(8) $(A-\lambda I)v_2 = v_1$

Example: Generalized Eigenvectors (3/3)

- Equations (7), (8), and (11) can be rewritten as

$$\mathbf{A}\mathbf{v}_1 = \lambda\mathbf{v}_1, \qquad \mathbf{A}\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1, \qquad \mathbf{A}\mathbf{v}_3 = \lambda\mathbf{v}_3 + \mathbf{v}_2, \tag{14}$$

- We obtain

$$\mathbf{A}\underbrace{\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}}_{\boldsymbol{\mathcal{V}}} = \underbrace{\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}}_{\boldsymbol{\mathcal{V}}}\underbrace{\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{bmatrix}}_{\boldsymbol{\mathcal{J}}}. \tag{15}$$

1

λ

$AV = VJ$

- Since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent, the matrix $\boldsymbol{\mathcal{V}}$ is invertible. We have

$$\mathbf{A} = \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{V}}^{-1}. \tag{16}$$

$J_k =$

- $\boldsymbol{\mathcal{J}}$ is the Jordan canonical form of $\mathbf{A}$.

$$\begin{cases} \underline{\underline{A}}\,\underline{v}_1 = \boxed{\lambda \underline{v}_1}_{3\times 1} = \boxed{\begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{bmatrix}_{3\times 3}} \begin{bmatrix} \lambda \\ 0 \\ 0 \end{bmatrix}_{3\times 1} \\[2ex] \underline{\underline{A}}\,\underline{v}_2 = \boxed{\lambda \underline{v}_2 + \underline{v}_1}_{3\times 1} = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 \\ \lambda \\ 0 \end{bmatrix}_{3\times 1} \\[2ex] \underline{\underline{A}}\,\underline{v}_3 = \boxed{\lambda \underline{v}_3 + \underline{v}_2} = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \lambda \end{bmatrix}_{3\times 1} \end{cases}$$

$$\underline{\underline{A}}\,\underline{\underline{U}} = \begin{bmatrix} \underline{\underline{A}}\,\underline{v}_1 & \underline{\underline{A}}\,\underline{v}_2 & \underline{\underline{A}}\,\underline{v}_3 \end{bmatrix}_{3\times 3} = \underline{\underline{U}} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

The Jordan Canonical Form

block diagonal

blkdiag

- We decompose the matrix $\mathbf{A} \in \mathbb{C}^{N \times N}$ into $\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{J}}\boldsymbol{\mathcal{V}}^{-1}$.
- The matrix $\boldsymbol{\mathcal{V}}$ contains the (generalized) eigenvectors.
- The Jordan canonical form $\boldsymbol{\mathcal{J}}$ of $\mathbf{A}$ is a block diagonal matrix of the form

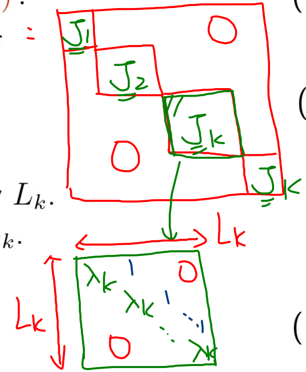
$$\boldsymbol{\mathcal{J}} = \operatorname{blkdiag}\left(\boldsymbol{\mathcal{J}}_1, \boldsymbol{\mathcal{J}}_2, \ldots, \boldsymbol{\mathcal{J}}_K\right). \tag{17}$$

- For every $k \in [K]$, the Jordan block $\boldsymbol{\mathcal{J}}_k$ has the form of

$$\boldsymbol{\mathcal{J}}_k = \underbrace{\lambda_k \mathbf{I}_{L_k}}_{\text{diagonal}} + \mathbf{U}_{L_k}, \tag{18}$$

  for some $L_k \in [N]$.
- The matrix $\mathbf{I}_{L_k}$ denotes the identity matrix of size L_k by L_k.
- The matrix $\mathbf{U}_{L_k}$ is an upper shift matrix of size L_k by L_k.
- Let $(i, j) \in [L_k]^2$. The (i, j)th entry of $\mathbf{U}_{L_k}$ is

$$\left[\mathbf{U}_{L_k}\right]_{i,j} = \delta_{i+1,j}. \tag{19}$$



"Block" Diagonal:

$$\operatorname{diag}([1, 2]) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\operatorname{blkdiag}\left(\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}, 5 \right)$$

$$= \begin{bmatrix} \boxed{\begin{matrix} 2 & 0 \\ 3 & 1 \end{matrix}} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \end{matrix} & \boxed{5} \end{bmatrix}$$

