}\left[\exp\left(-3|x|\right)\right] = \frac{6}{9+4\pi^2 f^2}$$

$x_0=1$ page 351 (4)

$$\mathcal{F}\left[\exp\left(-3|x-1|\right)\right] = \frac{6}{9+4\pi^2 f^2} e^{-j2\pi f}$$

time shifting property

$$\mathcal{F}\left[\exp\left(-3|x-1| + j6\pi x\right)\right] = \frac{6}{9+4\pi^2 (f-3)^2} e^{-j2\pi (f-3)}$$

modulation property

page 351 (3)

$f_0=3$

[Example 4] Determine the Fourier transform of the following signal.

$$g(x) = \cos(6\pi x) \qquad \text{for } 0 < x < 8,$$

$$g(x) = 0 \qquad \text{otherwise}$$

from page 339

$\cos(6\pi x)$

0 8

center: 4

width: 8

$\Pi\left(\frac{x-4}{8}\right)$

(Solution): Note that

$$g(x) = \cos(6\pi x)\Pi\left(\frac{x-4}{8}\right)$$

$$= \frac{1}{2}\exp(j6\pi x)\Pi\left(\frac{x-4}{8}\right) + \frac{1}{2}\exp(-j6\pi x)\Pi\left(\frac{x-4}{8}\right)$$

Since

$$\mathcal{F}[\Pi(x)] = \text{sinc}(f)$$

$$\mathcal{F}\left[\Pi\left(\frac{x}{8}\right)\right] = 8\,\text{sinc}(8f) \qquad \text{(scaling)}$$

351(5)

$$\mathcal{F}\left[\Pi\left(\frac{x-4}{8}\right)\right] = 8e^{-j8\pi f}\,\text{sinc}(8f) \qquad \text{(time shifting)}$$

351(4) $f_0 = 4$

$$\mathcal{F}\left[\Pi\left(\frac{x-4}{8}\right)\right]=8e^{-j8\pi f}\,\mathrm{sinc}(8f)$$

351(3) $f_0=3$

$$\mathcal{F}\left[\exp(j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right]=8e^{-j8\pi(f-3)}\,\mathrm{sinc}(8(f-3))\quad\text{(modulation)}$$

$f_0=-3$

$$\mathcal{F}\left[\exp(-j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right]=8e^{-j8\pi(f+3)}\,\mathrm{sinc}(8(f+3))$$

Therefore,

$$\mathcal{F}[g(x)]$$

$$=\frac{1}{2}\mathcal{F}\left[\exp(j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right]+\frac{1}{2}\mathcal{F}\left[\exp(-j6\pi x)\Pi\left(\frac{x-4}{8}\right)\right]$$

$$=4e^{-j8\pi(f-3)}\,\mathrm{sinc}(8(f-3))+4e^{-j8\pi(f+3)}\,\mathrm{sinc}(8(f+3))$$

4.3.2 Real, Imaginary, Even, and Odd Parts

Moreover, from Properties (7), (8), (9), we can conclude that

(i) $\Im\left[\mathcal{Re}\{g(x)\}\right]=\frac{1}{2}\left(G(f)+G^*(-f)\right)$

(ii) $\Im\left[j\mathcal{Im}\{g(x)\}\right]=\frac{1}{2}\left(G(f)-G^*(-f)\right)$

(Practice to prove them)

$\mathfrak{F}(Re(g(x)) = \mathfrak{F}\left(\frac{g(x)+g^*(x)}{2}\right) = \frac{G(f)+G^*(-f)}{2}$

Also, any function can be decomposed into

(1) $$g(x) = g_e(x) + g_o(x)$$

where $$g_e(x) = \frac{1}{2}\left(g(x) + g(-x)\right)$$

$$g_o(x) = \frac{1}{2}\left(g(x) - g(-x)\right)$$

(2) $$g(x) = g_{e,r}(x) + g_{e,i}(x) + g_{o,r}(x) + g_{o,i}(x)$$

where

$$g_{e,r}(x) = \mathfrak{Re}\left\{\frac{1}{2}\left(g(x) + g(-x)\right)\right\} = \frac{g(x) + g^*(x) + g(-x) + g^*(-x)}{4}$$

$$g_{e,i}(x) = j\mathfrak{Im}\left\{\frac{1}{2}\left(g(x) + g(-x)\right)\right\}$$

$$g_{o,r}(x) = \mathfrak{Re}\left\{\frac{1}{2}\left(g(x) - g(-x)\right)\right\}$$

$$g_{o,i}(x) = j\mathfrak{Im}\left\{\frac{1}{2}\left(g(x) - g(-x)\right)\right\}$$

One can prove that

$$\mathfrak{I}\left[g_e(x)\right]=G_e(f) \qquad \mathfrak{I}\left[g_o(x)\right]=G_o(f)$$

$$\mathfrak{I}\left[g_{e,r}(x)\right]=G_{e,r}(f) \qquad \mathfrak{I}\left[g_{e,i}(x)\right]=G_{e,i}(f)$$

$$\mathfrak{I}\left[g_{o,r}(x)\right]=G_{o,i}(f) \qquad \mathfrak{I}\left[g_{o,i}(x)\right]=G_{o,r}(f)$$

note

where

$$G_e(f)=\frac{1}{2}\left(G(f)+G(-f)\right) \qquad G_o(f)=\frac{1}{2}\left(G(f)-G(-f)\right)$$

$$G_{e,r}(f)=\boldsymbol{\mathscr{R}e}\left\{\frac{1}{2}\left(G(f)+G(-f)\right)\right\}$$

$$G_{e,i}(f)=j\boldsymbol{\mathscr{I}m}\left\{\frac{1}{2}\left(G(f)+G(-f)\right)\right\}$$

$$G_{o,r}(f)=\boldsymbol{\mathscr{R}e}\left\{\frac{1}{2}\left(G(f)-G(-f)\right)\right\}$$

$$G_{o,i}(f)=j\boldsymbol{\mathscr{I}m}\left\{\frac{1}{2}\left(G(f)-G(-f)\right)\right\}$$

$$\begin{tikzcd}
g(x) = & g_{e,r}(x) + \arrow[d, "\mathcal{F}"'] & g_{e,i}(x) + \arrow[d, "\mathcal{F}"'] & g_{o,r}(x) + \arrow[dr, "\mathcal{F}"'] & g_{o,i}(x) \arrow[dl, "\mathcal{F}"] \\
G(f) = & G_{e,r}(f) + & G_{e,i}(f) + & G_{o,r}(f) + & G_{o,i}(f)
\end{tikzcd}$$

4.3.3 Parseval's Theorem

$$\int_{-\infty}^{\infty} |g(x)|^2 dx = \int_{-\infty}^{\infty} |G(f)|^2 df$$

Parseval's theorem is also called the energy preservation property, Rayleigh's Theorem, or Plancheral's Theorem.

$|g(x)|$: amplitude

$|g(x)|^2$: power

$\int |g(x)|^2 dx$: energy

(Proof):

$$\int_{-\infty}^{\infty} G(f)G^*(f)df = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}dx \overline{\int_{-\infty}^{\infty} g(\tau)e^{-j2\pi f\tau}d\tau} df$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)g^*(\tau)\left[\int_{-\infty}^{\infty} e^{j2\pi f(\tau-x)}df\right]dxd\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)g^*(\tau)\delta(\tau-x)d\tau dx$$ (from page 346)

$$= \int_{-\infty}^{\infty} g(x)g^*(x)dx$$ (from the sifting property)

$$= \int_{-\infty}^{\infty} |g(x)|^2 dx$$

Generalized Parseval's Theorem

$$\int_{-\infty}^{\infty} g(x)h^*(x)\,dx = \int_{-\infty}^{\infty} G(f)H^*(f)\,df$$

It also called the power theorem.

[Example 5] Determine the following integral:

$$\int_{-\infty}^{\infty} \operatorname{sinc}^2(x)\,dx = \int |\Pi(f)|^2 df = \int_{-1/2}^{1/2} 1\, df = 1$$

[Example 6] Determine the following integral:

$$\int_{-\infty}^{\infty} \cos(8\pi x)\operatorname{sinc}(3x)\,dx$$

$$\int_{-\infty}^{\infty} \cos(2\pi x)\operatorname{sinc}(3x)dx$$

$$= \frac{1}{3}\cos(2\pi x)$$

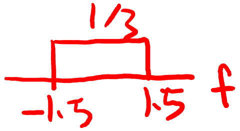
(Solution): Since

page 336 (5) $k=4$

$$\mathcal{F}[\cos(8\pi x)] = \frac{1}{2}\left(\delta(f-4)+\delta(f+4)\right)$$

$$\mathcal{F}[\operatorname{sinc}(3x)] = \frac{1}{3}\Pi\left(\frac{f}{3}\right)$$

page 351 (5)
$a=3$



$$\int_{-\infty}^{\infty} \cos(8\pi x)\operatorname{sinc}(3x)\,dx = \int_{-\infty}^{\infty} \frac{1}{2}\left(\delta(f-4)+\delta(f+4)\right)\frac{1}{3}\Pi\left(\frac{f}{3}\right)df$$

$$= \frac{1}{6}\int_{-3/2}^{3/2}\left(\delta(f-4)+\delta(f+4)\right)df = 0$$

測不準

4.4 Uncertainty Principles

4.4.1 Uncertainty Principles from Different Views

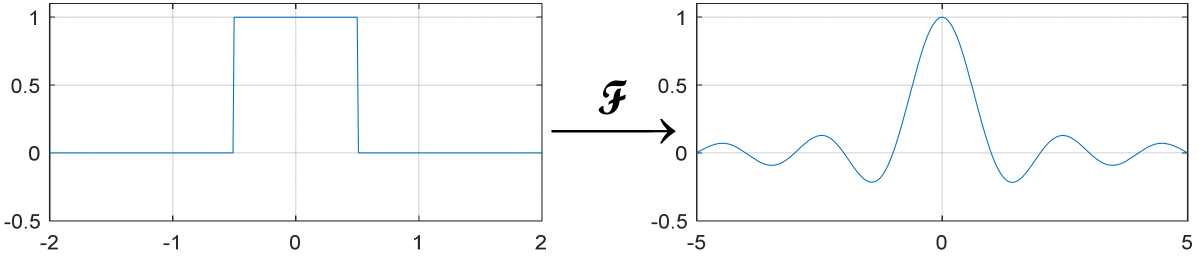
(1) From the Point of View of the Scaling Property

$$\mathcal{F}\left[g(ax)\right]=\frac{1}{|a|}G\left(\frac{f}{a}\right)$$

wide in the time domain → narrow in the frequency domain

narrow in the time domain → wide in the frequency domain

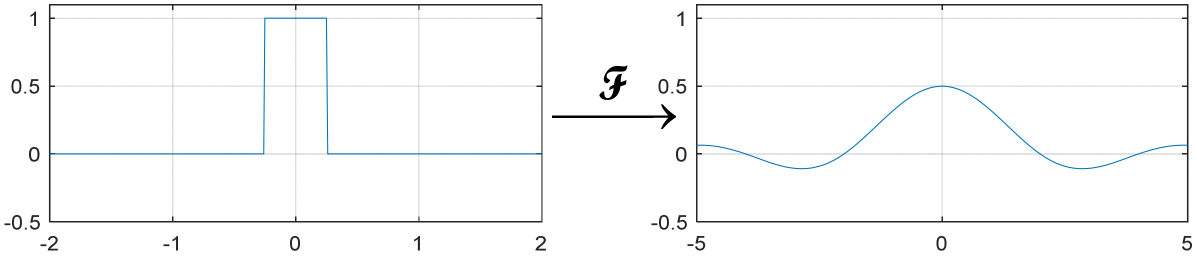
$\Pi(x)$



$\mathrm{sinc}(f)$

$\Pi(2x)$

a=2



$\frac{1}{2}\mathrm{sinc}\left(\frac{f}{2}\right)$

(2) From the Point of View of Equivalent Width

Equivalent Width in the Time Domain:

$$W_g = \frac{\int_{-\infty}^{\infty} g(x)\,dx}{g(0)}$$

from the integration property $W_g = \dfrac{G(0)}{g(0)}$

Equivalent Width in the Frequency Domain:

$$W_G = \frac{\int_{-\infty}^{\infty} G(f)\,df}{G(0)} = \frac{g(0)}{G(0)}$$

Product of the Two Equivalent Widths:

$$W_g W_G = \frac{G(0)}{g(0)}\frac{g(0)}{G(0)} = 1$$

1927

(3) Heisenberg's Uncertainty Principle

For a signal $g(x)$, if $\sqrt{x}\, g(x) = 0$ when $|x| \longrightarrow \infty$, then

$$\boxed{\sigma_x \sigma_f \geq 1/4\pi}$$

$\sigma_x = \sqrt{\text{variance}}$ = standard deviation

where $\sigma_x^2 = \int (x-\mu_x)^2 P_g(x)\,dx$, $\quad \sigma_f^2 = \int (f-\mu_f)^2 P_G(f)\,df$,

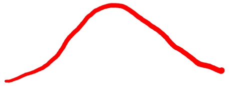
variance

$\mu_x = \int x\, P_g(x)\,dx$, $\quad \mu_f = \int f\, P_G(f)\,df$

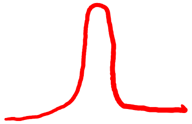
mean

$$P_g(x) = \frac{|g(x)|^2}{\int |g(x)|^2\,dx}, \qquad P_G(f) = \frac{|G(f)|^2}{\int |G(f)|^2\,df},$$

probability = $\dfrac{\text{power at } x}{\text{energy}}$



large standard deviation



small standard deviation

(Proof of Henseinberg's uncertainty principle):

For simplification, we consider the case where $\mu_x = \mu_f = 0$

Then, use Parseval's theorem

$$\sigma_x^2 \sigma_f^2 = \frac{\int x^2 |g(x)|^2 \, dx}{\int |g(x)|^2 \, dx} \frac{\int f^2 |G(f)|^2 \, df}{\int |G(f)|^2 \, df}$$

$$= \frac{1}{4\pi^2} \frac{\int x^2 |g(x)|^2 \, dx}{\int |g(x)|^2 \, dx} \frac{\int |g'(x)|^2 \, dx}{\int |g(x)|^2 \, dx}$$

Here, we apply the fact that

$$\int |g(x)|^2 \, dx = \int |G(f)|^2 \, df$$

$$\int |g'(x)|^2 \, dx = 4\pi^2 \int f^2 |G(f)|^2 \, df$$

$$\mathcal{F}\left[g'(x)\right] = j2\pi f \, G(f)$$

From Schwarz's inequality $\langle g(x), g(x)\rangle\langle h(x), h(x)\rangle \geq \left|\langle g(x), h(x)\rangle\right|^2$

$$\int x^2 \,|\, g(x)\,|^2 \, dx \int |\, g'(x)\,|^2 \, dx \geq \left(\left| \int x g^*(x) \frac{d}{dx} g(x) dx \right|^2 + \left| \int x g(x) \frac{d}{dx} g^*(x) dx \right|^2 \right) / 2$$

$$\geq \left| \int \left(x g^*(x) \frac{d}{dx} g(x) + x g(x) \frac{d}{dx} g^*(x) \right) dx \right|^2 / 4$$ (using $|a+b|^2 + |a-b|^2 \geq 2|a|^2$)

$$= \left| \int x \frac{d}{dx} \left[g(x) g^*(x) \right] dx \right|^2 / 4 = \left| x g(x) g^*(x) \Big|_{-\infty}^{\infty} - \int g^*(x) g(x) \, dx \right|^2 / 4$$

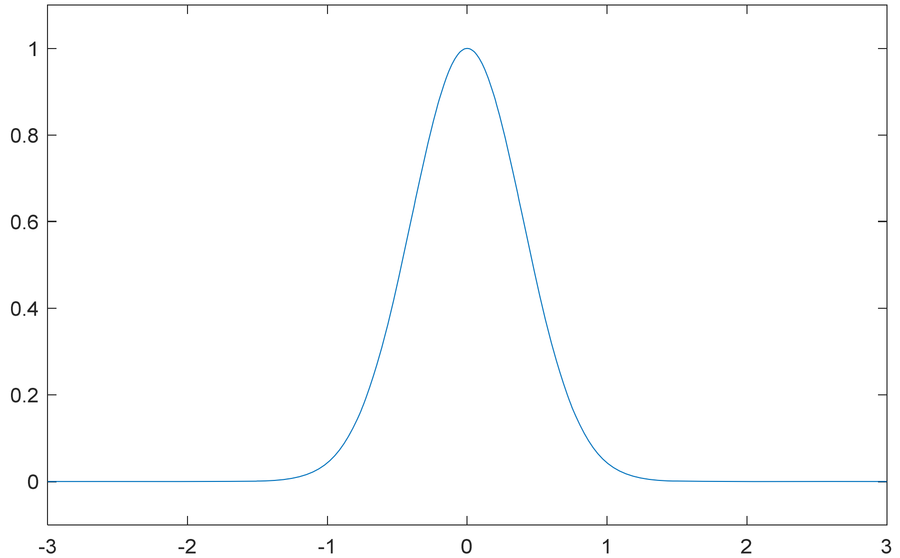
$$= \left| \left[x g(x) g^*(x) \Big|_{x \to \infty} - x g(x) g^*(x) \Big|_{x \to -\infty} \right] - \int g^*(x) g(x) \, dx \right|^2 / 4$$

$$= \left| \int |g(x)|^2 \, dx \right|^2 / 4$$

$$\sigma_t^2 \sigma_f^2 \geq \frac{1}{16\pi^2} \Longrightarrow \sigma_t \sigma_f \geq \frac{1}{4\pi}$$

4.4.2 Gaussian and Hermite-Gaussian Functions

Gaussian function: $\exp\left(-\pi x^2\right)$



$Ax = \lambda x$

The Gaussian function is an eigenfunction of the Fourier transform with eigenvalue = 1:

★ $$\Im\left[\exp\left(-\pi x^2\right)\right] = \exp\left(-\pi f^2\right)$$

$$\Im\left[\exp\left(-\pi x^2\right)\right] = \exp\left(-\pi f^2\right)$$

(Proof): From the fact that

$$\int_{-\infty}^{\infty} e^{-(at^2+bt)}dt = \sqrt{\pi / a}\cdot e^{b^2/4a}$$

M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 3rd Ed., 2009.

we have

$$\mathcal{F}\left\{e^{-\pi x^2}\right\} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-j2\pi fx} dx = \sqrt{\frac{\pi}{\pi}} e^{\frac{(j2\pi f)^2}{4\pi}} = e^{-\pi f^2}$$

The Gaussian function is not the only eigenfunction of the Fourier transform.