Examples: The Jordan Blocks

- If $\underline{k = 1}$ and $\boxed{L_k = 1}$, then

(18) $\quad [1] \quad [0]$

$$\mathcal{J}_1 = \lambda_1 \mathbf{I}_1 + \mathbf{U}_1 = \underline{\lambda_1}.$$

scalar

($\mathcal{J}_1$ becomes a scalar)

- If $k = 2$ and $\underline{L_k = 2}$, then

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(18)

2, 2, 2x2

$$\mathcal{J}_2 = \lambda_2 \mathbf{I}_2 + \mathbf{U}_2 = \begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix}.$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- If $k = 3$ and $\underline{L_k = 3}$, then

$$\mathcal{J}_3 = \underbrace{\lambda_3 \mathbf{I}_3} + \mathbf{U}_3 = \begin{bmatrix} \lambda_3 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example: The Jordan Canonical Form of a 4-by-4 Matrix (1/4)

≈ Eigen-decomposition "except" generalized eigenvector

- We consider the matrix $\mathbf{A}$

$\underline{\underline{V}}, \underline{\underline{J}}$

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & -2 \\ 0 & 4 & -2 & 0 \\ -1 & -1 & 4 & 0 \\ -1 & -1 & 0 & 4 \end{bmatrix}_{4\times4} \tag{20}$$

- From the characteristic equation, the eigenvalues of $\mathbf{A}$ are $\lambda = \underline{2, 4, 4, 6}$.

$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

- For $\lambda = 2$, it can be shown that $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ is an eigenvector.
- For $\lambda = 6$, it can be shown that $\begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}^T$ is an eigenvector.

$\underline{\underline{A}} - 2\underline{\underline{I}}$

Example: The Jordan Canonical Form of a 4-by-4 Matrix (2/4)

- For $\lambda = 4$, the eigenvector is assumed to be $\mathbf{v}_1 = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \end{bmatrix}^{\mathsf{T}}$.
- The equation $(\mathbf{A} - \underset{4}{\lambda}\mathbf{I})\,\mathbf{v}_1 = \mathbf{0}$ becomes

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}}_{A-4I} \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \\ \delta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{21}$$

- For $\lambda = 4$, there is only one linearly independent eigenvector:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}. \tag{22}$$

eigenvector

$c\underline{v}_1$

Example: The Jordan Canonical Form of a 4-by-4 Matrix (3/4)

- As a result, we need to find the generalized eigenvector $\mathbf{v}_2 = \begin{bmatrix} \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{bmatrix}^\mathsf{T}$.
- The equation $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v}_2 = \mathbf{v}_1$ can be expressed as

$$\begin{bmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \\ \delta_2 \end{bmatrix}}^{v_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}. \tag{23}$$

- For $\lambda = 4$, the generalized eigenvector $\mathbf{v}_2$ is

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ -1/2 \end{bmatrix}. \tag{24}$$

Example: The Jordan Canonical Form of a 4-by-4 Matrix (4/4)

- Based on the discussions on pages 15, 16, and 17, we obtain

$$\mathbf{A} = \underbrace{\mathcal{V}}_{4\times 4}\,\underbrace{\mathcal{J}}_{4\times 4}\,\mathcal{V}^{-1}, \tag{25}$$

where

$$\begin{array}{ccccc} & \text{eigenvector } \lambda=2 & & & \text{eigenvector } \lambda=6 \\ \mathcal{V} = & \left[\begin{matrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1/2 & -1 \\ 1 & 0 & -1/2 & -1 \end{matrix}\right] & & & 4\times4 \\ & & v_1 & v_2 & \end{array}, \quad \mathcal{J} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}. \tag{26}$$

J_1 J_2 J_3

$\therefore$ generalized eigenvector

$$\operatorname{blkdiag}\left(2,\ \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix},\ 6\right)$$

Example: The Jordan Canonical Form of a 5-by-5 Matrix (1/5)

- As an example, let the matrix $\mathbf{A}$ be

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 & 2 & 1 \\ 0 & 4 & 1 & -2 & 1 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix}. \quad (27)$$

5×5

- Solving the characteristic equation of $\mathbf{A}$ leads to the eigenvalues

$$\lambda = 2, \quad \boxed{4, \quad 4, \quad 4, \quad 4.} \quad (28)$$

- For $\lambda = 2$, it can be shown that $\begin{bmatrix} 0 & 0 & 1 & 0 & -1 \end{bmatrix}^T$ is an eigenvector.

Example: The Jordan Canonical Form of a 5-by-5 Matrix (2/5)

- For $\lambda = 4$, the eigenvector is assumed to be $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix}^\mathsf{T}$.
- From the equation $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{v} = \mathbf{0}$, we obtain

A-4I

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{29}$$

- For $\lambda = 4$, there are only two linearly independent solutions, denoted by ϕ_1 and ψ_1:

E-vec $\quad \phi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}, \longleftrightarrow \quad \underline{\psi_1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}. \quad (30)$

E-vec

G E-vec $\qquad\qquad$ G E-vec

Example: The Jordan Canonical Form of a 5-by-5 Matrix (3/5)

- For $\lambda = 4$ and the eigenvector $\phi_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}}$, we solve the equation $\boxed{(\mathbf{A} - \lambda \mathbf{I})\, \phi_2 = \phi_1}$ for the generalized eigenvector.
- We obtain

$\phi_1 \longrightarrow \phi_2$

$A - 4I$

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \phi_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{31}$$

- A solution to (31) is

$$\phi_2 = \begin{bmatrix} 0 & 0 & 1/4 & 1/4 & 1/4 \end{bmatrix}^{\mathsf{T}}, \tag{32}$$

where the first and the second entries of ϕ_2 are set to zero for simplicity.

Example: The Jordan Canonical Form of a 5-by-5 Matrix (4/5)

E -vec

- For $\lambda = 4$ and the eigenvector $\psi_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^\mathsf{T}$, we solve the equation $(\mathbf{A} - \lambda \mathbf{I})\, \psi_2 = \psi_1$ for the generalized eigenvector.

E -vec

- We obtain

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \psi_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{33}$$

- A solution to (33) is

$$\psi_2 = \begin{bmatrix} 0 & 0 & 1/4 & -1/4 & 1/4 \end{bmatrix}^\mathsf{T}. \tag{34}$$

Example: The Jordan Canonical Form of a 5-by-5 Matrix (5/5)

E-values $2, \underline{4, 4, 4, 4}$

- Therefore, we can decompose the matrix $\mathbf{A}$ into

vector $\Phi_1 \to \Phi_2$, $\Psi_1 \to \Psi_2$

$$\mathbf{A} = \boldsymbol{\mathcal{V}} \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{V}}^{-1}, \tag{35}$$

where

$$\boldsymbol{\mathcal{V}} = \begin{array}{c} \begin{array}{ccccc} \text{E-vec} & \text{G-E-vec} & & & \text{E-vec} \\ \Phi_1 & \Phi_2 & & & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 1 \\ 0 & 1/4 & 0 & -1/4 & 0 \\ 0 & 1/4 & 0 & 1/4 & -1 \end{bmatrix} \\ \begin{array}{ccccc} & & \Psi_1 & \to \Psi_2 & \\ & & \text{(E-vec)} & \text{(G-E-vec)} & \end{array} \end{array}, \qquad \boldsymbol{\mathcal{J}} = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{36}$$

E-val

$$\lambda = 4, 4, 4, 4$$

$$J = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{4} & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Outline

A^{20} $\qquad$ J^{20}

1. Motivations
2. Jordan Canonical Form
   - Definition and Examples
   - The Integer Power of a Matrix

$A = V J V^{-1}$

3. Singular Value Decomposition (SVD)
   - Definition and Properties
   - Matrix Norms and SVD
4. Principal Component Analysis (PCA)

The Integer Power of a Matrix

- We consider the Jordan canonical form of a matrix $\mathbf{A} \in \mathbb{C}^{N \times N}$,

$$\mathbf{A} = \boldsymbol{\mathcal{V}} \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{V}}^{-1}. \tag{37}$$