$e^{-\pi C x^2}$ also satisfies $\sigma_x \sigma_f = \frac{1}{4\pi}$

The Gaussian function satisfies the lower bound of Heisenberg's uncertainty principle.

$$\mathcal{F}\left[e^{-\pi x^2}\right]=e^{-\pi f^2}$$

$$\int_{-\infty}^{\infty}\left|g(x)\right|^2 dx=\int_{-\infty}^{\infty}e^{-2\pi x^2}dx=\sqrt{1/2}$$

use $\int_{-\infty}^{\infty}e^{-(at^2+bt)}dt=\sqrt{\pi/a}\cdot e^{b^2/4a}$

$$\int_{-\infty}^{\infty}x^2\left|g(x)\right|^2 dx=\int_{-\infty}^{\infty}x^2e^{-2\pi x^2}dx=2\int_{0}^{\infty}x^2e^{-2\pi x^2}dt=$$

$$=\frac{2\Gamma[3/2]}{2(2\pi)^{3/2}}=\frac{\sqrt{\pi}/2}{(2\pi)^{3/2}}=\frac{1}{4\sqrt{2}\pi}$$

use $\int_{0}^{\infty}x^m e^{-ax^2}dx=\frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$

$\Gamma(1/2)=\sqrt{\pi}$ $\qquad$ $\Gamma(n+1)=n\Gamma(n)$

$$\mu_x=\frac{\int_{-\infty}^{\infty}x\left|g(x)\right|^2 dx}{\int_{-\infty}^{\infty}\left|g(x)\right|^2 dx}=0 \qquad \sigma_x^2=\frac{\int_{-\infty}^{\infty}x^2\left|g(x)\right|^2 dx}{\int_{-\infty}^{\infty}\left|g(x)\right|^2 dx}=\frac{1}{4\pi}$$

$$\sigma_x^2 = \frac{1}{4\pi} \qquad \sigma_x = \sqrt{\frac{1}{4\pi}}$$

Since $G(f) = g(f)$,

$$\sigma_f = \sqrt{\frac{1}{4\pi}}$$

Therefore,

$$\sigma_x \sigma_f = \frac{1}{4\pi} \quad \text{if} \quad g(x) = e^{-\pi x^2}$$

Note: Other Hermite Gaussian functions do not satisfy the lower bound of Heisenberg's uncertainty principle.

$$\Im\left[\overset{\nearrow\ \text{eigenfunctions}}{H_n\left(\sqrt{2\pi}\,x\right)\exp\left(-\pi x^2\right)}\right]=\underset{\hookrightarrow\ \text{eigenvalues}}{\underline{\underline{(-j)^n}}}\exp\left(-\pi f^2\right)H_n\left(\sqrt{2\pi}\,f\right)$$

$H_n(x)$: The Hermite polynomial of order n (see page 318).

$$H_0(x)=1 \qquad H_1(x)=2x$$

$$H_2(x)=4x^2-2 \qquad H_3(x)=8x^3-12x$$

$$H_4(x)=16x^4-48x^2+12 \qquad H_5(x)=32x^5-160x^3+120x$$

附錄八 Convolution

Convolution: $g(x)*h(x)=\int_{-\infty}^{\infty} g(x-\tau)h(\tau)d\tau=\int_{-\infty}^{\infty} g(\tau)h(x-\tau)d\tau$

Specially, if $g(x) = 0$ for $x < 0$ and $h(x) = 0$ for $x < 0$, then

Convolution (causal form):

$$g(x)*h(x)=\int_{0}^{\infty} g(x-\tau)h(\tau)d\tau=\int_{0}^{\infty} g(\tau)h(x-\tau)d\tau$$

Physical meaning: The effect of the input on the output is determined by their time difference.

$$\underset{\substack{\uparrow\\ \text{output}}}{y(x)}=\int_{0}^{\infty} \underset{\substack{\uparrow\\ \text{input}}}{g(\tau)}\,\underset{\substack{\nwarrow\\ \text{effect of } g(\tau) \text{ on } y(x)}}{h(x-\tau)}\,d\tau$$

$$\underset{\substack{\uparrow\\ \text{output}}}{y(x)} = \int_0^\infty \underset{\substack{\uparrow\\ \text{input}}}{g(\tau)}\, \underset{\substack{\uparrow\\ \text{effect of } g(\tau) \text{ on } y(x)}}{h(x-\tau)}\, d\tau$$

$$y(x) = g(0)h(x)\Delta + g(\Delta)h(x-\Delta)\Delta + g(2\Delta)h(x-2\Delta)\Delta$$
$$+\cdots g(x)h(0)\Delta$$

Any linear time-invariant system can be expressed as the convolution form.

$$z(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(\tau) h(x-\tau) d\tau$$

[Support Theorem]:

If the support of $g(x)$ is $x \in [x_1, x_2]$

(i.e., $g(x) = 0$ for $x < x_1$ and $x > x_2$)

and the support of $h(x)$ is $x \in [x_3, x_4]$,

then the support of $z(x)$ is

$$x \in [x_1 + x_3,\ x_2 + x_4]$$

(Proof):

$$z(x) = \int_{-\infty}^{\infty} g(\tau) h(x-\tau) d\tau = \int_{x_1}^{x_2} g(\tau) h(x-\tau) d\tau$$

$h(x-\tau) \neq 0$ when $x - \tau \in [x_3, x_4]$,

$$x \in [x_3 + \min(\tau), x_4 + \max(\tau)] = [x_1 + x_3,\ x_2 + x_4]$$

附錄九 Change of Independent Variables for Integrals

$$\iint \cdots\cdots\cdots dxdy = \iint \cdots\cdots\cdots C^{-1} dwdv$$

where $C = \det\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det\begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial v} \end{bmatrix}^{-1}$, $C^{-1} = \det\begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial v} \end{bmatrix}$

For the indefinite integral case

$$C = \left|\det\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}\right| = \left|\det\begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial v} \end{bmatrix}\right|^{-1}$$

4.5 Convolution and Correlation

4.5.1 Convolution Property

If $z(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(\tau) h(x-\tau) d\tau,$

then $Z(f) = G(f) H(f)$

convolution ⟹ multiplication

(Proof): If $z(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(\tau) h(x-\tau) d\tau$,

$$\mathcal{F}^{-1}\left[G(f)H(f)\right] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-j2\pi f\tau} g(\tau) d\tau \int_{-\infty}^{\infty} e^{-j2\pi ft} h(t) dt\right] e^{j2\pi fx} df$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi f\tau} g(\tau) e^{-j2\pi ft} h(t) e^{j2\pi fx} d\tau dt df$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-j2\pi f\tau} e^{-j2\pi ft} e^{j2\pi fx} df\right] g(\tau) h(t) d\tau dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-\tau-t) g(\tau) h(t) d\tau dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} \delta(t+\tau-x) h(t) dt\right] g(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} h(x-\tau) g(\tau) d\tau = z(x)$$

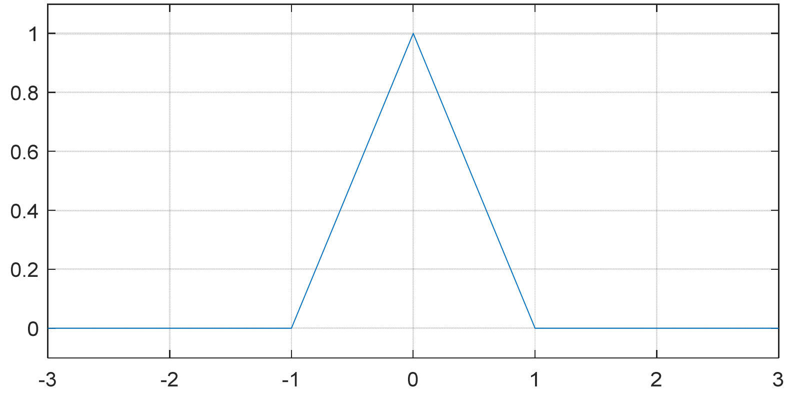
Therefore,

$$\mathcal{F}\left[z(x)\right] = G(f)H(f) \qquad\qquad Z(f) = G(f)H(f)$$

[Example 1] Determine the Fourier transform of

$$\Lambda(x)=\begin{cases} x+1 & for\ -1\le x\le 0 \\ 1-x & for\ 0\le x\le 1 \\ 0 & otherwise \end{cases}$$

We call it the triangular function.



(Solution):

Note that $\Lambda(x)=\Pi(x)*\Pi(x)$

$$\Pi(x)*\Pi(x)=\int_{-\infty}^{\infty}\Pi(\tau)\Pi(x-\tau)d\tau=\int_{-1/2}^{1/2}\Pi(x-\tau)d\tau$$

$$\Pi(x-\tau)=1 \quad for\ -\frac{1}{2}<x-\tau<\frac{1}{2},\ i.e.,\ x-\frac{1}{2}<\tau<x+\frac{1}{2}$$

When $x-1/2>1/2$ or $x+1/2<-1/2$, i.e., $x>1$ or $x<-1$,

$$\int_{-1/2}^{1/2}\Pi(x-\tau)d\tau=0$$

When $-1<x<0$ $\quad \int_{-1/2}^{1/2}\Pi(x-\tau)d\tau=\int_{-1/2}^{x+1/2}1\cdot d\tau=x+1$

When $0<x<1$ $\quad \int_{-1/2}^{1/2}\Pi(x-\tau)d\tau=\int_{x-1/2}^{1/2}1\cdot d\tau=1-x$

Therefore,

$$\begin{aligned}\mathcal{F}\left[\Lambda(x)\right] &= \mathcal{F}\left[\Pi(x) * \Pi(x)\right] \\ &= \mathcal{F}\left[\Pi(x)\right]\mathcal{F}\left[\Pi(x)\right] = \operatorname{sinc}^2 x\end{aligned}$$

[Example 2] Determine the inverse Fourier transform of

$$G(f)=\frac{1}{1+\pi^2 f^2}\frac{1}{j\pi f+1}$$

(Solution): Note that

$$G(f)=2\frac{2\cdot 2}{4+4\pi^2 f^2}\frac{1}{j2\pi f+2}$$

From page 336 (10) $k=2$ (9) $k=2$

$$\mathcal{F}^{-1}\left[\frac{2\cdot 2}{4+4\pi^2 f^2}\right]=\exp(-2|x|) \qquad \mathcal{F}^{-1}\left[\frac{1}{j2\pi f+2}\right]=\exp(-2x)U(x)$$

$$g(x)=2\exp(-2|x|)*\exp(-2x)U(x)$$

$$=2\int_{-\infty}^{\infty}\exp(-2|\tau|)\exp(-2x+2\tau)U(x-\tau)\,d\tau$$

$x-\tau>0$

$\tau<x$

$$=2\int_{-\infty}^{x}\exp(-2|\tau|)\exp(-2x+2\tau)\,d\tau$$

$$g(x) = 2\int_{-\infty}^{x} \exp(-2|\tau|)\exp(-2x+2\tau)\,d\tau$$

When $x \le 0$, since $\tau \in (-\infty, x]$, $\tau \le 0$ is always satisfied,

$$\exp(-2|\tau|) = \exp(2\tau)$$

$$g(x) = 2\exp(-2x)\int_{-\infty}^{x} \exp(4\tau)\,d\tau = \frac{\exp(2x)}{2}$$

When $x > 0$,

$$g(x) = 2\exp(-2x)\left[\int_{-\infty}^{0} \exp(4\tau)\,d\tau + \int_{0}^{x} d\tau\right]$$

$$= \exp(-2x)\left[\frac{1}{2} + 2x\right]$$

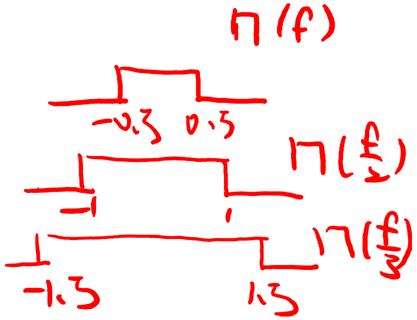
[Example 3] Determine

$$\operatorname{sinc}(t) * \operatorname{sinc}(t) = \operatorname{sinc}(t)$$

FT ↓ $\qquad$ ↑ IFT

$$\Pi(f)\,\Pi(f) = \Pi(f)$$

$$\Pi(f)\Pi\left(\frac{f}{2}\right)\Pi\left(\frac{f}{3}\right) = \Pi(f)$$



[Example 4] Determine

$$\operatorname{sinc}(t) * \operatorname{sinc}(2t) * \operatorname{sinc}(3t) \qquad \frac{1}{6}\operatorname{sinc}(t)$$

page 351 (5)
a = 2, 3

FT ↓ $\qquad$ ↑ IFT

$$\Pi(f)\ \frac{1}{2}\Pi\left(\frac{f}{2}\right)\ \frac{1}{3}\Pi\left(\frac{f}{3}\right) = \frac{1}{6}\Pi(f)$$

[Example 5] Determine

$$\delta(t) * \delta(t) = \delta(t)$$

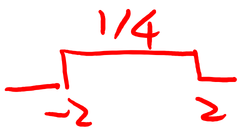
FT ↓ $\qquad$ ↑ IFT

$$1 \cdot 1 = 1$$

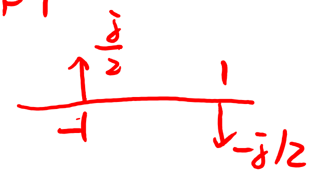
[Example 6] Determine

$$\operatorname{sinc}(4t) * \sin(2\pi t) = \frac{1}{4}\sin(2\pi t)$$

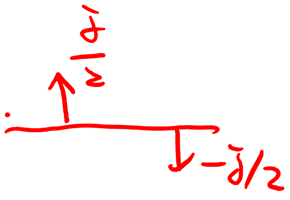
↓ FT $\qquad$ IFT ↑



$$\frac{1}{4}\Pi\left(\frac{f}{4}\right) \cdot$$



$$= \frac{1}{4} \cdot$$



[Theorem 4.5.1]

$$g(x)*\delta(x-x_0)=g(x-x_0)$$

Specially,

$$g(x)*\delta(x)=g(x)$$

(Proof):

$$\mathcal{F}[g(x)*\delta(x-x_0)]=\mathcal{F}[g(x)]\mathcal{F}[\delta(x-x_0)]$$

$$=G(f)\exp(-j2\pi x_0 f)$$ 336(3)

$$\mathcal{F}^{-1}\{\mathcal{F}[g(x)*\delta(x-x_0)]\}=\mathcal{F}^{-1}[G(f)\exp(-j2\pi x_0 f)]$$

$$g(x)*\delta(x-x_0)=g(x-x_0)$$

(from the time-shifting property)

page 351 (4)

4.5.2 Multiplication Property

If $z(x) = g(x)h(x)$

then $Z(f) = G(f) * H(f) = \int_{-\infty}^{\infty} G(s)H(f-s)\,ds$

multiplication ⟹ convolution

(Proof): $\mathcal{F}\left[z(x)\right]=\mathcal{F}\left[g(x)h(x)\right]$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}e^{j2\pi sx}G(s)ds\int_{-\infty}^{\infty}e^{j2\pi ux}H(u)du\right]e^{-j2\pi fx}dx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{j2\pi sx}G(s)e^{j2\pi ux}H(u)e^{-j2\pi fx}dsdudx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}e^{j2\pi sx}e^{j2\pi ux}e^{-j2\pi fx}dx\right]G(s)H(u)dsdu$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(s+u-f)G(s)H(u)dsdu$$

$$=\int_{-\infty}^{\infty}G(s)H(f-s)ds$$

Therefore,

$$Z(f)=G(f)*H(f)=\int_{-\infty}^{\infty}G(s)H(f-s)ds$$

[Example 7] Determine the Fourier transform of

$$g(x) = \cos(4\pi x)\operatorname{rect}\left(\frac{x}{6}\right)$$

(Solution):

$$\mathcal{F}\left\{\operatorname{rect}\left(\frac{x}{6}\right)\right\} = 6\operatorname{sinc}(6f)$$

$$\mathcal{F}\left\{\cos(4\pi x)\right\} = \frac{1}{2}\left[\delta(f-2) + \delta(f+2)\right]$$

Therefore,

$$G(f) = 6\operatorname{sinc}(6f) * \frac{1}{2}\left[\delta(f-2) + \delta(f+2)\right]$$

$$G(f) = 3\operatorname{sinc}\left(6(f-2)\right) + 3\operatorname{sinc}\left(6(f+2)\right)$$

4.5.3 Correlation

Correlation

$$z(x) = corr\big(g(x), h(x)\big) = \int_{-\infty}^{\infty} g(\tau + x) h^*(\tau) d\tau$$

Auto-Correlation

$$a_g(x) = corr\big(g(x), g(x)\big) = \int_{-\infty}^{\infty} g(\tau + x) g^*(\tau) d\tau$$

Applications: Matched filter, communication, pattern recognition, signal detection ……

[Theorem 4.5.2] In fact, correlation is equivalent to convolution with the conjugate + time reverse of a signal.

$$\underline{corr\left(g(x),h(x)\right)} = \underline{g(x)*h^*(-x)}$$

(Proof):

$$g(x)*h^*(-x) = \int_{-\infty}^{\infty} g(x-\tau)h_1(\tau)d\tau \quad \text{where} \quad h_1(x) = h^*(-x)$$

$$= \int_{-\infty}^{\infty} g(x-\tau)h^*(-\tau)d\tau$$

$$= -\int_{\infty}^{-\infty} g(x+\tau)h^*(\tau)d\tau \qquad \tau_{new} = -\tau_{old}$$

$$= \int_{-\infty}^{\infty} g(x+\tau)h^*(\tau)d\tau$$

$$= corr\left(g(x),h(x)\right)$$

Since

$$corr\left(g\left(x\right),h\left(x\right)\right)=g\left(x\right)*h^{*}\left(-x\right)$$

we have

$$\mathcal{F}\left[corr\left(g\left(x\right),h\left(x\right)\right)\right]=G\left(f\right)H^{*}\left(f\right)$$

Specially, if $a_g(x)$ is the auto-correlation of $g(x)$:

$$a_{g}\left(x\right)=corr\left(g\left(x\right),g\left(x\right)\right)$$

then

$$\mathcal{F}\left[a_{g}\left(x\right)\right]=\left|G\left(f\right)\right|^{2}$$

4.6 Two-Dimensional Fourier Transform

4.6.1 Rectangular Coordinate

Two Dimensional Fourier Transform

$$\Im_{2D}\left[g(x,y)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x,y)e^{-j2\pi fx}e^{-j2\pi hy}dxdy=G(f,h)$$

Two Dimensional Inverse Fourier Transform

$$\Im_{2D}^{-1}\left[G(f,h)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(f,h)e^{j2\pi fx}e^{j2\pi hy}dfdh=g(x,y)$$

Possible Applications: Image processing, optics, electromagnet wave propagation analysis, ….

[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

Physical meaning: Express a signal by a linear combination of

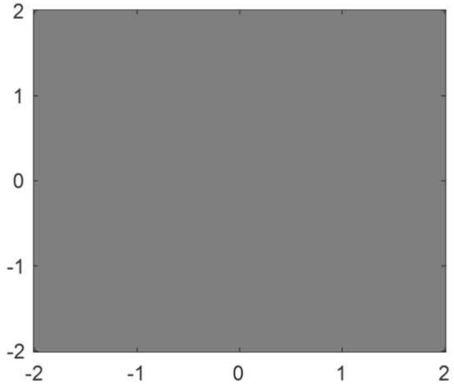
$$e^{j2\pi fx}e^{j2\pi hy}$$

f is the number of periods per unit of x

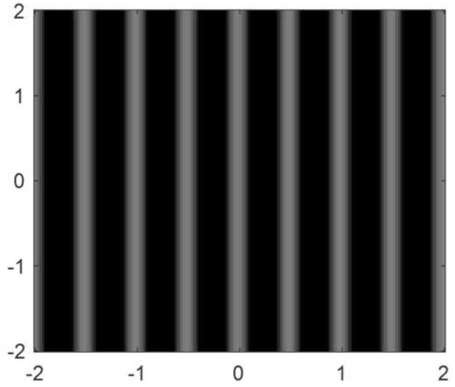
h is the number of periods per unit of y

real part of $e^{j2\pi fx}e^{j2\pi hy}$

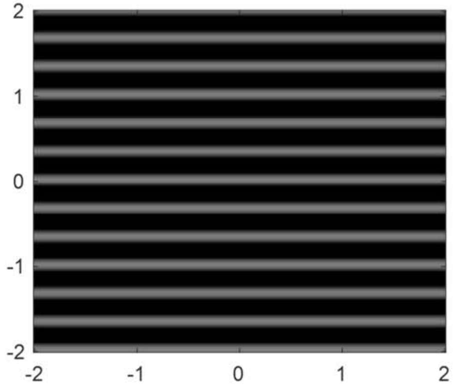
Bright colors mean higher values and dark colors mean lower values.