- For a non-negative integer α, the matrix power $\mathbf{A}^{\alpha}$ becomes

$$\mathbf{A}^{\alpha} = \underbrace{\left(\boldsymbol{\mathcal{V}} \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{V}}^{-1}\right)\left(\boldsymbol{\mathcal{V}} \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{V}}^{-1}\right) \cdots \left(\boldsymbol{\mathcal{V}} \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{V}}^{-1}\right)}_{\alpha \text{ terms}} \tag{38}$$

$$= \boldsymbol{\mathcal{V}} \boldsymbol{\mathcal{J}} \underbrace{\left(\boldsymbol{\mathcal{V}}^{-1} \boldsymbol{\mathcal{V}}\right)}_{\mathbf{I}} \boldsymbol{\mathcal{J}} \underbrace{\left(\boldsymbol{\mathcal{V}}^{-1} \boldsymbol{\mathcal{V}}\right)}_{\mathbf{I}} \boldsymbol{\mathcal{J}} \cdots \boldsymbol{\mathcal{J}} \boldsymbol{\mathcal{V}}^{-1} \tag{39}$$

$$= \boldsymbol{\mathcal{V}} \boldsymbol{\mathcal{J}}^{\alpha} \boldsymbol{\mathcal{V}}^{-1}. \tag{40}$$

- (Question) How do you determine $\boldsymbol{\mathcal{J}}^{\alpha}$?

The Power of $\mathcal{J}$

$$\underline{J} = \begin{bmatrix} \underline{J_1} & & & \\ & \underline{J_2} & & \\ & & \ddots & \\ & & & \underline{J_K} \end{bmatrix}$$

- From (17), we obtain

$$\mathcal{J}^{\alpha} = \operatorname{blkdiag}\left(\mathcal{J}_1^{\alpha}, \mathcal{J}_2^{\alpha}, \ldots, \mathcal{J}_K^{\alpha}\right). \tag{41}$$

- After dropping the subscript L_k in (18) for simplicity, we rewrite the matrix $\mathcal{J}_k^{\alpha}$ as

$$\mathcal{J}_k^{\alpha} = (\lambda_k \mathbf{I} + \mathbf{U})^{\alpha} = \sum_{\ell=0}^{\alpha} \binom{\alpha}{\ell} (\lambda_k \mathbf{I})^{\alpha-\ell} \mathbf{U}^{\ell} \tag{42}$$

$$= \sum_{\ell=0}^{\alpha} \binom{\alpha}{\ell} \lambda_k^{\alpha-\ell} \mathbf{U}^{\ell}. \tag{43}$$

scalar

- (Cross reference) The binomial expansion for scalars

C^N_n

$$(x+y)^N = \sum_{n=0}^{N} \binom{N}{n} x^{N-n} y^n, \qquad \binom{N}{n} = \frac{N!}{(N-n)!n!}. \tag{44}$$

$$\underline{\underline{J}} = \begin{array}{c} \\ 2 \\ 5 \end{array}\!\!\begin{array}{cc} 2 & 5 \\ \multicolumn{2}{c}{\left[\begin{array}{cc} \underline{\underline{J_1}} & 0 \\ \underline{\underline{0}} & \underline{\underline{J_2}} \end{array}\right]} \end{array}$$

$$\underline{\underline{J}}^2 = \begin{bmatrix} \underline{\underline{J_1}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{J_2}} \end{bmatrix} \begin{bmatrix} \underline{\underline{J_1}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{J_2}} \end{bmatrix}$$

$$= \begin{array}{c} \\ 2 \\ 5 \end{array}\!\!\begin{array}{cc} 2 & 5 \\ \multicolumn{2}{c}{\left[\begin{array}{cc} \underline{\underline{J_1}}^2 & \underline{\underline{J_1}}\,\underline{\underline{0}} + \underline{\underline{0}}\,\underline{\underline{J_2}} = \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{J_2}}^2 \end{array}\right]} \end{array}$$

$$(\lambda \underline{\underline{I}} + \underline{\underline{U}})^2 = (\lambda \underline{\underline{I}} + \underline{\underline{U}})(\lambda \underline{\underline{I}} + \underline{\underline{U}})$$

$$= \lambda^2 \underline{\underline{I}}\,\underline{\underline{I}} + \lambda \underline{\underline{I}}\,\underline{\underline{U}} + \lambda \underline{\underline{U}}\,\underline{\underline{I}} + \underline{\underline{U}}^2$$

$$= \lambda^2 \underline{\underline{I}} + 2\lambda \underline{\underline{U}} + \underline{\underline{U}}^2$$