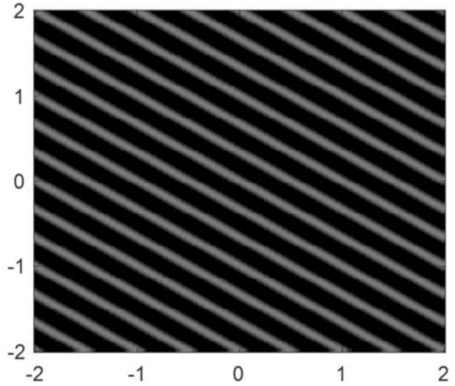
(a) $f = h = 0$



(b) $f = 2$, $h = 0$



(c) $f = 0$, $h = 3$



(d) $f = 2$, $h = 3$

[**Example 1**] Find the 2D Fourier transform of

$$g(x,y) = \mathrm{sinc}(x)\mathrm{sinc}(y)$$

(Solution):

$$G(f,h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi fx} e^{-j2\pi hy}\,\mathrm{sinc}(x)\mathrm{sinc}(y)\,dxdy$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-j2\pi fx}\,\mathrm{sinc}(x)\,dx\right] e^{-j2\pi hy}\,\mathrm{sinc}(y)\,dy$$

$$= \int_{-\infty}^{\infty} \Pi(f) e^{-j2\pi hy}\,\mathrm{sinc}(y)\,dy$$

$$= \Pi(f)\int_{-\infty}^{\infty} e^{-j2\pi hy}\,\mathrm{sinc}(y)\,dy = \Pi(f)\Pi(h)$$

[**Example 2**] Find the 2D Fourier transform of

$$g(x,y)=\Pi(x) \qquad \Pi(x)\cdot 1$$

(Solution): $G(f,h)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-j2\pi fx}e^{-j2\pi hy}\Pi(x)\,dxdy$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}e^{-j2\pi fx}\Pi(x)\,dx\right]e^{-j2\pi hy}dy=\mathrm{sinc}(f)\int_{-\infty}^{\infty}e^{-j2\pi hy}dy$$

$$=\mathrm{sinc}(f)\,\delta(h)$$

[**Example 3**] Find the 2D Fourier transform of

$$g(x,y)=\sin(2\pi(x+2y))$$

$$G(f,h)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin(2\pi(x+2y))\,e^{-j2\pi fx}\,e^{-j2\pi hy}dxdy$$

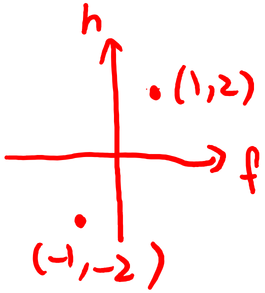
$$=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\sin(2\pi(x+2y))e^{-j2\pi fx}dx\;e^{-j2\pi hy}dy\right.$$

$$=\int_{-\infty}^{\infty}e^{j4\pi fy}\left(\frac{-j}{2}\delta(f-1)+\frac{j}{2}\delta(f+1)\right)e^{-j2\pi hy}dy$$

(page 336(6))

$$=\delta(h-2f)\left(\frac{-j}{2}\delta(f-1)+\frac{j}{2}\delta(f+1)\right)=\frac{-j}{2}\delta(h-2)\delta(f-1)+\frac{j}{2}\delta(h+2)\delta(f+1)$$

page 336 (4) page 345



4.6.2 Circular Coordinate Conversion

$$g(x,y) \Longrightarrow g(r,\theta) \qquad x = r\cos\theta, \quad y = r\sin\theta \qquad r:\sqrt{x^2+y^2}$$

$$G(f,h) \Longrightarrow G(s,\phi) \qquad f = s\cos\phi, \quad h = s\sin\phi \qquad s:\sqrt{f^2+h^2}$$

$$G(f,h) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi fx} e^{-j2\pi hy} g(x,y)\,dxdy$$

$$\Big\Downarrow \qquad\qquad\qquad\qquad \Big\Downarrow$$

$$G(s,\phi) = \int_{0}^{\infty}\int_{0}^{2\pi} e^{-j2\pi sr\cos\phi\cos\theta} e^{-j2\pi sr\sin\phi\sin\theta} g(r,\theta) C^{-1} d\theta dr$$

where $C^{-1} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r$

(page 381)

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$G(s,\phi) = \int_{0}^{\infty}\int_{0}^{2\pi} e^{-j2\pi sr\cos(\phi-\theta)} g(r,\theta) r d\theta dr$$

$$G(s,\phi)=\int_0^\infty\int_0^{2\pi} e^{-j2\pi sr\cos(\phi-\theta)}g(r,\theta)rd\theta dr$$

Specially, if *g*(*r*, *θ*) is independent of *θ*

$$g(r,\theta)=g(r)$$

$$G(s,\phi)=\int_0^\infty\left[\int_0^{2\pi} e^{-j2\pi sr\cos(\phi-\theta)}d\theta\right]rg(r)dr$$

From the fact that

$$(\varphi=\phi-\theta)$$

$$\underline{\int_0^{2\pi} e^{-jx\cos(\phi-\theta)}d\theta}=-\int_\phi^{\phi-2\pi} e^{-jx\cos\varphi}d\varphi=\int_{\phi-2\pi}^{\phi} e^{-jx\cos\varphi}d\varphi$$

$$=\int_0^{2\pi} e^{-jx\cos(\varphi)}d\varphi=\underline{2\pi J_0(x)}$$

Bessel function of the 1st kind of zero order, see pages 201-203

($\cos\varphi$ has a period of 2π)

We have $$G(s,\phi)=2\pi\int_0^\infty J_0(2\pi sr)rg(r)dr$$

(Note that it is independent of ϕ)

$$G(s)=2\pi\int_0^\infty J_0(2\pi sr)rg(r)dr$$

Hankel Transform

$$G(s) = 2\pi \int_0^\infty J_0(2\pi sr) r g(r) dr$$

It is in fact the 2D Fourier transform for a rotationally symmetric signal.

$$g(r,\theta) = g(r)$$

Inverse Hankel Transform

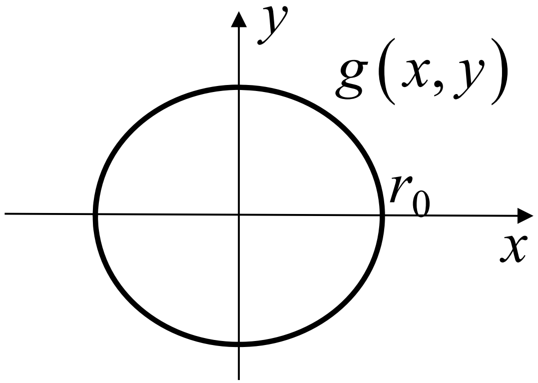
$$g(r) = 2\pi \int_0^\infty J_0(2\pi sr) s G(s) ds$$

It has the same form as the forward transform.

Several Hankel Transform Pairs

(i) If $g(r) = \delta(r - r_0)$

then $G(s) = 2\pi r_0 J_0(2\pi r_0 s)$



Note that

$$2\pi \int_0^\infty J_0(2\pi s r) r \delta(r - r_0) dr$$

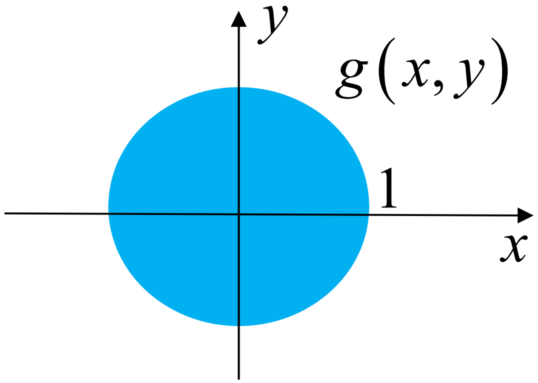
$$= 2\pi J_0(2\pi s r_0) r_0$$

(from page 344(2))

$k = r_0$

(ii) If $g(r) = \text{circ}(r)$

similar to $\Pi(x)$



$$\text{circ}(r) = \begin{cases} 1 & \textit{for } r < 1 \\ 0 & \textit{for } r > 1 \end{cases}$$

then $G(s) = \dfrac{J_1(2\pi s)}{s}$

Bessel function of the 1st kind of 1st order,
See pages 201-203.

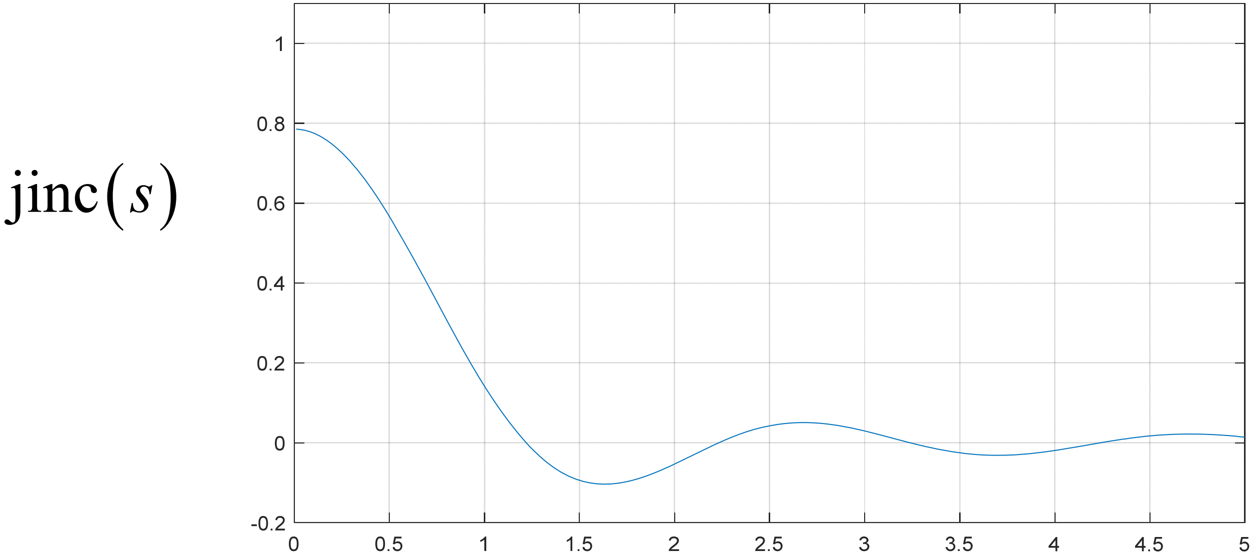
sinc

Jinc function (also called the Besinc function).

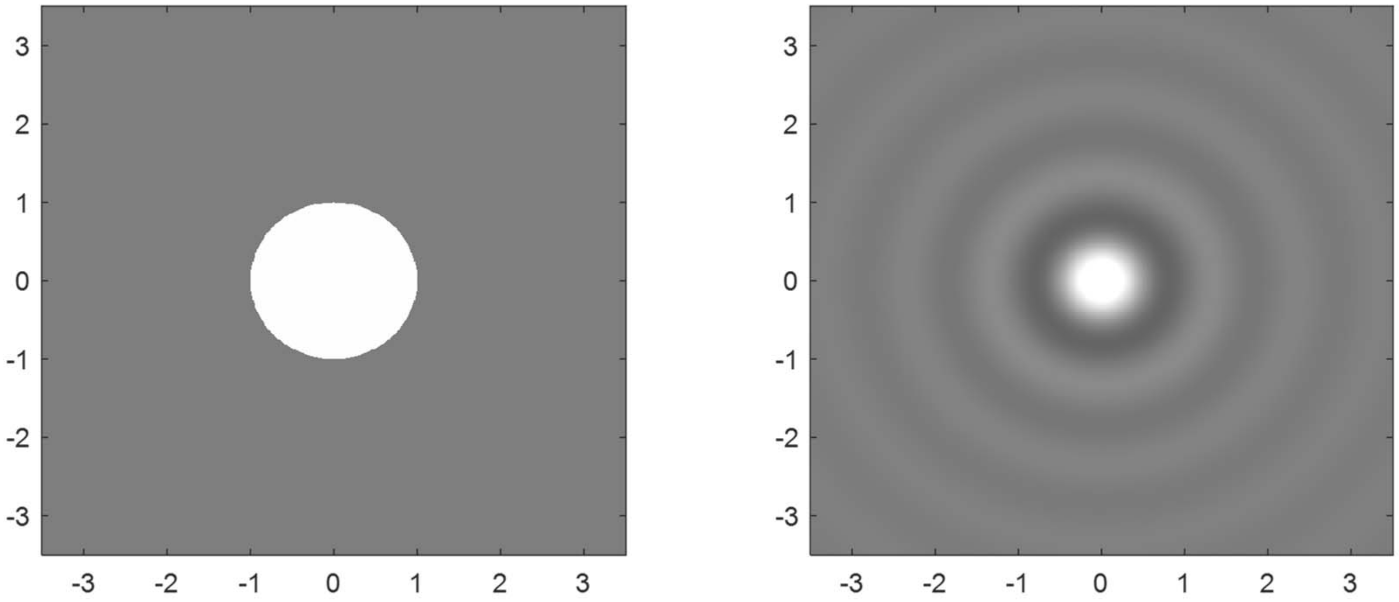
$$\mathrm{jinc}(s)=\frac{J_1(\pi s)}{2s}$$

$$\text{If}\quad g(r)=\mathrm{circ}(r)\xrightarrow[\text{2D FT}]{\text{Hankel}}G(s)=\frac{J_1(2\pi s)}{s}=4\,\mathrm{jinc}(2s)$$

It plays a similar role as the sinc function.



$$g(r)=\mathrm{circ}(r) \xrightarrow{\text{Hankel transform}} G(s)=4\,\mathrm{jinc}(2s)$$



4.7 The Operations Closely Related to the Fourier Transform (只教不考)

(1) Two-Sided Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t)dt$$

Note that when

$$s = j2\pi f$$

it is reduced to the Fourier transform. When

$$s = \sigma + j2\pi f$$

it is equivalent to the Fourier transform of exp($-\sigma t$)$f(t)$.

$$\mathcal{L}\{f(t)\}_{s=\sigma+j2\pi f} = \int_{-\infty}^{\infty} e^{-j2\pi ft} e^{-\sigma t} f(t)dt$$

$$\mathcal{L}\{f(t)\}_{s=\sigma+j2\pi f} = \Im\left[e^{-\sigma t} f(t)\right]$$

(2) One-Sided Laplace Transform

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

When

$$s = \sigma + j2\pi f$$

it is equivalent to the Fourier transform of exp($-\sigma t$)$f(t)U(t)$.

$$\mathcal{L}\{f(t)\}_{s=\sigma+j2\pi f} = \Im\left[e^{-\sigma t} f(t) U(t)\right]$$

$U(t)$: unit step function

It has less physical meaning, but the probability that the transform exists is higher. It is suitable for solving the initial value problem.

FT $\mathfrak{I}\left[g(x)\right]=\int_{-\infty}^{\infty}g(x)e^{-j2\pi fx}dx=G(f)$

inverse FT $\mathfrak{I}^{-1}\left[G(f)\right]=\int_{-\infty}^{\infty}G(f)e^{j2\pi fx}df=g(x)$

(3) Fourier Cosine Transform

When $g(x)$ is even, the FT is reduced to the Fourier cosine transform.

$$\mathfrak{I}_c\left[g(x)\right]=\int_0^{\infty}g(x)\cos(2\pi fx)dx=G_c(f)$$

$$\mathfrak{I}_c^{-1}\left[G_c(f)\right]=4\int_0^{\infty}G_c(f)\cos(2\pi fx)df=g(x)$$

(4) Fourier Sine Transform

When $g(x)$ is odd, the FT is reduced to the Fourier sine transform.

$$\mathfrak{I}_s\left[g(x)\right]=\int_0^{\infty}g(x)\sin(2\pi fx)dx=G_s(f)$$

$$\mathfrak{I}_s^{-1}\left[G_s(f)\right]=4\int_0^{\infty}G_s(f)\sin(2\pi fx)df=g(x)$$

(5) Hartley Transform

$$\Im_{ha}\left[g(x)\right]=\int_{-\infty}^{\infty}g(x)\mathrm{cas}(2\pi fx)\,dx=G_{ha}(f)$$

where cas(x) = cos(x) + sin(x)

$$\Im_{ha}^{-1}\left[G_{ha}(f)\right]=\int_{-\infty}^{\infty}G_{ha}(f)\mathrm{cas}(2\pi fx)\,dx=g(x)$$

real input → real output

(6) Mellin Transform

$$\Im_M\left[g(x)\right]=\int_0^{\infty} g(x)x^{s-1}dx=G_M(s)$$

If we set x = exp(-t), $\dfrac{dx}{dt}=-x,\quad dt=-x^{-1}dx$, then

$$G_M(s)=\int_{-\infty}^{\infty} g\left(e^{-t}\right)e^{-st}dt$$

It is the two-sided Laplace transform of $g\left(e^{-t}\right)$

It is suitable to deal with the fractal (碎形), since if

$$g(cx)=\lambda g(x)$$

then

$$g\left(e^{-t+\ln c}\right)=\lambda g\left(e^{-t}\right)$$

(7) Hilbert Transform

$$g_H(x) = \mathfrak{I}^{-1}\left\{\mathfrak{I}\left[g(x)\right]H(f)\right\}$$

where

$$H(f) = \begin{cases} -j & if\ f > 0 \\ 0 & if\ f = 0 \\ j & if\ f < 0 \end{cases}$$

Note that $\mathcal{F}^{-1}\left[H(f)\right] = j\int_{-\infty}^{0} e^{j2\pi fx} df - j\int_{0}^{\infty} e^{j2\pi fx} df$

$$= \lim_{\sigma \to 0}\left[j\int_{-\infty}^{0} e^{\sigma f} e^{j2\pi fx} df - j\int_{0}^{\infty} e^{-\sigma f} e^{j2\pi fx} df \right]$$

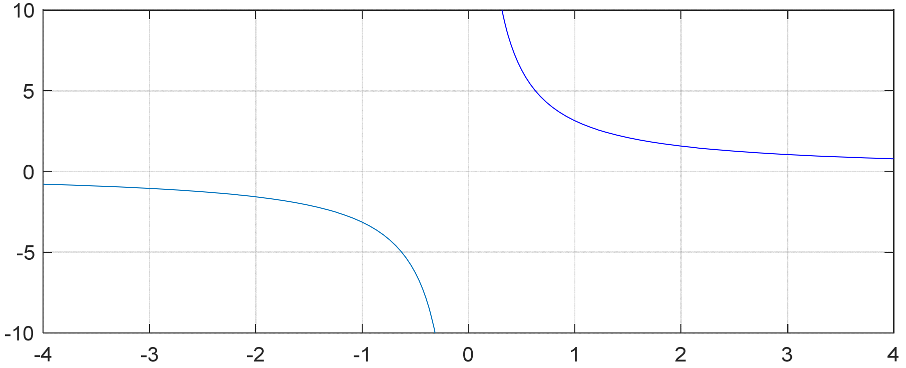
$$= \lim_{\sigma \to 0}\left[j\left.\frac{e^{\sigma f} e^{j2\pi xf}}{\sigma + j2\pi x}\right|_{-\infty}^{0} - j\left.\frac{e^{-\sigma f} e^{j2\pi xf}}{-\sigma + j2\pi x}\right|_{0}^{\infty} \right]$$

$$= \lim_{\sigma \to 0}\left[j\frac{1}{\sigma + j2\pi x} + j\frac{1}{-\sigma + j2\pi x} \right] = \lim_{\sigma \to 0}\frac{4\pi x}{\sigma^2 + 4\pi^2 x^2} = \frac{1}{\pi x}$$

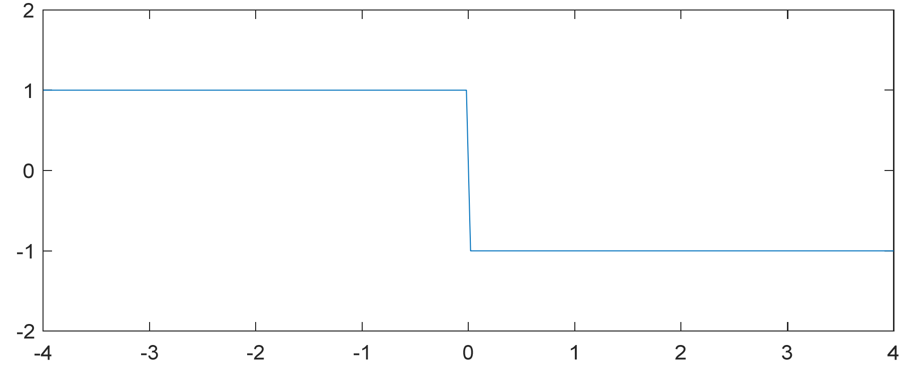
Therefore,

$$g_H\left(x\right)=g\left(x\right)*\overset{\overset{h(x)}{\swarrow}}{\frac{1}{\pi x}}=\int_{-\infty}^{\infty}\frac{g\left(\tau\right)}{\pi(x-\tau)}d\tau$$

$h(x)$



$\mathcal{Im}\left[H\left(f\right)\right]$



$$g(x)=\cos(2\pi kx) \xrightarrow{\text{Hilbert}} g_H(x)=\sin(2\pi kx)$$

$$k\neq 0$$

$$g(x)=\sin(2\pi kx) \xrightarrow{\text{Hilbert}} g_H(x)=-\cos(2\pi kx)$$

$$k\neq 0$$

(Proof): If $g(x)=\cos(2\pi kx)$

then $G(f)=\frac{1}{2}\delta(f-k)+\frac{1}{2}\delta(f+k)$

$$H(f)G(f)=\frac{-j}{2}\delta(f-k)+\frac{j}{2}\delta(f+k)$$

$$g_H(x)=\Im^{-1}\{G(f)H(f)\}=\sin(2\pi kx)$$

(8) Analytic Signal

reconstruction:

$$g_a(x) = g(x) + jg_H(x)$$

$$g(x) = \mathcal{Re}\{g_a(x)\}$$

if $g(x)$ is real

Since

$$\mathcal{F}[g_a(x)] = \mathcal{F}[g(x)] + j\mathcal{F}[g_H(x)]$$

$$G_a(f) = G(f) + jH(f)G(f) = (1 + jH(f))G(f)$$

$$1 + jH(f) = \begin{cases} 2 & if\ f > 0 \\ 1 & if\ f = 0 \\ 0 & if\ f < 0 \end{cases}$$

we have

$$G_a(f) = \begin{cases} 2G(f) & if\ f > 0 \\ G(f) & if\ f = 0 \\ 0 & if\ f < 0 \end{cases}$$

(It is called the 'single sided band signal')

(halve the bandwidth)

(9) Fractional Fourier Transform

$$X_\phi(u)=\sqrt{1-j\cot\phi}\; e^{j\pi\cot\phi\cdot u^2}\int_{-\infty}^{\infty} e^{-j2\pi\cdot\csc\phi\cdot u\,t}\, e^{j\pi\cdot\cot\phi\cdot t^2}\, x(t)dt$$

When $\phi = 0.5\pi$, the FRFT becomes the FT.