$$(\lambda \underline{\underline{I}} + \underline{\underline{U}})^3 = \lambda^3 \underline{\underline{I}} + 3\lambda^2 \underline{\underline{U}} + 3\lambda \underline{\underline{U}}^2 + \underline{\underline{U}}^3$$

$$=$$

Examples of the Powers of U

J^{α} $\underline{\underline{J}}^{k}$ $\underline{\underline{U}}^{\ell}$

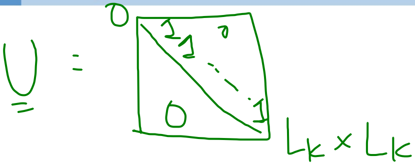
- For instance, we assume that $\mathbf{U} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. 5×5
- The powers of $\mathbf{U}$ are

$$\mathbf{U}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{U}^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{U}^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- It can be shown that $\mathbf{U}^{\ell} = \mathbf{0}$ for $\ell \geq 5$.

$\underline{\underline{U}}^5 = \underline{\underline{0}}$ $\quad \underline{\underline{U}}^6 = \underline{\underline{0}}$

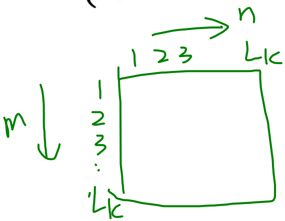
The General Form of $\mathbf{U}^\ell$



- If $\ell < L_k$, then $\mathbf{U}_{L_k}^\ell$ satisfies

$U^1, U^2, \ldots$

$$\left[\mathbf{U}_{L_k}^\ell\right]_{m,n} = \delta_{n-m,\ell} = \begin{cases} 1, & \text{if } \boxed{n - m = \ell,} \\ 0, & \text{otherwise.} \end{cases} \tag{45}$$



- If $\ell \geq L_k$, then $\mathbf{U}_{L_k}^\ell = \mathbf{0}$.

The General Form of $\mathcal{J}_k^\alpha$

Powers of a Jordan block

The kth eigenvalue is denoted by λ_k. Let α be a non-negative integer. Let $\mathcal{J}_k$ be the kth Jordan block. Then

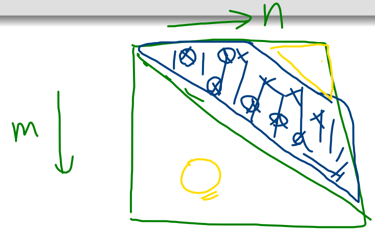
λ_1^α

$m=1,\ n=2$

$n-m=1$

$$[\mathcal{J}_k^\alpha]_{m,n} = \begin{cases} \lambda_k^\alpha, & \text{if } m = n, \\ \binom{\alpha}{n-m}\lambda_k^{\alpha-n+m}, & \text{if } n > m \text{ and } \alpha \geq n-m, \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

$\alpha=3$



$\binom{\alpha}{1}\lambda_k^{3-2+1}$

$\binom{\alpha}{1}\lambda_k^{2}$

An Example of $\mathcal{J}_k^\alpha$

$$J_1 = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ & \lambda_1 & 1 & 0 & 0 \\ & & \lambda_1 & 1 & 0 \\ 0 & & & \lambda_1 & 1 \\ & & & & \lambda_1 \end{bmatrix}$$

- We assume that $k = 1$, $L_k = 5$, and $\alpha = 3$
- Then

$n \longrightarrow$

$m \downarrow$

J_1^{100}

$$\mathcal{J}_1^3 = \begin{array}{c} \\ m=1 \\ m=2 \\ m=3 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} n=1 & n=2 & n=3 & n=4 & \end{array} \\ \begin{bmatrix} \lambda_1^3 & \binom{\alpha}{1}\lambda_1^2 & \binom{\alpha}{2}\lambda_1^1 & \binom{\alpha}{3}\lambda_1^0 & 0 \\ 0 & \lambda_1^3 & \binom{\alpha}{1}\lambda_1^2 & \binom{\alpha}{2}\lambda_1^1 & \binom{\alpha}{3}\lambda_1^0 \\ 0 & 0 & \lambda_1^3 & \binom{\alpha}{1}\lambda_1^2 & \binom{\alpha}{2}\lambda_1^1 \\ 0 & 0 & 0 & \lambda_1^3 & \binom{\alpha}{1}\lambda_1^2 \\ 0 & 0 & 0 & 0 & \lambda_1^3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow U^3 \\ \leftarrow U^2 \\ \leftarrow U^1 \\ \leftarrow \lambda_1^\alpha \end{array}.$$