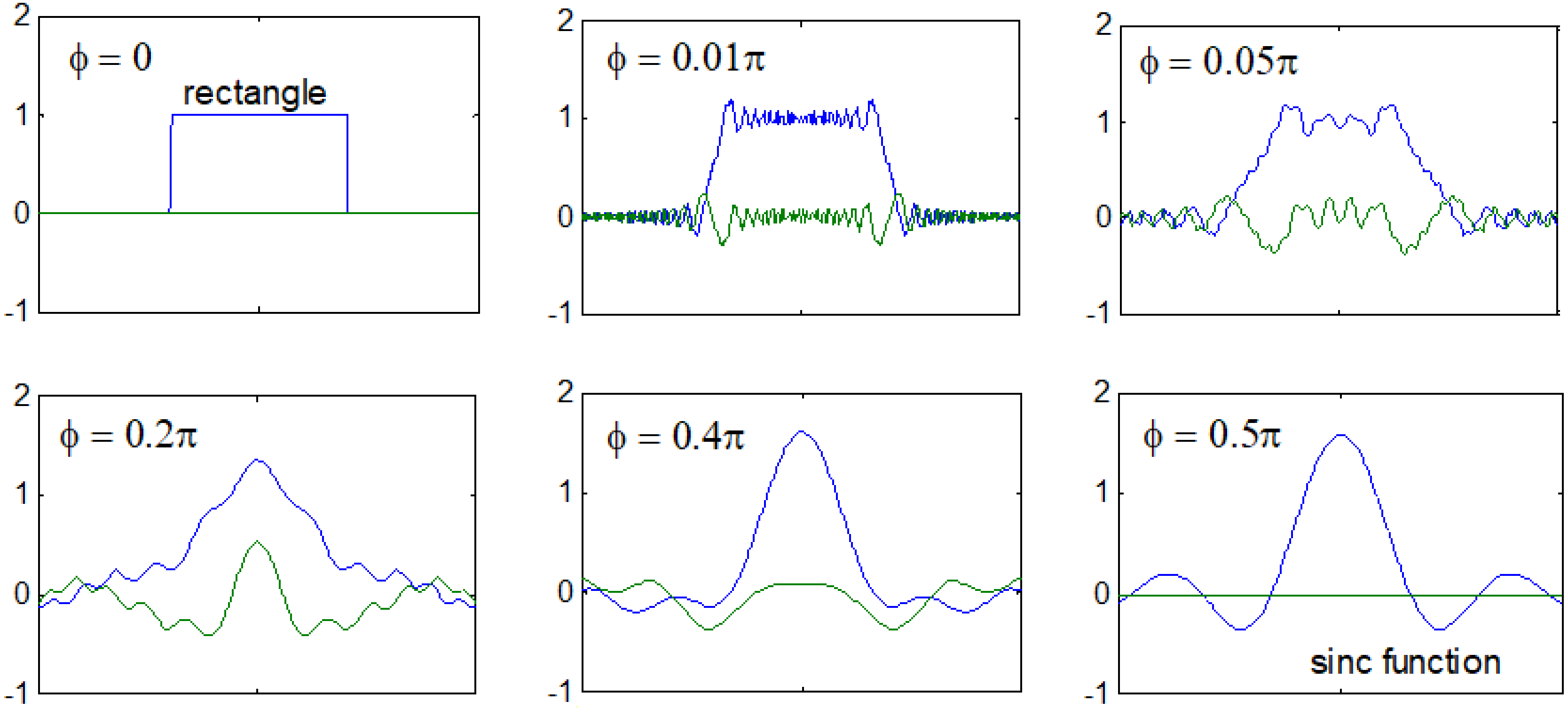
Physical meaning: Performing the FT a times, $\phi = 0.5a\pi$

[Ref] L. B. Almeida, “The fractional Fourier transform and time-frequency representations,” *IEEE Trans. Signal Processing*, vol. 42, no. 11, pp. 3084-3091, Nov. 1994.

[Ref] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, New York, John Wiley & Sons, 2000.

[Ref] V. Namias, “The fractional order Fourier transform and its application to quantum mechanics,” *J. Inst. Maths. Applics.*, vol. 25, pp. 241-265, 1980.

Fractional Fourier transforms for a rectangular function



blue lines: real parts; green lines: imaginary part

(10) Linear Canonical Transform

$$X_{(a,b,c,d)}(u) = \sqrt{\frac{1}{jb}}\, e^{j\pi \frac{d}{b}u^2} \int_{-\infty}^{\infty} e^{-j2\pi \frac{1}{b}ut}\, e^{j\pi \frac{a}{b}t^2}\, x(t)\,dt$$

where $ad - bc = 1$

when $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ ⟹ Fourier transform

when $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$ ⟹ fractional Fourier transform

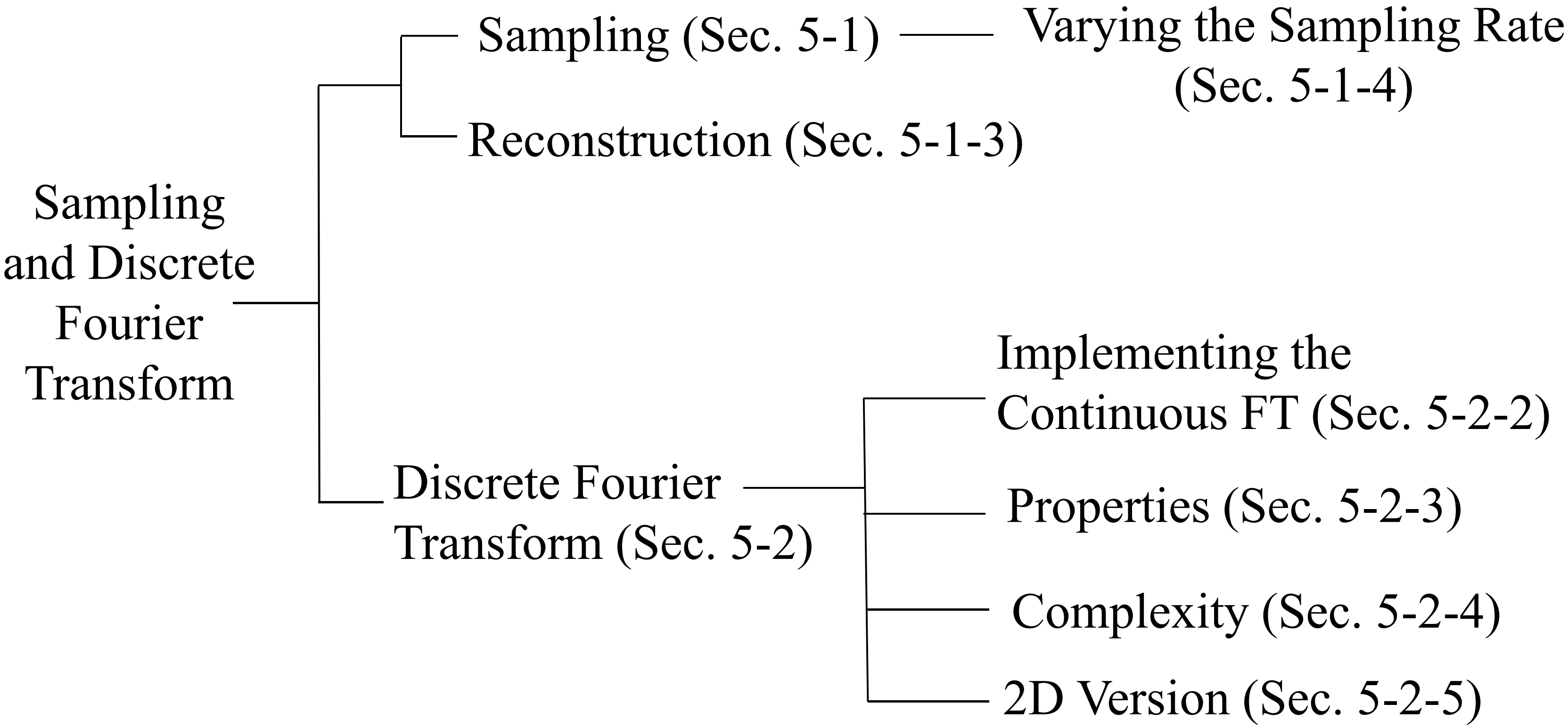
[Ref] K. B. Wolf, "*Integral Transforms in Science and Engineering*," Ch. 9: Canonical transforms, New York, Plenum Press, 1979.

Summary of Transforms

Fourier Transform	$G(f)=\int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}dx$
2D Fourier Transform	$G(f,h)=$ $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi fx}e^{-j2\pi hy}dxdy$
Hankel Transform	$G(s)=2\pi\int_{0}^{\infty} J_0(2\pi sr)rg(r)dr$
Two-Sided Laplace Transform	$G(s)=\int_{-\infty}^{\infty} e^{-st}g(t)dt$
One-Sided Laplace Transform	$G(s)=\int_{0}^{\infty} e^{-st}g(t)dt$
Fourier Cosine Transform	$G_c(f)=\int_{0}^{\infty} g(x)\cos(2\pi fx)dx$
Fourier Sine Transform	$G_s(f)=\int_{0}^{\infty} g(x)\sin(2\pi fx)dx$

Hartley Transform	$G_{ha}(f)=\int_{-\infty}^{\infty} g(x)\operatorname{cas}(2\pi fx)\,dx$
Mellin Transform (Laplace for a scaled signal)	$G_M(s)=\int_{0}^{\infty} g(x)x^{s-1}dx$
Hilbert Transform	$g_H(x)=\mathfrak{I}^{-1}\{\mathfrak{I}[g(x)]H(f)\}$ $H(f)=-j\ for\ f>0,$ $H(f)=j\ for\ f<0,\quad H(0)=0$
Analytic Signal	$g_a(x)=g(x)+jg_H(x)$
Fractional Fourier Transform	$X_{\phi}(u)=\sqrt{1-j\cot\phi}\ e^{j\pi\cot\phi\cdot u^2}$ $\int_{-\infty}^{\infty} e^{-j2\pi\cdot\csc\phi\cdot u\,t}\,e^{j\pi\cdot\cot\phi\cdot t^2}\ x(t)dt$ α times of FT $\alpha=\frac{\phi}{\pi/2}$
Linear Canonical Transform	$X_{(a,b,c,d)}(u)=\sqrt{1/jb}\ e^{j\pi d\,u^2/b}$ $\int_{-\infty}^{\infty} e^{-j2\pi\,ut/b}\,e^{j\pi a t^2/b}\ x(t)dt$

5. Sampling and Discrete Fourier Transform



[1] R. N. Bracewell, The Fourier Transform and Its Applications, 3rd ed., McGraw Hill, Boston, 2000.

[2] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*, London: Prentice-Hall, 3rd ed., 2010.

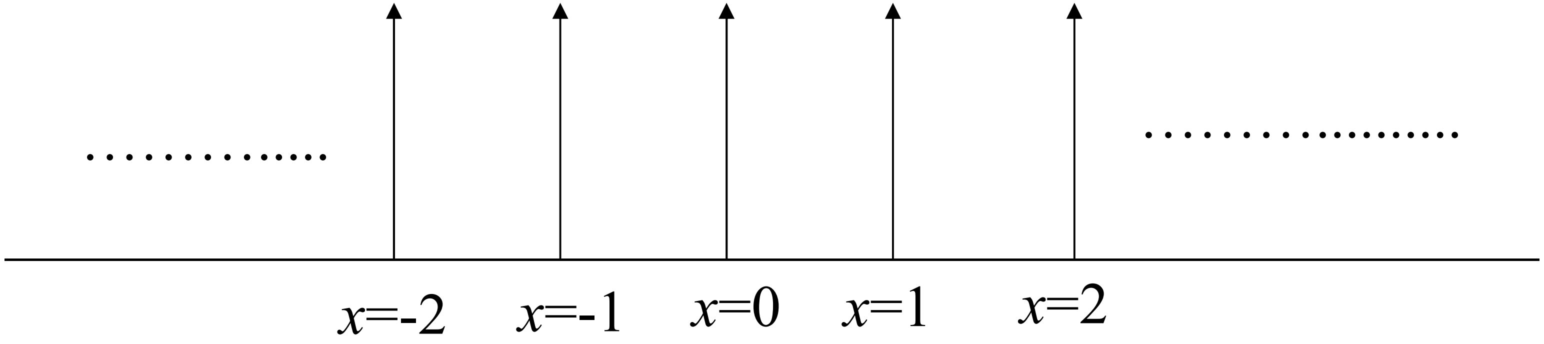
5.1 Sampling

5.1.1 Impulse Train

Impulse Train

$$p(x)=\sum_{n}\delta(x-n)=\cdots+\delta(x+1)+\delta(x)+\delta(x-1)+\delta(x-2)+\cdots\cdots$$

It is also called the comb function.



Signal sampling can be express in terms of the impulse train

Δ_x: sampling interval

$$g(x) \xrightarrow{sampling} g_s(x) = g(x)\sum_n \delta(x - n\Delta_x)$$

$$= \sum_n g_n \delta(x - n\Delta_x)$$

where $g_n = g(n\Delta_x)$

page 343

Since

$$\sum_n \delta(x - n\Delta_x) = \frac{1}{\Delta_x}\sum_n \delta\left(\frac{x}{\Delta_x} - n\right) = \frac{1}{\Delta_x} p\left(\frac{x}{\Delta_x}\right)$$

page 345

the sampled signal $g_s(x)$ can be expressed in terms of

$$g_s(x) = \frac{1}{\Delta_x} g(x) p\left(\frac{x}{\Delta_x}\right)$$

[**Theorem 5.1.1**] The impulse train is also an eigenfunction of the Fourier transform, i.e.,

$$P(f)=\Im\{p(x)\}=\sum_{n}\delta(f-n)$$

(Proof): Note that the impulse train is a periodic function

$$p(x)=p(x+1)$$

Therefore, it can be expanded by the Fourier series (page 323) of the complex form with $T = 1$

$$p(x)=\sum_{n}c_n\exp(j2\pi nx)$$

where

$$c_n=\frac{1}{1}\int_{-1/2}^{1/2}p(x)\exp(-j2\pi nx)\,dx=\int_{-1/2}^{1/2}\delta(x)\exp(-j2\pi nx)\,dx=1$$

(page 344(2))

k=0

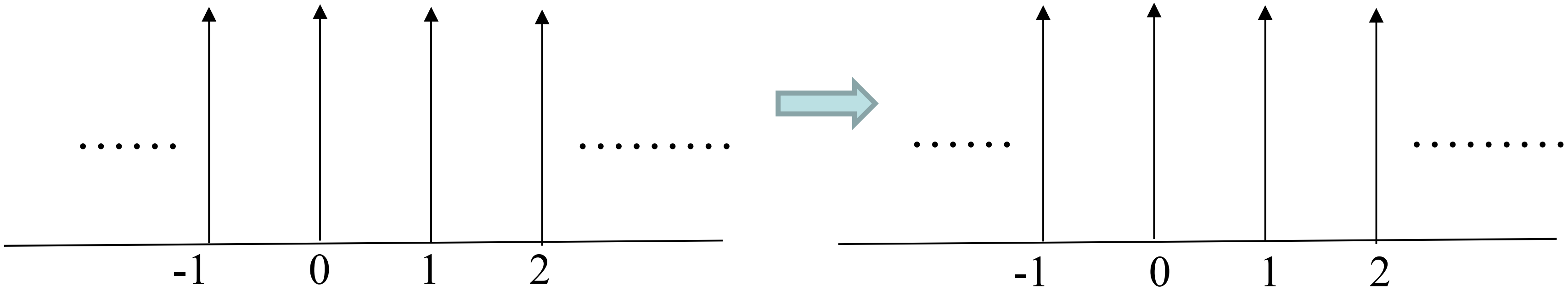
Therefore,

$$p(x) = \sum_n \exp(j2\pi nx)$$

$$\mathcal{F}\left[p(x)\right] = \sum_n \mathcal{F}\left[\exp(j2\pi nx)\right]$$

$$= \sum_n \delta(f-n) = p(f)$$

$$\Im\left\{\sum_n \delta(x-n)\right\} = \sum_n \delta(f-n)$$



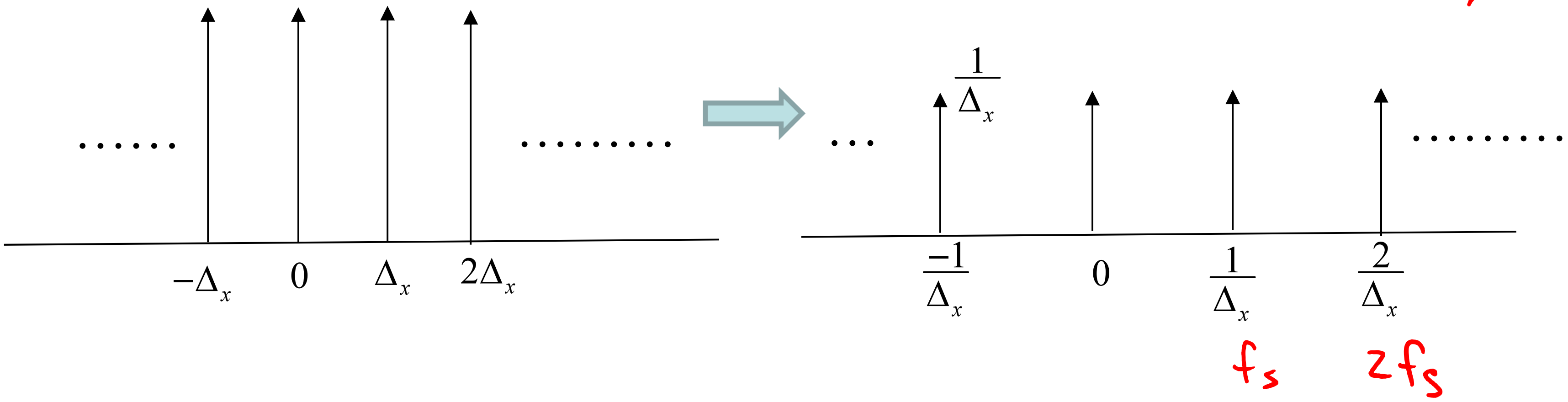
Varying the interval of the impulse train

$$\sum_n \delta(x - n\Delta_x) = \frac{1}{\Delta_x}\sum_n \delta\left(\frac{x}{\Delta_x} - n\right) = \frac{1}{\Delta_x}\, p\left(\frac{x}{\Delta_x}\right) \qquad \text{(page 345(4))}$$

$$\mathcal{F}\left[\sum_n \delta(x - n\Delta_x)\right] = \mathcal{F}\left[\frac{1}{\Delta_x}\, p\left(\frac{x}{\Delta_x}\right)\right] = p(\Delta_x f) \qquad \text{(page 351(5))}$$

$$= \sum_n \delta(\Delta_x f - n) = \frac{1}{\Delta_x}\sum_n \delta\left(f - \frac{n}{\Delta_x}\right)$$