Example: The Power of a 4-by-4 Matrix (1/2)

- Find the matrix power $\mathbf{A}^5$, where

$$\underset{=}{V}\,\underset{=}{J}\,\underset{=}{V}^{-1} = \mathbf{A} = \begin{bmatrix} 4 & 0 & 0 & -2 \\ 0 & 4 & -2 & 0 \\ -1 & -1 & 4 & 0 \\ -1 & -1 & 0 & 4 \end{bmatrix}_{4\times 4}. \tag{47}$$

- According to the example on pages 15 to 18, the matrix power $\mathbf{A}^5$ becomes

$$\mathbf{A}^5 = \boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{J}}^5\boldsymbol{\mathcal{V}}^{-1} = \boldsymbol{\mathcal{V}}\mathrm{blkdiag}\left(\boldsymbol{\mathcal{J}}_1^5, \boldsymbol{\mathcal{J}}_2^5, \boldsymbol{\mathcal{J}}_3^5\right)\boldsymbol{\mathcal{V}}^{-1}. \tag{48}$$

- The Jordan blocks are

$$\boldsymbol{\mathcal{J}}_1 = 2, \qquad \boldsymbol{\mathcal{J}}_2 = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \qquad \boldsymbol{\mathcal{J}}_3 = 6, \tag{49}$$

Example: The Power of a 4-by-4 Matrix (2/2)

- The powers of the Jordan blocks can be expressed as

$$\mathcal{J}_1^5 = 2^5 = 32, \tag{50}$$

$$\mathcal{J}_2^5 = \begin{bmatrix} 4^5 & \binom{5}{1} \times 4^4 \\ 0 & 4^5 \end{bmatrix} = \begin{bmatrix} 1024 & 1280 \\ 0 & 1024 \end{bmatrix}, \tag{51}$$

$$\mathcal{J}_3^5 = 6^5 = 7776. \tag{52}$$

- Substituting (50), (50), and (50) into (48) yields

$$\mathbf{A}^5 = \begin{bmatrix} 2464 & 1440 & -656 & -3216 \\ 1440 & 2464 & -3216 & -656 \\ -1936 & -1936 & 2464 & 1440 \\ -1936 & -1936 & 1440 & 2464 \end{bmatrix}. \tag{53}$$

Outline

EVD

$\begin{cases} \text{E-vec} \\ \text{E-val.} \end{cases}$

Outline

EVD.

The Eigen-Decomposition of Hermitian Matrices

EVD

$A v = \lambda v$

- Let $\mathbf{A} \in \mathbb{C}^{N\times N}$ and $\mathbf{A}^{\mathsf{H}} = \mathbf{A}$ (Hermitian matrices).
- The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ are real numbers. $\in \mathbb{R}$
- After normalization, the set of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{N-1}, \mathbf{v}_N\}$ is complete and orthonormal.

  $(V^H = V^{-1})$

- The eigen-decomposition of a Hermitian matrix $\mathbf{A}$ is

$[v_1 \; v_2 \; \cdots \; v_N]$ $\quad V^{-1}$

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{\mathsf{H}} = \boxed{\sum_{n=1}^{N} \lambda_n \mathbf{v}_n \mathbf{v}_n^{\mathsf{H}}}. \tag{54}$$

$N \times N$ $\quad \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}$

(N components) N terms $\boxed{\lambda_n v_n v_n^H}$

A_j SVD

Motivating Questions

1. How do we extend the decomposition to M-by-N (non-square) matrices?

The Singular Value Decomposition [HJ2013, pp. 150], [GVL2013, pp. 76]

M N M N M N

- We assume that $\mathbf{A} \in \mathbb{C}^{M\times N}$, $q = \min\{M, N\}$, and $\operatorname{rank}(\mathbf{A}) = r$. 3
- There are unitary matrices $\mathbf{U} \in \mathbb{C}^{M\times M}$ and $\mathbf{V} \in \mathbb{C}^{N\times N}$, and a square diagonal matrix

$U^H U = I$ $V^H V = I$

$$\boldsymbol{\Sigma}_q = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_q). \tag{55}$$

σ_1 0 0 σ_q $q \times q$

such that

σ_1 σ_2 σ_3 > 0

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_q, \tag{56}$$

Given

N M

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{H}}, \qquad \boldsymbol{\Sigma} = \begin{cases} \boldsymbol{\Sigma}_q \in \mathbb{R}^{M\times N} & \text{if } M = N, \\ \begin{bmatrix} \boldsymbol{\Sigma}_q & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{M\times N} & \text{if } M < N, \\ \begin{bmatrix} \boldsymbol{\Sigma}_q \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{M\times N} & \text{if } M > N, \end{cases} \tag{57}$$

M M

10 5

$$\underline{\underline{A}} = \begin{array}{c} 5 \\ 10 \left[\quad\quad\quad \right] \end{array} = \begin{array}{c} 10 \\ 10 \left[\;\underline{\underline{U}}\; \right] \end{array} \begin{array}{c} 5 \\ 10 \left[\;\underline{\underline{\Sigma}}\; \right] \end{array} \begin{array}{c} 5 \\ 5 \left[\;\underline{\underline{V}}^H\; \right] \end{array}$$

$$\begin{cases} M = 10 \\ N = 5 \end{cases}$$

$$\begin{array}{c} 5 \\ 10 \left[\begin{array}{ccccc} \sigma_1 & & & & \\ & \sigma_2 & & \underline{\underline{0}} & \\ & & \sigma & & \\ & \underline{\underline{0}} & & \sigma & \\ & & & & \sigma_5 \\ \hdashline & & \underline{\underline{0}} & & \end{array} \right] \end{array}$$

$$\begin{array}{c} 5 \\ 5 \left[\;\underline{\underline{V}}\; \right] \end{array}$$

Terminologies

≥ 0

- The scalars $\sigma_1, \sigma_2, \ldots, \sigma_q$ are the singular values of $\mathbf{A}$.
- The largest singular value of $\mathbf{A}$ is denoted by $\sigma_{\max}(\mathbf{A}) = \sigma_1$.
- Let

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_M \end{bmatrix} \in \mathbb{C}^{M \times M}. \tag{58}$$

  The column vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M$ are the left singular vectors of $\mathbf{A}$.
- Let

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_N \end{bmatrix} \in \mathbb{C}^{N \times N}. \tag{59}$$

  The column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ are the right singular vectors of $\mathbf{A}$.

EVD

(Hermitian matrices)

$\square_{M \times M}$

$$\underline{\underline{A}} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_M \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_M \end{bmatrix} \underbrace{\begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_M \end{bmatrix}^H}_{\text{unitary}}$$

SVD

$\underline{\underline{A}} \in \mathbb{C}^{M \times N}$

$$\underline{\underline{A}} = \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_M \end{bmatrix}}_{\underline{\underline{U}}\ \text{unitary}} \underbrace{\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & \\ & & \ddots \\ & & \sigma_N \\ \hline & 0 & \end{bmatrix}}_{\underline{\underline{\Sigma}}} \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}^H}_{\underline{\underline{V}}^H\ \text{unitary}}$$

An Example of the SVD

$A A^H$

$A^H A$

- It can be verified that

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \\ -1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}}_{\mathbf{U}\ 3\times 3} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}}_{\mathbf{\Sigma}\ 3\times 2} \underbrace{\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^{\mathsf{H}}}_{\mathbf{V}^{\mathsf{H}}\ 2\times 2}$$

??

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{8} & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^{\mathsf{H}}.$$

- (Questions) How do we find the singular values and singular vectors for a matrix $\mathbf{A}$?

$\sigma_1 \sim \sigma_q$ $\quad u_1 \sim u_M$ $\quad v_1 \sim v_N$

SVD and Eigen-Decompositions (1/2)

$\underline{U}\,\underline{\Sigma}\,\underbrace{\underline{V}^H\,\underline{V}}_{=I}\,\underline{\Sigma}^H\,\underline{U}$

- Assume that $\boxed{\mathbf{A} = \mathbf{U\Sigma V}^{\mathsf{H}}}$ is the SVD of $\mathbf{A} \in \mathbb{C}^{M\times N}$.
- The matrix $\boxed{\mathbf{AA}^{\mathsf{H}}}$ can be expressed as