$f_s = \frac{1}{\Delta_x}$



5.1.2 Sampling Theory

[**Theorem 5.1.2**] Suppose that we perform sampling for a continuous signal with sampling interval Δ_x

$$g(x)\xrightarrow{sampling} g_s(x)=g(x)\sum_n \delta(x-n\Delta_x)$$

$$=\sum_n g_n\delta(x-n\Delta_x)$$

where

$$g_n=g(n\Delta_x)$$

then

$$\boxed{G_s(f)=\frac{1}{\Delta_x}\sum_n G\left(f-\frac{n}{\Delta_x}\right)}$$

(Proof): Since

$$g_s(x)=g(x)\frac{1}{\Delta_x}p\left(\frac{x}{\Delta_x}\right)$$

$p(x) \xrightarrow{FT} P(f)$

page 351 (3)

$$\mathcal{F}\left[g_s(x)\right]=\mathcal{F}\left[g(x)\right]*\underline{\mathcal{F}\left[\frac{1}{\Delta_x}p\left(\frac{x}{\Delta_x}\right)\right]}$$

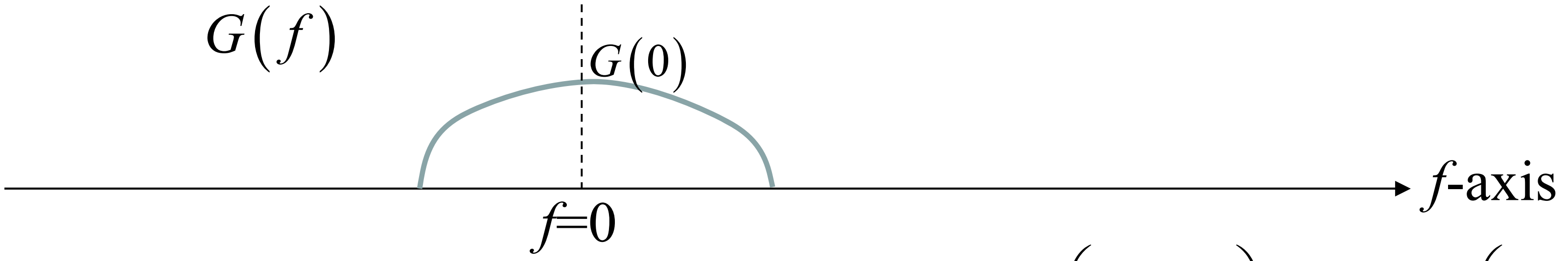
$P(\Delta_x f)$

$$G_s(f)=G(f)*p(\Delta_x f)=\frac{1}{\Delta_x}\sum_n G(f)*\delta\left(f-\frac{n}{\Delta_x}\right)=\frac{1}{\Delta_x}\sum_n G\left(f-\frac{n}{\Delta_x}\right)$$

↳ $\Sigma\,\delta(\Delta_x f-n)$ page345 (page 389)

If we set $f_s = \dfrac{1}{\Delta_x}$ (f_s is call the sampling frequency)

then $\boxed{G_s(f) = f_s \sum_n G(f - n f_s)}$

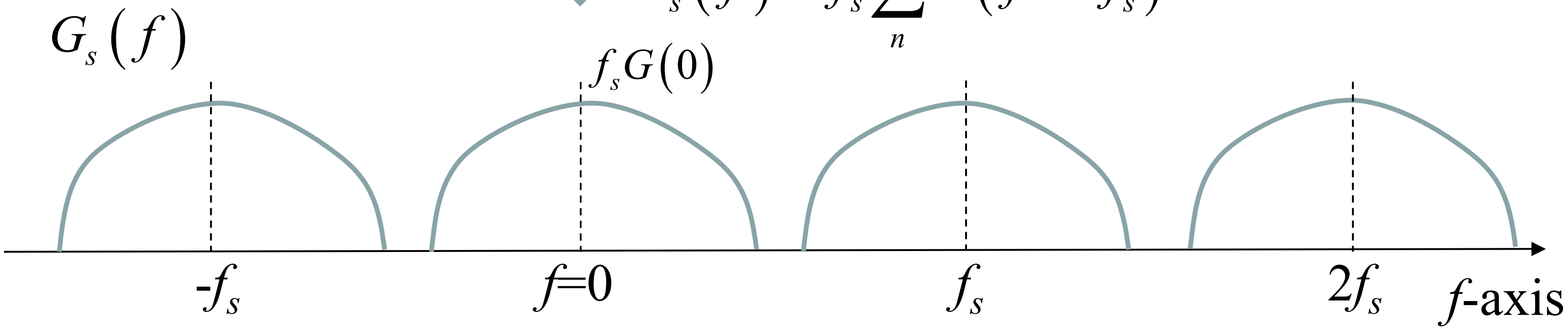


$g_n = g(n\Delta_x)$

sampling

$$g_s(x) = g(x)\sum_n \delta\left(x - \frac{n}{f_s}\right) = \sum_n g_n \delta\left(x - \frac{n}{f_s}\right)$$

$$G_s(f) = f_s \sum_n G(f - n f_s)$$



[Sampling Theory]

The sampling frequency $f_s = \dfrac{1}{\Delta_x}$ should be larger than twice of the bandwidth of the original continuous function:

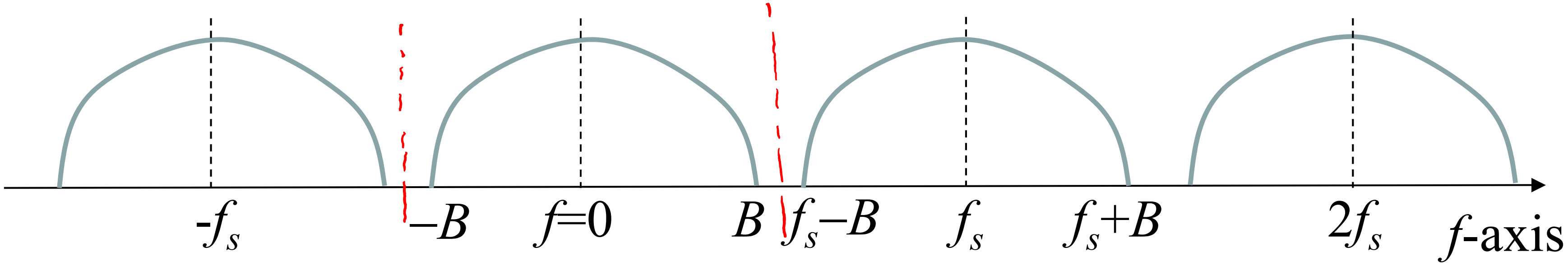
$$f_s - B > B$$

$$f_s > 2B \qquad \text{(Nyquist criterion)}$$

where

$$G(f) = 0 \qquad \text{when } f > B.$$

Otherwise, the original function cannot be reconstructed and the aliasing effect is led.



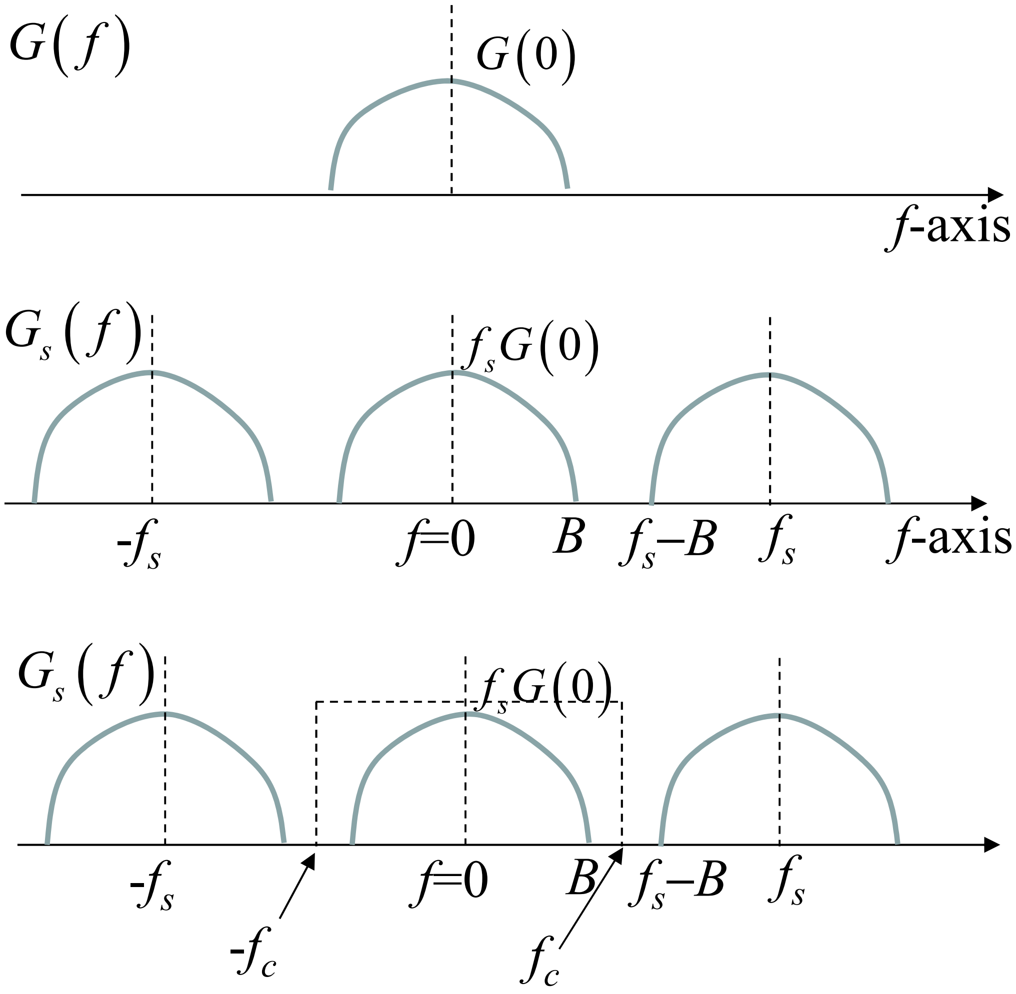
Q: What will happen if $f_s = 2B$?

Even component with frequency $f = \pm B \rightarrow$ preserved

Odd component with frequency $f = \pm B \rightarrow$ destroyed

5.1.3 Reconstruction (Digital to Analogous)

When the Nyquist criterion is satisfied, one can apply the lowpass filter to reconstruct the original signal.



Frequency Domain

$$G(f)$$

$$\downarrow$$

$$G_s(f) = f_s \sum_n G(f - nf_s)$$

$$\downarrow$$

$$G(f) = \frac{1}{f_s}\Pi\left(\frac{f}{2f_c}\right)G_s(f)$$

where $B < f_c < f_s – B$

Time Domain

$$g(x)$$

$$\Big\downarrow \text{ sampling}$$

$$g_n = g(n\Delta_x)$$

$$g_s(x) = \sum_n g_n \delta\left(x - \frac{n}{f_s}\right)$$

D/A conversion

$$\Big\downarrow \text{ reconstruction}$$

A/D conversion

$$g(x) = g_s(x) * \frac{2f_c}{f_s}\operatorname{sinc}(2f_c x)$$

Frequency Domain

$$G(f)$$

$$\Big\downarrow$$

$$G_s(f) = f_s \sum_n G(f - nf_s)$$

$$\Big\downarrow$$

$$G(f) = \frac{1}{f_s}\Pi\left(\frac{f}{2f_c}\right)G_s(f)$$

where $B < f_c < f_s - B$

$$g(x) = \frac{2f_c}{f_s}\int g_s(\tau)\,\mathrm{sinc}(2f_c(x-\tau))\,d\tau$$

$$= \frac{2f_c}{f_s}\int \sum_n g_n \delta\left(\tau - \frac{n}{f_s}\right)\mathrm{sinc}(2f_c(x-\tau))\,d\tau$$

$$g(x) = \frac{2f_c}{f_s}\sum_n g_n\,\mathrm{sinc}\left(2f_c(x-\frac{n}{f_s})\right)$$

Specially, when $f_c = f_s/2$

$$g(x) = \sum_n g_n\,\mathrm{sinc}(f_s x - n)$$

Signal Reconstruction Formula:

$$\boxed{g(x) = \sum_n g_n\,\mathrm{sinc}\left(\frac{x}{\Delta_x} - n\right)}$$

where $g_n = g(n\Delta_x)$

constraint: $\frac{1}{\Delta_x} > 2B$

[Example 1] Suppose that

$$g_n = g\left(\frac{n}{2}\right)$$

$g_{-1} = g_1 = 1, \quad g_0 = 2, \qquad\qquad g_n = 0 \quad$ otherwise

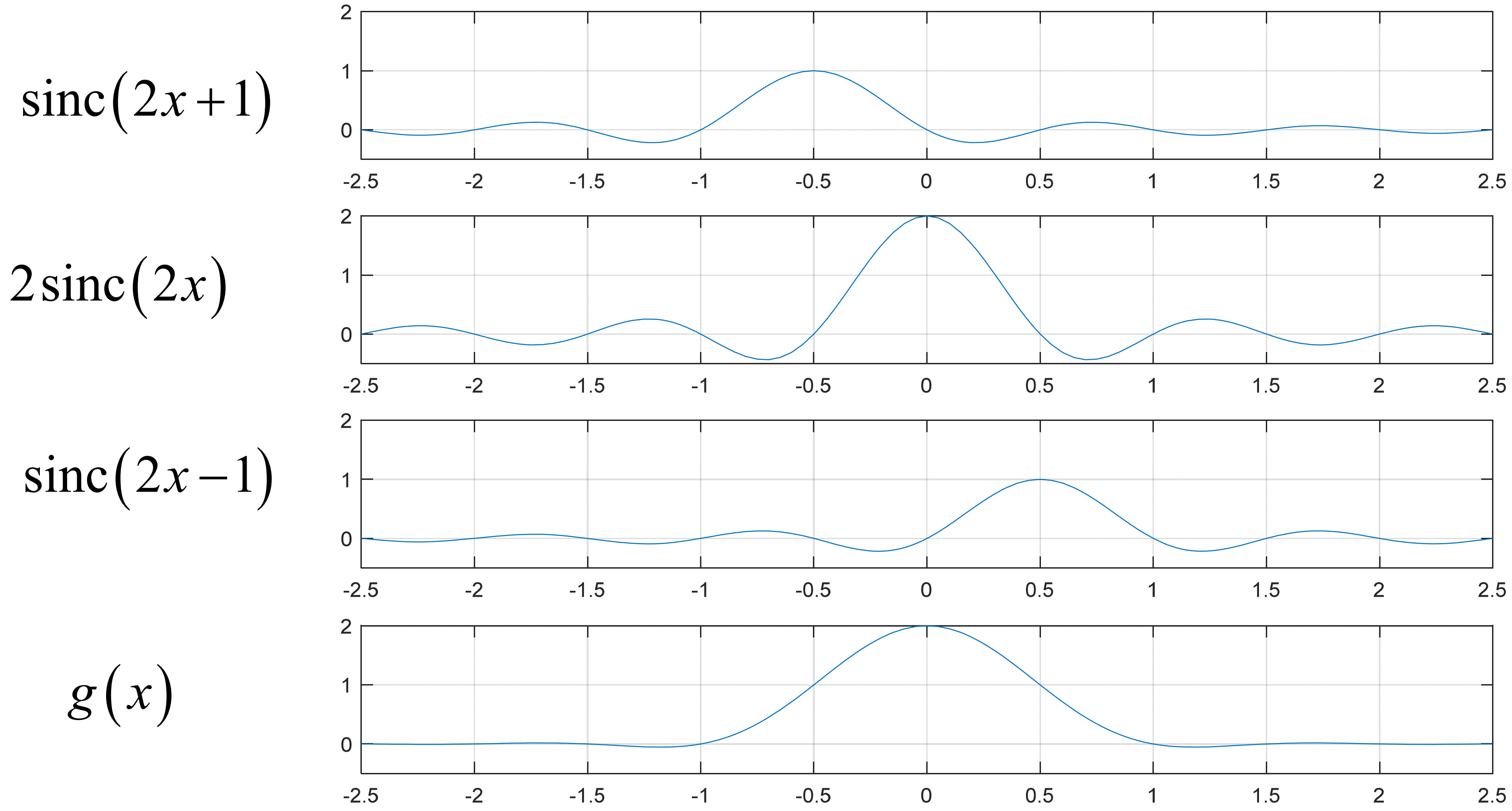
$G(f) = 0 \quad$ for $f \geq 1$

Try to reconstruct $g(x)$.

(Solution): $\Delta_x = 1/2$

$$g(x) = \operatorname{sinc}(2x+1) + 2\operatorname{sinc}2x + \operatorname{sinc}(2x-1)$$

$$g(x)=\mathrm{sinc}(2x+1)+2\,\mathrm{sinc}\,2x+\mathrm{sinc}(2x-1)$$



(Note): $\mathrm{sinc}(2x+1),\ 2\,\mathrm{sinc}(2x),\ \mathrm{sinc}(2x-1)$

do not interfere with one another at $x = n/2$.

5.1.4 Varying the Sampling Rate

(i) D/A conversion

$$g(x) = \sum_n g_n \operatorname{sinc}\left(\frac{x}{\Delta_x} - n\right)$$

(ii) Re-sampling

$$\hat{g}_n = g(n\Delta_{new}) = \sum_m g_m \operatorname{sinc}\left(\frac{n\Delta_{new}}{\Delta_x} - m\right)$$

Note: When $\Delta_{new} = k\Delta_x$ and k is an integer

$$\hat{g}_n = g_{kn}$$

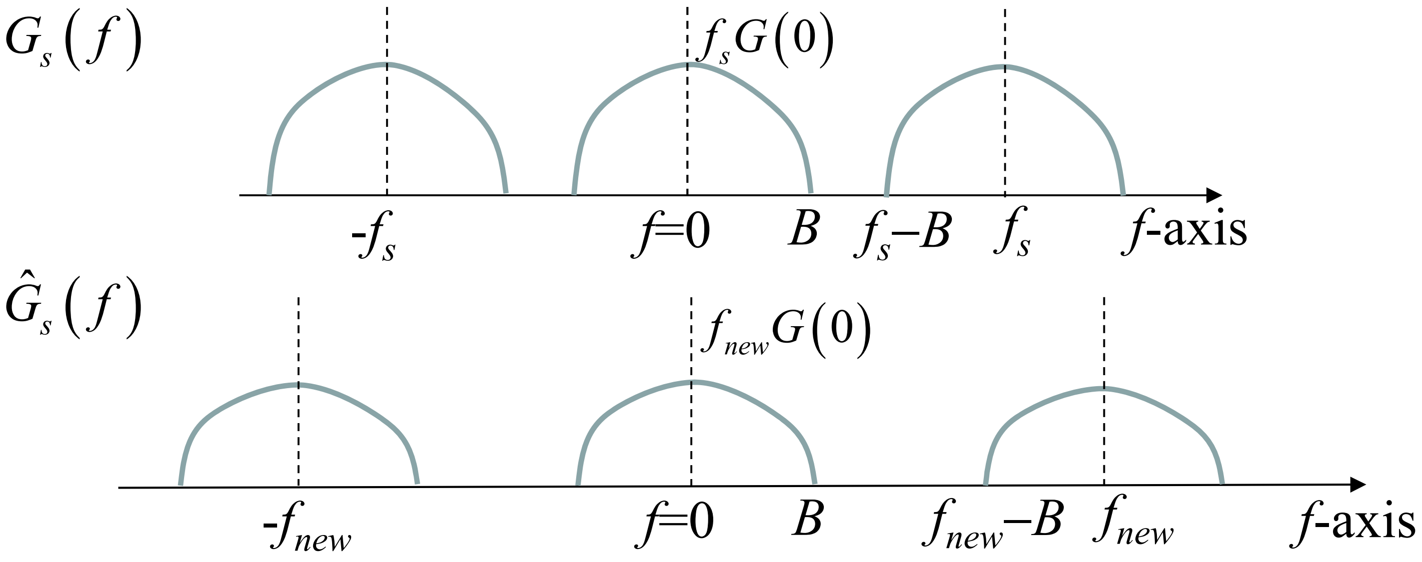
From the view point of the spectrum, if

$$g_s(x) = \sum_n g_n \delta(x - n\Delta_x) \qquad \hat{g}_s(x) = \sum_n \hat{g}_n \delta(x - n\Delta_{new})$$

then

$$G_s(f) = f_s \sum_n G(f - nf_s) \qquad \hat{G}_s(f) = f_{new} \sum_n G(f - nf_{new})$$

where $f_s = 1/\Delta_x, \quad f_{new} = 1/\Delta_{new}$



5.2 Discrete Fourier Transform

5.2.1 Derivation and Definitions of the Discrete Fourier Transform

To process discrete functions, the continuous Fourier transform should be converted into the discrete version.

Continuous Fourier transform:

$$G(f)=\int_{-\infty}^{\infty} g(x)e^{-j2\pi fx}\boxed{dx}$$

If we set

$$f=m\Delta_f,\quad x=n\Delta_x$$

then

$$G\left(m\Delta_f\right)=\sum_n g(n\Delta_x)e^{-j2\pi mn\Delta_f\Delta_x}\boxed{\Delta_x}$$

$$G\left(m\Delta_f\right)=\sum_n g\left(n\Delta_x\right)e^{-j2\pi mn\Delta_f\Delta_x}\Delta_x$$

Specially, if

$$\underline{\underline{\Delta_f\Delta_x}}=\underline{\underline{1/N}}$$

then

$$G\left(m\Delta_f\right)=\sum_n g\left(n\Delta_x\right)e^{-j\frac{2\pi mn}{N}}\underline{\underline{\Delta_x}}$$

Similarly, for the continuous inverse Fourier transform:

$$g\left(x\right)=\int_{-\infty}^{\infty}G\left(f\right)e^{j2\pi fx}df$$

$$g\left(n\Delta_x\right)=\sum_n G\left(m\Delta_f\right)e^{j2\pi mn\Delta_f\Delta_x}\Delta_f$$

$$g\left(n\Delta_x\right)=\sum_m G\left(m\Delta_f\right)e^{j\frac{2\pi mn}{N}}\Delta_f$$

Discrete Fourier Transform (DFT)

$$G[m] = DFT\{g[n]\} = \sum_{n=0}^{N-1} g[n] e^{-j\frac{2\pi mn}{N}}$$

$m, n = 0, 1, 2, \ldots, N\text{-}1$

Inverse Discrete Fourier Transform (IDFT)

$$g[n] = IDFT\{G[m]\} = \frac{1}{N}\sum_{m=0}^{N-1} G[m] e^{j\frac{2\pi mn}{N}}$$

Note that the output of $g[n]$ is periodic

$$\underline{G[m]} = G[m+N]$$

$$G[m+N] = \sum_{n=0}^{N-1} g[n] e^{-j\frac{2\pi}{N} n(m+N)}$$

$$= \sum_{n=0}^{N-1} g[n] e^{-j\frac{2\pi}{N} mn} \cancel{e^{-j2\pi n}}$$

$$= G[m]$$

Also note that, on page 440,

$$\underline{G(m\Delta_f)} = DFT(g(n\Delta_x))\Delta_x$$

The DFT and the IDFT form a transform pair since

$$\frac{1}{N}\sum_{m=0}^{N-1}G[m]e^{j\frac{2\pi mn}{N}}=\frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1}g[k]e^{-j\frac{2\pi mk}{N}}e^{j\frac{2\pi mn}{N}}=\frac{1}{N}\sum_{k=0}^{N-1}g[k]\left(\sum_{m=0}^{N-1}e^{-j\frac{2\pi mk}{N}}e^{j\frac{2\pi mn}{N}}\right)$$

Because

$$\sum_{m=0}^{N-1}e^{j\frac{2\pi a}{N}m}=\frac{1-e^{j\frac{2\pi a}{N}N}}{1-e^{j\frac{2\pi a}{N}}}=\frac{1-e^{j2\pi a}}{1-e^{j\frac{2\pi a}{N}}}=0 \qquad \text{if } a\neq 0,$$

$$\sum_{m=0}^{N-1}e^{j\frac{2\pi a}{N}m}=N \qquad \text{if } a=0,$$

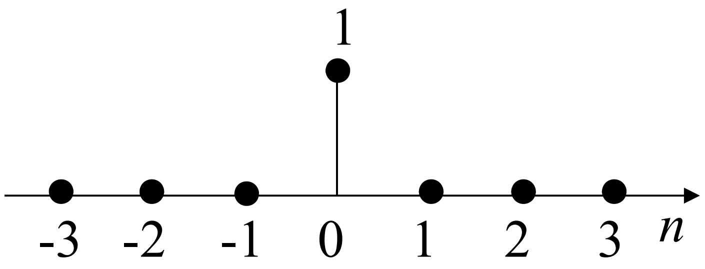
$$\sum_{m=0}^{N-1}e^{j\frac{2\pi m}{N}(n-k)}=N\delta_d[n-k]$$

unit impulse function
(discrete Dirac delta function)

$$\frac{1}{N}\sum_{m=0}^{N-1}G[m]e^{j\frac{2\pi mn}{N}}=\frac{1}{N}\sum_{k=0}^{N-1}g[k]N\delta_d[n-k]=g[n]$$

Unit Impulse Function (discrete Dirac delta function)

$$\delta_d[n] = \begin{cases} 1 & if\ n = 0 \\ 0 & if\ n \neq 0 \end{cases}$$



The unit impulse function has an explicit form. It does not a limitation of a distribution.

Compared to page 343,

$$\sum_n \delta_d[n] = 1$$

$$\delta_d[n] = 0 \quad if\ n \neq 0$$

$$\delta_d[n] = \delta_d[-n]$$

Other possible definitions of the DFT

DFT $$G[m]=\sum_{n=n_0}^{n_0+N-1} g[n]e^{-j\frac{2\pi mn}{N}}$$

IDFT $$g[n]=\frac{1}{N}\sum_{m=m_0}^{m_0+N-1} G[m]e^{j\frac{2\pi mn}{N}}$$

DFT $$G[m]=\sqrt{\frac{1}{N}}\sum_{n=0}^{N-1} g[n]e^{-j\frac{2\pi mn}{N}}$$

IDFT $$g[n]=\sqrt{\frac{1}{N}}\sum_{m=0}^{0\,N-1} G[m]e^{j\frac{2\pi mn}{N}}$$

5.2.2 Implementing Continuous FT by the DFT

Suppose that we want to calculate the continuous FT of $g(x)$ digitally and

$$g(x)=0 \qquad \text{for } x \notin [x_1, x_1+T]$$

(i) Shifting

$$g_1(x)=g(x+x_1) \qquad G_1(f)=G(f)e^{j2\pi f x_1}$$

Note: $$g_1(x)=0 \qquad \text{for } x \notin [0, T]$$

(ii) Sampling

$$g_d[n]=g_1(n\Delta_x)$$

(iii) DFT

$$G_d[m]=\sum_{n=0}^{N-1} g_d[n]e^{-j\frac{2\pi mn}{N}}$$

(iv) Mapping to the true frequency

$$G_1\left(m\Delta_f\right) = G_d\left[m\right]\Delta_x \qquad \text{(from pages 440 and 441)}$$

Since

$$\Delta_f\Delta_x = 1/N \qquad \underline{\Delta_f} = \frac{1}{N\Delta_x} = \underline{\frac{f_s}{N}}$$

Therefore,

$$\boxed{\begin{array}{ll} G_1\left(m\dfrac{f_s}{N}\right) = G_d\left[m\right]\Delta_x & \text{if } 0 \le m \le N/2, \\[2ex] G_1\left(m\dfrac{f_s}{N} - f_s\right) = G_d\left[m\right]\Delta_x & \text{if } N/2 \le m \le N\text{-}1 \end{array}}$$

(v) Using the modulation property

$$G\left(f\right) = e^{-j2\pi x_1 f}G_1\left(f\right)$$

[Example 1]：Suppose that

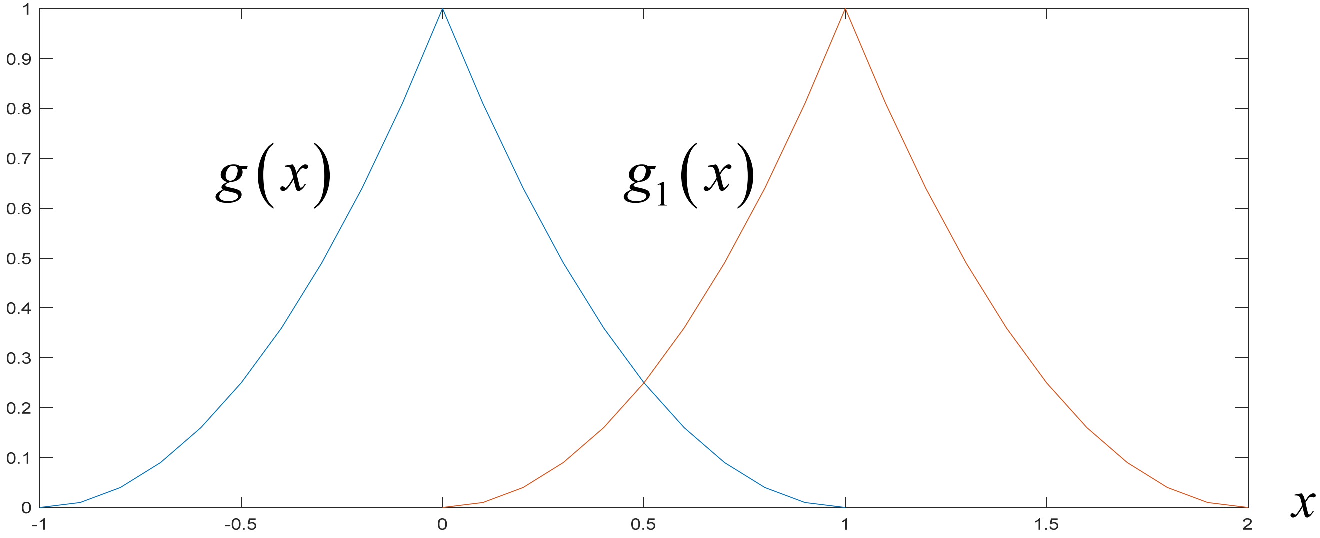
$g(x) = (1-|x|)^2$ for $-1 \leq x \leq 1$ $g(x) = 0$ otherwise

Sampling interval： $\Delta_x = 0.1$

How do we obtain the FT of $g(x)$ by the DFT?

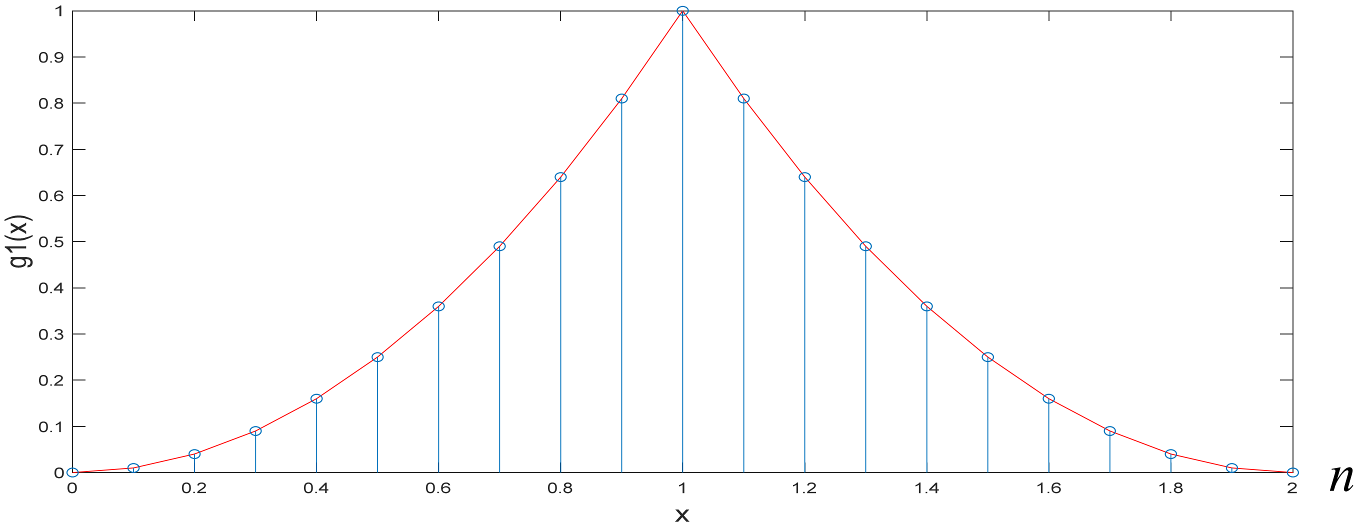
(Solution):

(i) $g_1(x) = g(x-1)$ $G_1(f) = G(f)e^{-j2\pi f}$



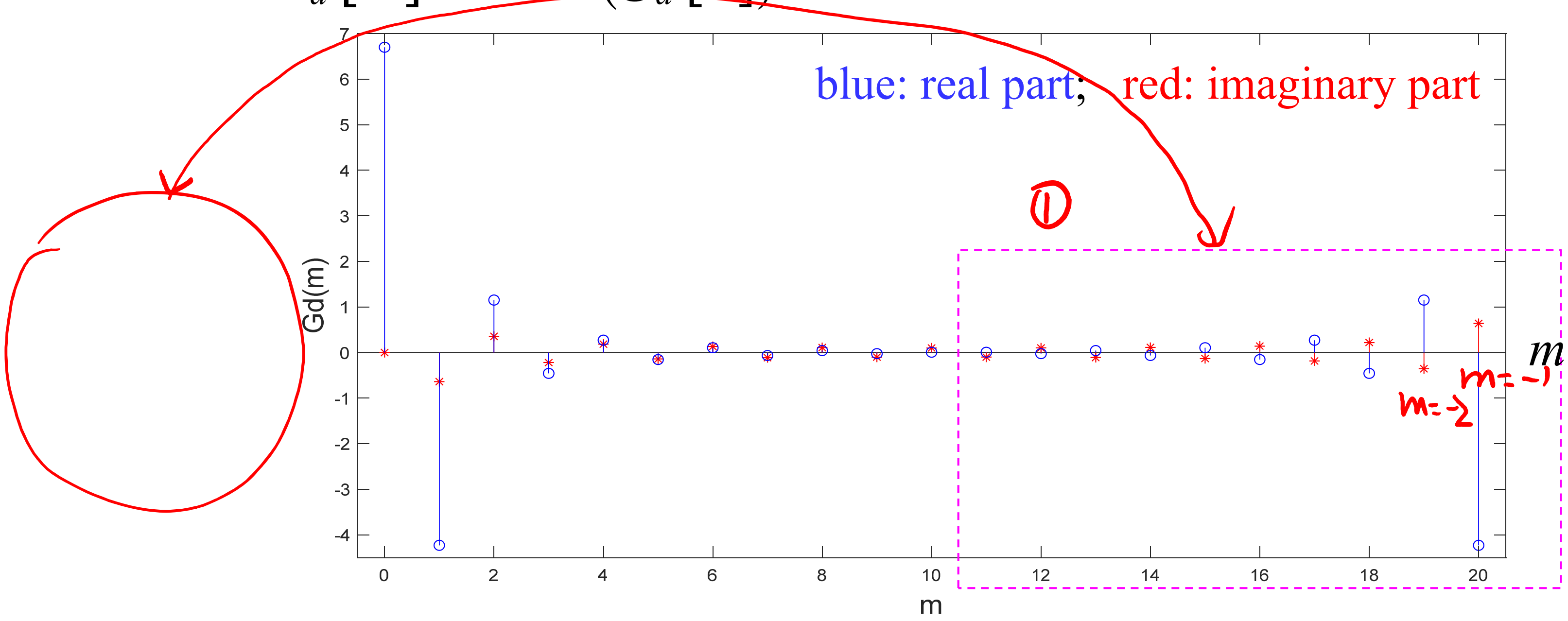
(ii) $g_d[n] = g_1(n\Delta_x)$

$n = 0, 1, \cdots 20$, $N = 21$



(iii) $G_d[m] = DFT(g_d[n])$

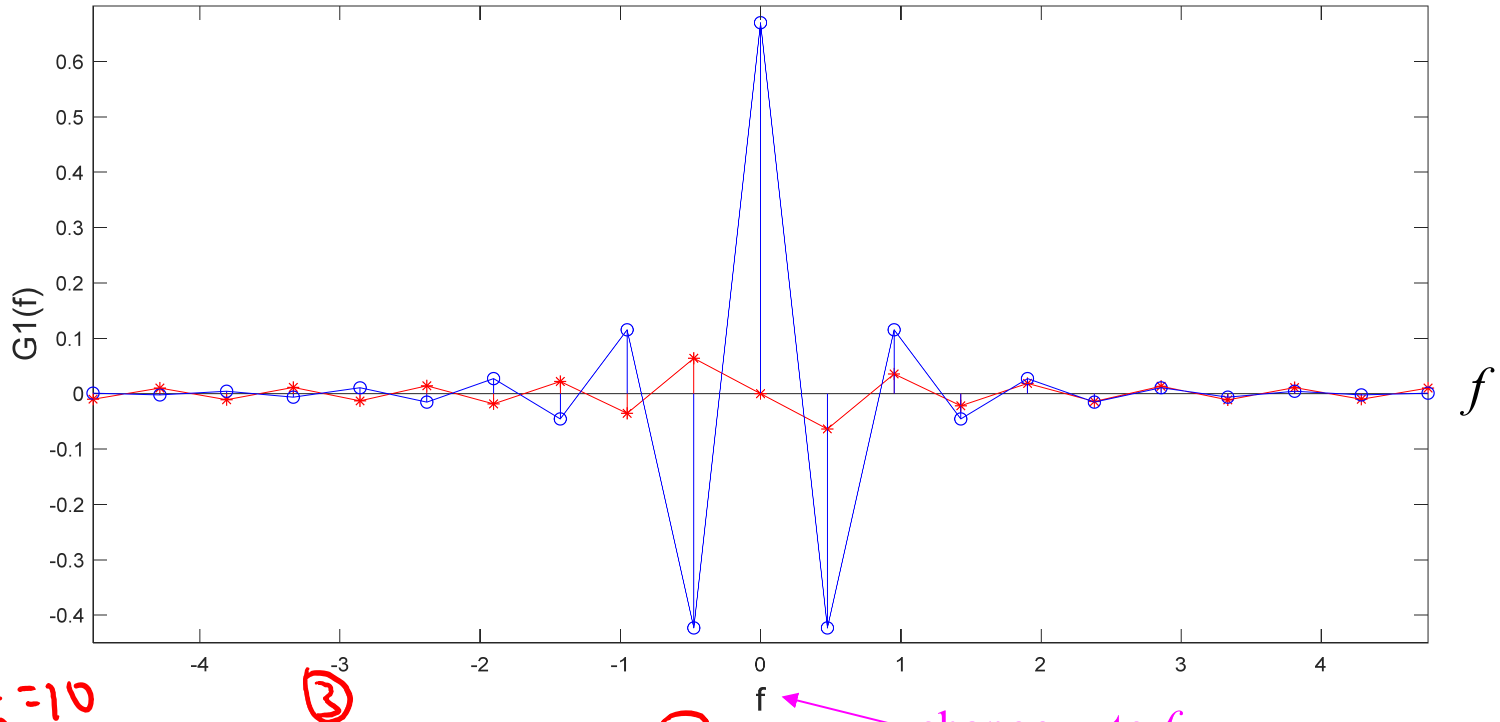
$G_d(m) = G_d(m+N]$ $N=21$



move to left

(iv) Mapping

blue: real part; red: imaginary part



change m to f

$f_s: \frac{1}{0,1} = 10$

$N: 21$

$\frac{f_s}{N} : 0.4762$

③ ②

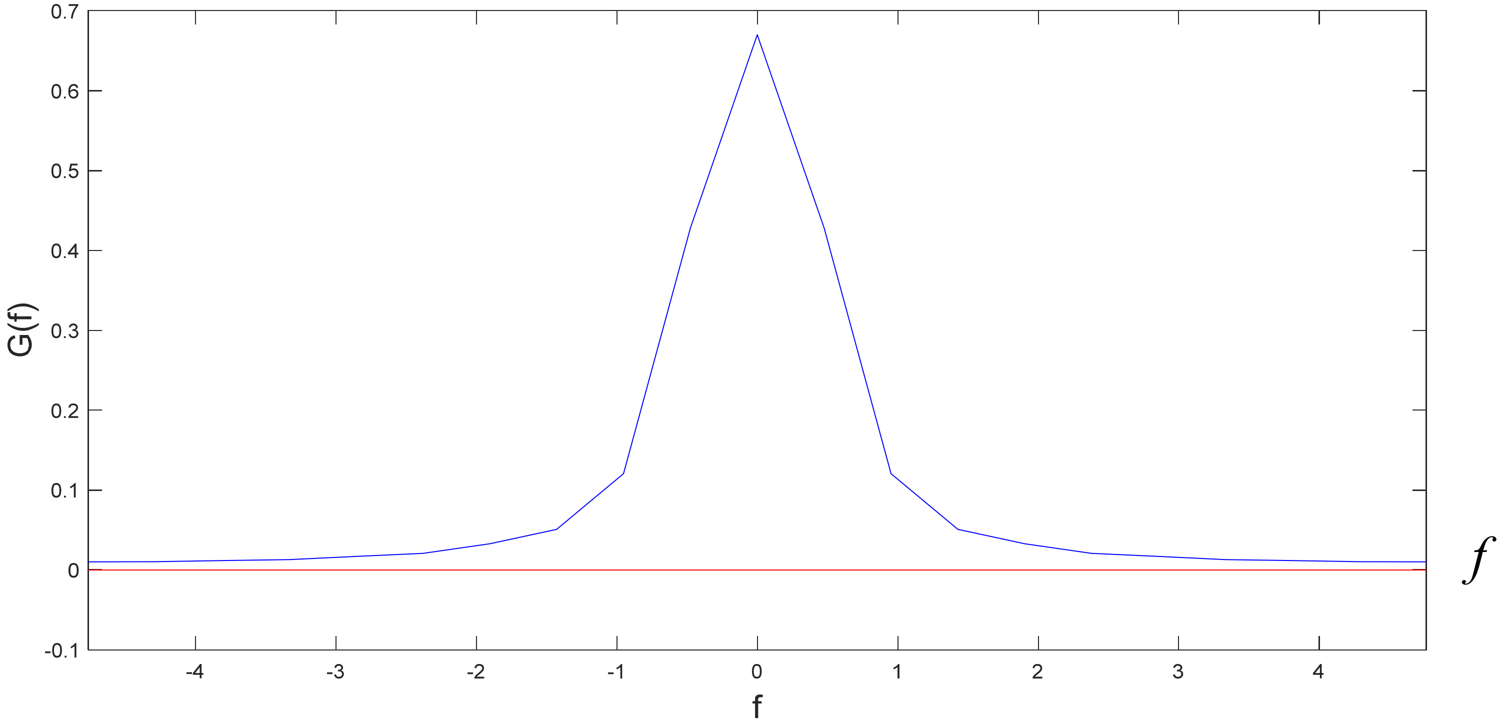
$$G_1\left(m\frac{f_s}{N}\right) = G_d[m]\Delta_x \qquad \text{if } 0 \le m \le N/2,$$

$$G_1\left(m\frac{f_s}{N} - f_s\right) = G_d[m]\Delta_x \qquad \text{if } N/2 \le m \le N\text{-}1$$

④

(v) $G(f) = e^{j2\pi f} G_1(f)$

blue: real part; red: imaginary part



5.2.3 Transform Pairs and Properties

[Duality Property]

If $G[m] = DFT\{g[n]\}$

then $g[((-m))_N] = \frac{1}{N} DFT\{G[n]\}$ $DFT\{G[n]\} = Ng[((-m))_N]$

$((a))_N$: a modulo N

the remainder of a after divided by N

$$g[((-m))_N] = \begin{cases} g[N-m] & if\ m = 1, 2, \cdots, N-1 \\ g[0] & if\ m = 0 \end{cases}$$

If $G[m]=DFT\{g[n]\}$ then $g[((-m))_N]=\frac{1}{N}DFT\{G[n]\}$

(Proof): $\frac{1}{N}DFT\{G[n]\}=\frac{1}{N}\sum_{n=0}^{N-1}e^{-j\frac{2\pi mn}{N}}G[n]$

$$=\frac{1}{N}\sum_{n=0}^{N-1}e^{-j\frac{2\pi mn}{N}}\sum_{k=0}^{N-1}e^{-j\frac{2\pi nk}{N}}g[k]=\frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{n=0}^{N-1}e^{-j\frac{2\pi mn}{N}}e^{-j\frac{2\pi nk}{N}}\right)g[k]$$

Since $\sum_{n=0}^{N-1}e^{j\frac{2\pi n}{N}a}=\delta_d[((a))_N]$

(proved on the next page)

$$\frac{1}{N}DFT\{G[n]\}=\frac{1}{N}\sum_{k=0}^{N-1}N\delta_d[((-m-k))_N]g[k]$$

$$=\sum_{k=0}^{N-1}\delta_d[((m+k))_N]g[k]=g[((-m))_N]$$

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi a}{N}n} = N\delta_d\left[((a))_N\right]$$

$((a))_N$: the remainder of a after divided by N

When $a \neq bN$ where b is some integer

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi a}{N}n} = \frac{1-e^{j\frac{2\pi a}{N}N}}{1-e^{j\frac{2\pi a}{N}}} = \frac{1-1}{1-e^{j\frac{2\pi a}{N}}} = 0$$

When $a = bN$ where b is some integer (i.e., $((a))_N = 0$)

$$\sum_{n=0}^{N-1} e^{j\frac{2\pi a}{N}n} = \sum_{n=0}^{N-1} e^{j2\pi bn} = \sum_{n=0}^{N-1} 1 = N$$

[Determine the IDFT by the DFT]

$$g_1[m] = DFT\{G[n]\}$$

$$g[n] = \frac{1}{N} g_1[-n]$$

Note:

(i) Computation loading of the IDFT = Computation loading of the DFT

(ii) In industry, only the chip of the DFT is required.

[Transform Pairs]

$g[n]$	$G[m]$
(1) $\delta_d[n]$ (see page 443)	1
(2) 1	$N\delta_d\left[m\right]$
(3) $\delta_d[n-k]$	$\exp[-j2\pi km/N]$
(4) $\exp[j2\pi kn/N]$	$N\delta_d\left[m-k\right]$
(5) $\cos[2\pi kn/N]$	$\frac{N}{2}\delta_d\left[m-k\right]+\frac{N}{2}\delta_d\left[m-(N-k)\right]$
(6) $\sin[2\pi kn/N]$	$-j\frac{N}{2}\delta_d\left[m-k\right]+j\frac{N}{2}\delta_d\left[m-(N-k)\right]$
(7) $g[n] = 1$ for $0 \leq n \leq W$ $g[n] = 0$ otherwise	$e^{-j\frac{\pi W}{N}m}\frac{\sin\left(\pi m(W+1)/N\right)}{\sin\left(\pi m/N\right)}$ $for\ m \neq 0,$ $W+1$ $for\ m=0$
(8) $\exp[-kn]$, $k \neq 0$	$\frac{1-e^{-Nk}}{1-e^{-k-j2\pi m/N}}$

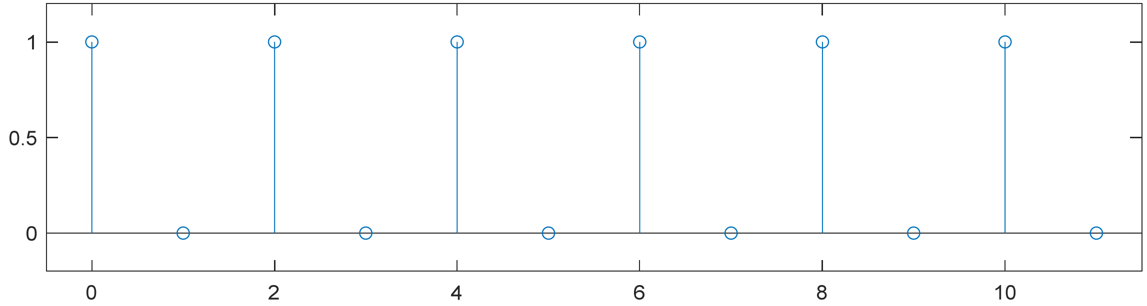
[Discrete Impulse Train]

$$p_c[n] = \begin{cases} 1 & if\ \ n\, is\ a\ multiple\ of\ \ c \\ 0 & otherwise \end{cases}$$

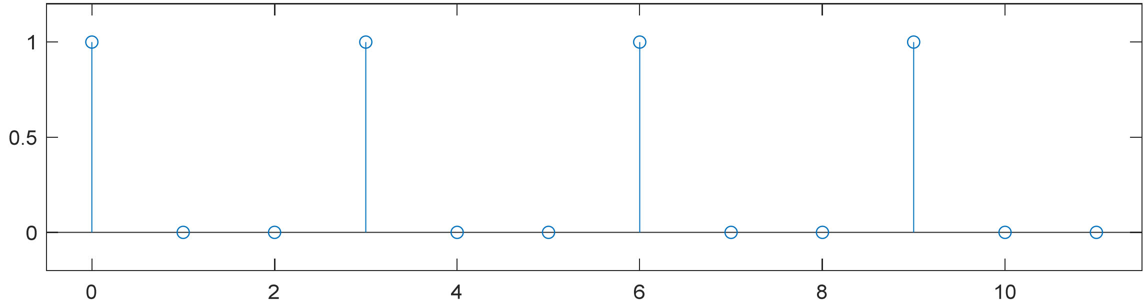
c is a factor of N

$N = 12$

$p_2[n]$



$p_3[n]$



[Example 2] Determine the DFT of $p_c[n]$

c is a factor of N

$n = ck$

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi m}{N}n} p_c[n] = \sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N}ck} = \sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N/c}k}$$

$$\sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N/c}k} = \frac{1-e^{-j\frac{2\pi m}{N/c}\frac{N}{c}}}{1-e^{-j\frac{2\pi m}{N/c}}} = 0$$

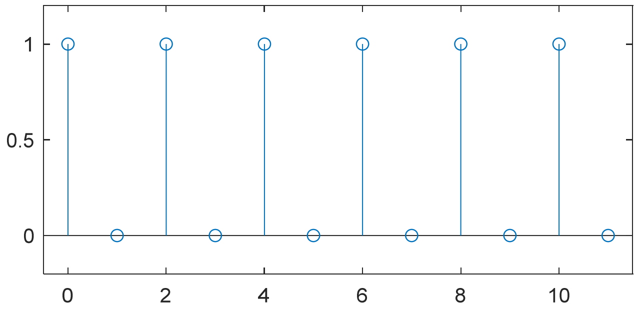
if m is not a multiple *of* N/c

$$\sum_{k=0}^{N/c-1} e^{-j\frac{2\pi m}{N/c}k} = \frac{N}{c}$$

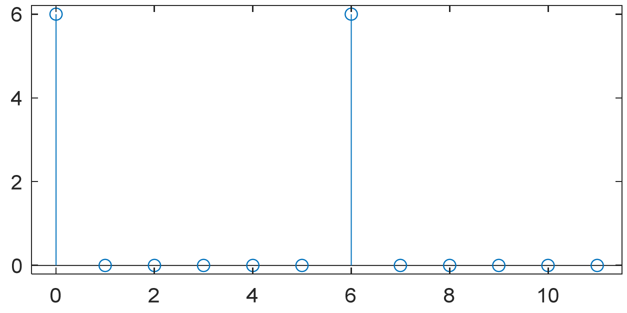
if m is a multiple *of* N/c

Therefore, $DFT\{p_c[n]\} = \frac{N}{c} p_{N/c}[m]$

$p_2[n]$

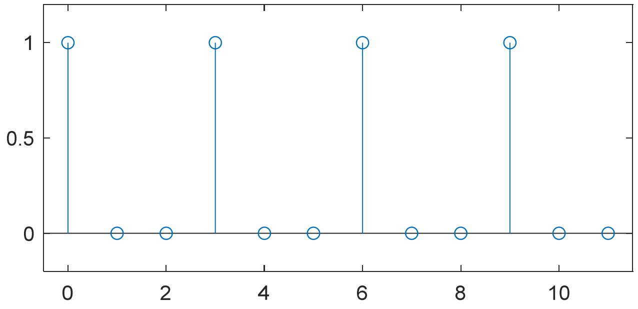


$\xrightarrow{\text{DFT}}$

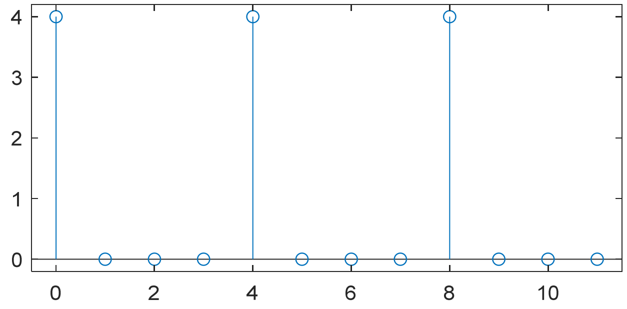


$6p_6[m]$

$p_3[n]$



$\xrightarrow{\text{DFT}}$



$4p_4[m]$

Properties

(1) Linear	$DFT\{ax[n] + by[n]\} = aX[m] + bY[m]$
(2) DC Values	$G[0] = \sum_{n=0}^{N-1} g[n], \quad g[0] = \frac{1}{N}\sum_{n=0}^{N-1} G[m]$
(3) Shifting	$DFT\{g[((n-k))_N]\} = W^{km}G[m]$ where $((n))_N = n$ if $0 \le n \le N$-1 $((n))_N = n+N$ if -$N \le n \le$ -1 $((n))_N = n-N$ if $N \le n \le 2N$-1 $W = \exp(-j2\pi / N)$
(4) Modulation	$DFT\{W^{kn}g[n]\} = G[((m+k))_N]$
(5) Time Reverse	$DFT\{g[((-n))_N]\} = G[((-m))_N]$
(6) Even /Odd Input	If $g[n] = g[((-n))_N]$, then $G[m] = G[((-m))_N]$; If $g[n] = -g[((-n))_N]$, then $G[m] = -G[((-m))_N]$;

(7) Conjugate	$DFT\{g^*[n]\} = G^*[((-m))_N]$
(8) Real/Imaginary Input	If $g[n]$ is real, then $G[m] = G^*[((-m))_N]$; If $g[n]$ is pure imaginary, then $G[m] = -G^*[((-m))_N]$;
(9) Circular Convolution	If $y[n] = g[n] *_c h[n] = \sum_{k=0}^{N-1} g[k]h[((n-k))_N]$ then $Y[m] = G[m]H[m]$
(10) Circular Correlation	If $y[n] = g[n] *_c h^*[((-n))_N]$ $= \sum_{k=0}^{N-1} g[((k+n))_N]h^*[k]$ then $Y[m] = G[m]H^*[m]$
(11) Parseval's Theorem (Energy Preservation)	$N\sum_{n=0}^{N-1} \lvert g[n] \rvert^2 = \sum_{m=0}^{N-1} \lvert G[m] \rvert^2$
(12) Generalized Parseval's Theorem	$N\sum_{n=0}^{N-1} g[n]h^*[n] = \sum_{m=0}^{N-1} G[m]H^*[m]$

5.2.4 Discrete Circular Convolution

[Discrete Circular Convolution]

$$g[n]*_c h[n]=\sum_{k=0}^{N-1}g[k]h[((n-k))_N]$$

(Proof of the convolution property)

$$\frac{1}{N}\sum_{m=0}^{N-1}G[m]H[m]e^{j\frac{2\pi m}{N}n}=\frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1}g[k]e^{-j\frac{2\pi m}{N}k}\sum_{s=0}^{N-1}h[s]e^{-j\frac{2\pi m}{N}s}e^{j\frac{2\pi m}{N}n}$$

$$=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{s=0}^{N-1}g[k]h[s]\sum_{m=0}^{N-1}e^{j\frac{2\pi(n-s-k)}{N}m}$$

$$=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{s=0}^{N-1}g[k]h[s]N\delta_d[((n-s-k))_N]$$

$$=\sum_{k=0}^{N-1}g[k]h[((n-k))_N]$$

Here we apply $\sum_{n=0}^{N-1}e^{j\frac{2\pi a}{N}n}=N\delta_d[((a))_N]$

$((a))_N$: the remainder of a after divided by N

[Discrete Circular Convolution and Discrete Linear Convolution]

A discrete linear time-invariant (LTI) system can always be expressed a discrete linear convolution:

$$y_1[n] = g[n] * h[n] = \sum_{k=0}^{N-1} g[k]h[n-k]$$

However, the convolution implemented by the DFT is the discrete circular convolution:

If

$$y[n] = IDFT\left(DFT\{g[n]\}\, DFT\{h[n]\}\right) = IDFT\left(G[m]H[m]\right)$$

then

$$y[n] = g[n] *_c h[n] = \sum_{k=0}^{N-1} g[k]h[((n-k))_N]$$

$((a))_N$: the remainder of a
after divided by N

linear convolution: $y_1[n]=g[n]*h[n]=\sum_{k=0}^{N-1} g[k]h[n-k]$

circular convolution: $y[n]=g[n]*_c h[n]=\sum_{k=0}^{N-1} g[k]h[((n-k))_N]$

For example,

$$y_1[2]=g[0]h[2]+g[1]h[1]+g[2]h[0]+g[3]h[-1]+g[4]h[-2]+\cdots\cdots$$

$$y[2]=g[0]h[2]+g[1]h[1]+g[2]h[0]+g[3]h[N-1]+g[4]h[N-2]$$
$$+\cdots\cdots+g[N-1]h[3]$$

The condition where the circular convolution is equal to the linear convolution:

(i) $g[n]=0 \quad for\ n<0\ \ or\ \ n\geq M$

(ii) $h[n]=0 \quad for\ n<0\ \ or\ \ n\geq L$

(iii) $N\geq M+L-1$

The condition where the circular convolution is equal to the linear convolution:

(i) $g[n]=0 \quad for\ n<0\ \ or\ \ n\geq M$

(ii) $h[n]=0 \quad for\ n<0\ \ or\ \ n\geq L$

(iii) $N\geq M+L-1$

(Proof): $y[n]=\sum_{k=0}^{N-1} g[k]h[((n-k))_N]$

$$y[n]=g[0]h[n]+g[1]h[n-1]+\cdots+g[n]h[0]+g[n+1]h[N-1]+$$
$$g[n+2]h[N-2]+\cdots+g[N+n+1-L]h[L-1]+\cdots+g[N-1]h[n+1]$$
$$=g[0]h[n]+g[1]h[n-1]+\cdots+g[n]h[0]$$
$$+g[N+n+1-L]h[L-1]+\cdots+g[N-1]h[n+1]$$
$$=g[0]h[n]+g[1]h[n-1]+\cdots+g[n]h[0]=y_1[n]$$

(Since $N+n+1-L \geq N+1-L \geq M$)

5.2.5 Complexity

$$G[m] = \sum_{n=0}^{N-1} g[n] e^{-j\frac{2\pi mn}{N}}$$

Direct implementation: Complexity = $O(N^2)$

With the fast algorithm: Complexity = $O(N\log_2 N)$

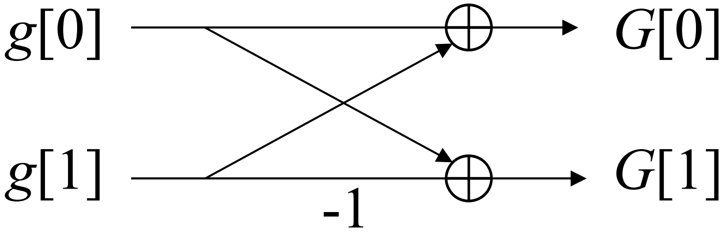
$$G[m]=\sum_{n=0}^{N-1} g[n]e^{-j\frac{2\pi mn}{N}}$$

When $N=2$

$$e^{-j\frac{2\pi}{2}mn} = e^{-j\pi mn}$$
$$= (-1)^{mn}$$

2-point DFT

$$\begin{bmatrix} G[0] \\ G[1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \end{bmatrix}$$



When $N = 2^k$

$$
\begin{aligned}
G[m] &= \sum_{n=0}^{N-1} g[n] e^{-j\frac{2\pi mn}{N}} \\
&= \sum_{n=0}^{N/2-1} g[2n] e^{-j\frac{2\pi m(2n)}{N}} + \sum_{n=0}^{N/2-1} g[2n+1] e^{-j\frac{2\pi m(2n+1)}{N}} \\
&= \sum_{n=0}^{N/2-1} g_1[n] e^{-j\frac{2\pi mn}{N/2}} + e^{-j\frac{2\pi m}{N}} \sum_{n=0}^{N/2-1} g_2[n] e^{-j\frac{2\pi mn}{N/2}}
\end{aligned}
$$

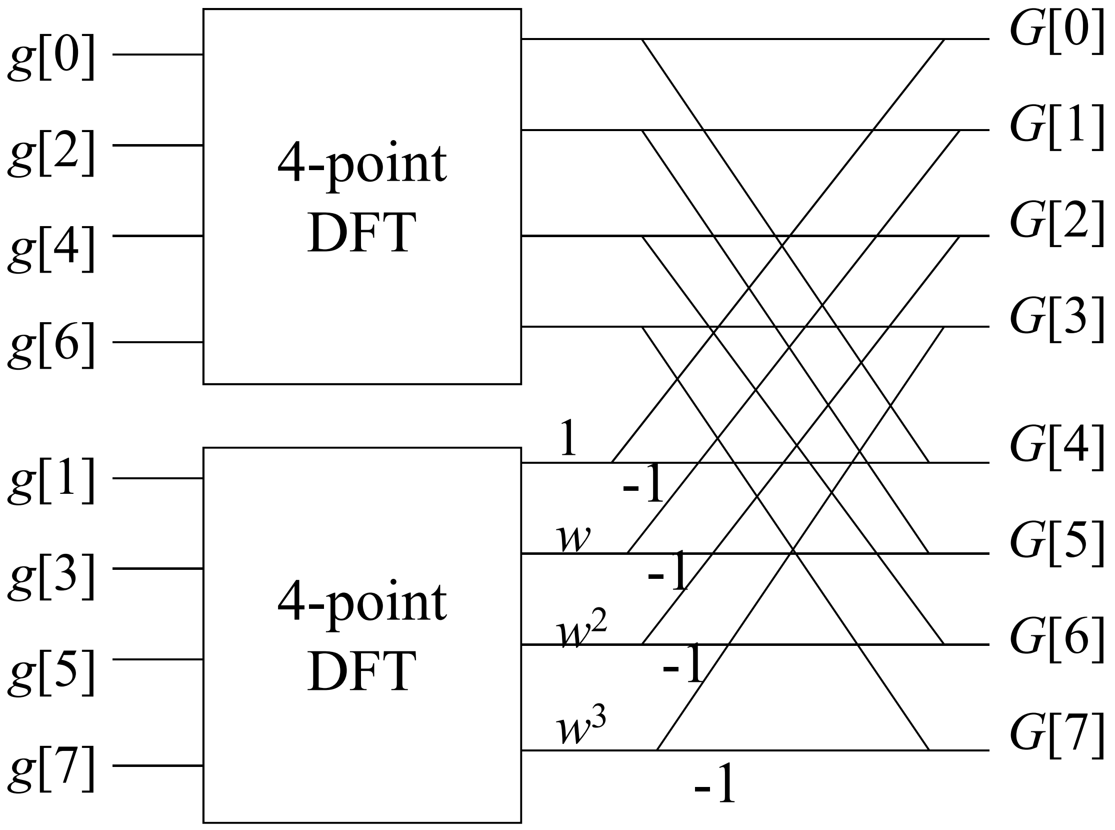
twiddle factors

$$g_1[n] = g[2n], \quad g_2[n] = g[2n+1]$$

Therefore,

one N-point DFT = two ($N/2$)-point DFT + twiddle factors

8-point DFT



When $N = 8$

$$w = e^{-j\frac{2\pi}{8}}$$

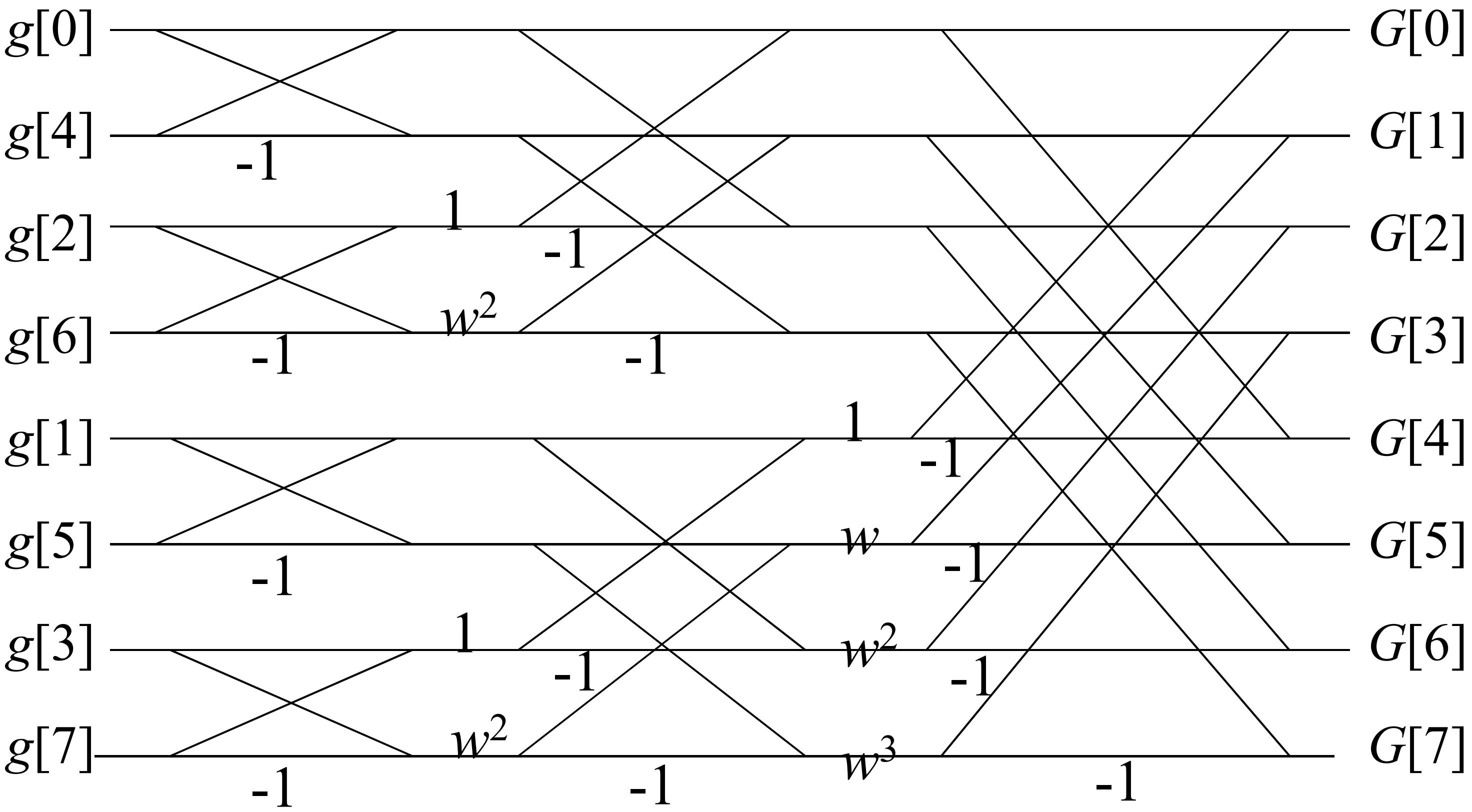
$$w^4 = -1$$

$$w^8 = 1$$

$$w = e^{-j\frac{2\pi}{N}}$$

$$w^{(m+\frac{N}{2})} = e^{-j\frac{2\pi}{N}(m+\frac{N}{2})} = e^{-j\frac{2\pi}{N}m} e^{-j\frac{2\pi}{N}\frac{N}{2}} = -e^{-j\frac{2\pi}{N}m} = -w^m$$

8-point DFT



$$w = e^{-j\frac{2\pi}{8}}$$

complexity: $N\log_2 N$

$N/2$ twiddle factors are required between two stages

Number of stages

- J. W. Cooley and J. W. Tukey, “An algorithm for the machine computation of complex Fourier series,” *Mathematics of Computation*, vol. 19, pp. 297-301, Apr. 1965. (Cooley-Tukey)
- C. S. Burrus, “Index Mappings for multidimensional formulation of the DFT and convolution,” *IEEE Trans. Acoustics, Speech, and Signal Processing*, vol. 25, pp. 1239-242, June 1977. (Prime factor)

5.2.6 2D DFTs

2-D Discrete Fourier Transform (2-D DFT)

$$G[p,q]=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}g[m,n]e^{-j\frac{2\pi mp}{M}}e^{-j\frac{2\pi nq}{N}}$$

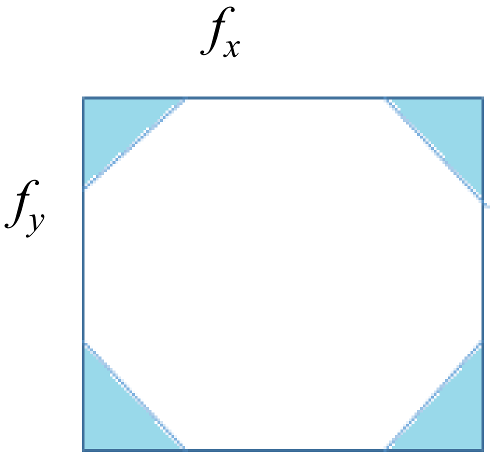
2-D Inverse Discrete Fourier Transform (2-D IDFT)

$$g[m,n]=\frac{1}{MN}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1}G[p,q]e^{j\frac{2\pi mp}{M}}e^{j\frac{2\pi nq}{N}}$$

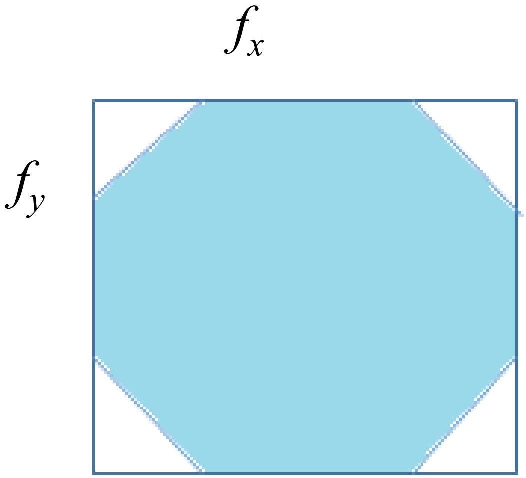
Low Frequency Part → Mild Variation → Plane

High Frequency Part → Large Variation → Edge and Noise

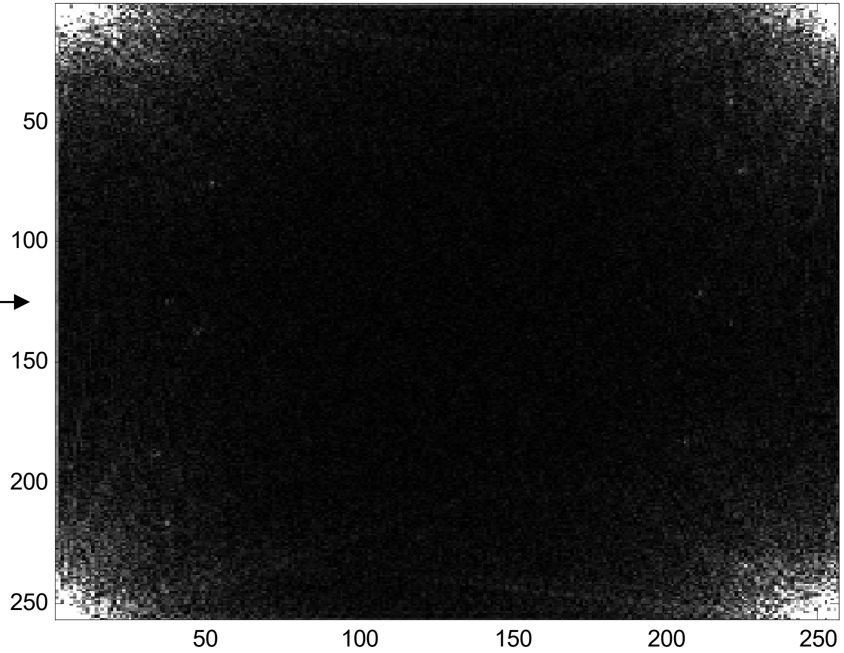
$G[m,n]$ low frequency part



high frequency part



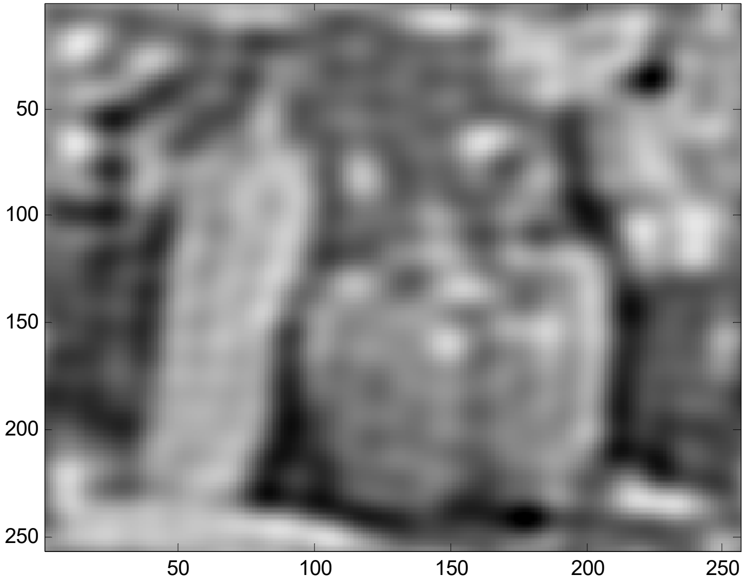
$$\xrightarrow[\text{DFT}]{\text{2D}}$$



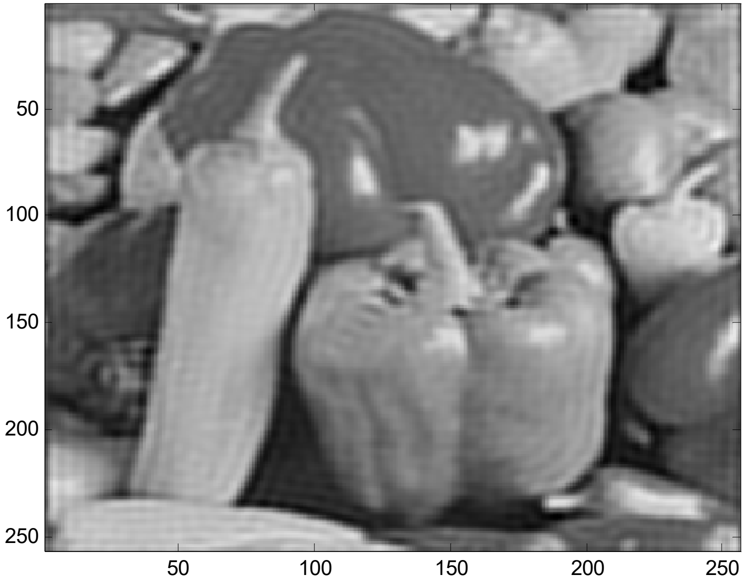
Using the gray level to show the intensity

image(…….)
colormap(gray(256))

Low Frequency Part (similar to the blurred version of the input image)

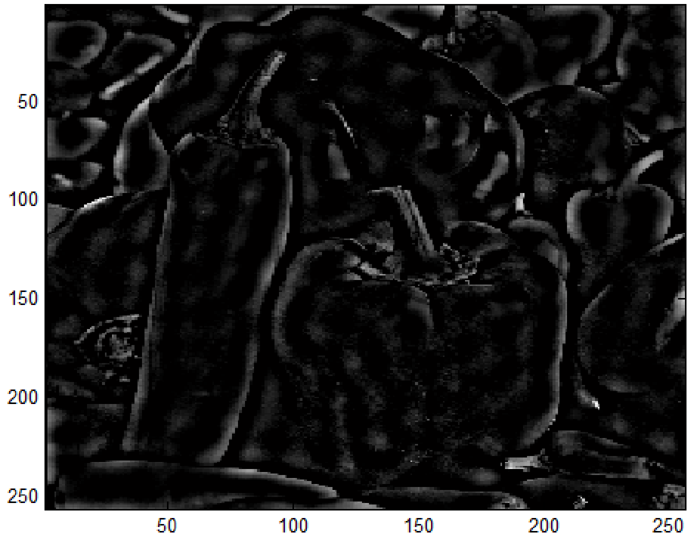


Passband: $|f_x| + |f_y| \le N/30$

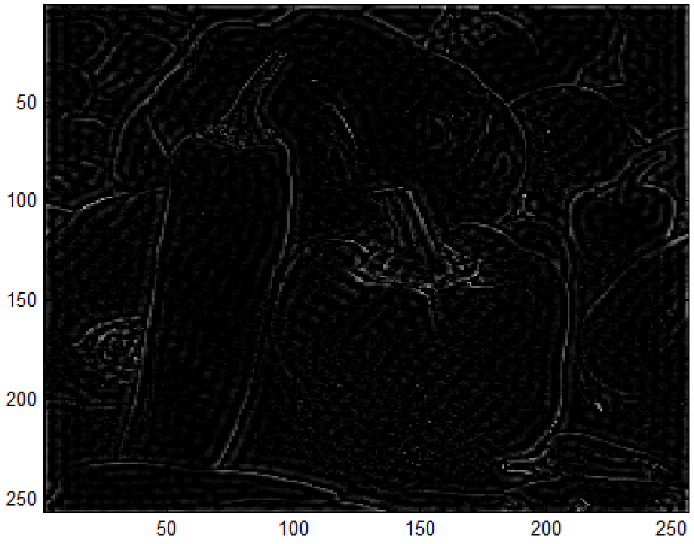


Passband: $|f_x| + |f_y| \le N/10$

High Frequency Part (similar to the edges)



Passband: $|f_x| + |f_y| > N/30$



Passband: $|f_x| + |f_y| > N/10$