Hermitian

$$\begin{aligned} \mathbf{AA}^{\mathsf{H}} &= \mathbf{U\Sigma V}^{\mathsf{H}} (\mathbf{U\Sigma V}^{\mathsf{H}})^{\mathsf{H}} \\ &= \boxed{\mathbf{U}(\mathbf{\Sigma\Sigma}^{\mathsf{H}})\mathbf{U}^{\mathsf{H}}.} \end{aligned} \tag{60}$$

→ EVD

M×M

- Remarks on (60):
  - The left singular vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_M$ are the eigenvectors of $\mathbf{AA}^{\mathsf{H}}$.
  - The matrix $\mathbf{\Sigma\Sigma}^{\mathsf{H}}$ contains the eigenvalues of $\mathbf{AA}^{\mathsf{H}}$.

EVD on $\underline{\underline{A}}\,\underline{\underline{A}}^H$

SVD and Eigen-Decompositions (2/2)

$A \in \mathbb{C}^{M \times N}$

- Similarly, the matrix $\boxed{\mathbf{A}^{\mathsf{H}}\mathbf{A}}$ can be expressed as

$N \times N$

Hermitian

$$\begin{aligned} \mathbf{A}^{\mathsf{H}}\mathbf{A} &= \left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{H}}\right)^{\mathsf{H}}\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{H}} \\ &= \mathbf{V}\left(\boldsymbol{\Sigma}^{\mathsf{H}}\boldsymbol{\Sigma}\right)\mathbf{V}^{\mathsf{H}}. \end{aligned} \tag{61}$$

- Remarks on (61):
  - The right singular vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N$ are the eigenvectors of $\mathbf{A}^{\mathsf{H}}\mathbf{A}$.
  - The matrix $\boldsymbol{\Sigma}^{\mathsf{H}}\boldsymbol{\Sigma}$ contains the eigenvalues of $\mathbf{A}\mathbf{A}^{\mathsf{H}}$.
- How do we find both the left and right singular vectors?

(60)
E-val: 3, E-vec .. → left singular vectors

(61): 3, E-vec → right singular vectors

Relations among $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$ (1/2)

A Property rephrased from [GVL2013, Corollary 2.4.2]

If $\boxed{\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathsf{H}}}$ is the SVD of $\mathbf{A} \in \mathbb{C}^{M\times N}$ and $M \geq N$, then for $i \in [N]$, we have

$$AV = U\Sigma(\underbrace{V^H V}_{=I}) \qquad \boxed{\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i,} \qquad \underline{\mathbf{A}^{\mathsf{H}}\mathbf{u}_i = \sigma_i \mathbf{v}_i.} \tag{62}$$

- Proof sketch (1/2): We rewrite the SVD as $\underline{\mathbf{A}\mathbf{V} = \mathbf{U}\mathbf{\Sigma}}$, which is

$$\mathbf{A}\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_N\end{bmatrix}$$

$$A v_1 = \sigma_1 u_1$$

$$= \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_N & \mathbf{u}_{N+1} & \dots & \mathbf{u}_M\end{bmatrix}\begin{bmatrix}\sigma_1 & 0 & \dots & 0\\ 0 & \sigma_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \sigma_N\\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0}\end{bmatrix}.$$

Relations among $\mathbf{U}$, $\boldsymbol{\Sigma}$, and $\mathbf{V}$ (2/2)

- Proof sketch (2/2): The SVD of $\mathbf{A}^{\mathsf{H}}$ can be expressed as

$$\mathbf{A}^{\mathsf{H}} = \left(\mathbf{U}\begin{bmatrix}\boldsymbol{\Sigma}_q \\ \mathbf{0}\end{bmatrix}\mathbf{V}^{\mathsf{H}}\right)^{\mathsf{H}} \tag{63}$$

$$= \mathbf{V}\begin{bmatrix}\boldsymbol{\Sigma}_q^{\mathsf{H}} & \mathbf{0}^{\mathsf{H}}\end{bmatrix}\mathbf{U}^{\mathsf{H}}. \tag{64}$$

$$= \mathbf{V}\begin{bmatrix}\cancel{\boldsymbol{\Sigma}_q} & \mathbf{0}\end{bmatrix}\mathbf{U}^{\mathsf{H}}. \tag{65}$$

  Comparing the columns of $\mathbf{A}^{\mathsf{H}}\mathbf{U} = \mathbf{V}\begin{bmatrix}\boldsymbol{\Sigma}_q & \mathbf{0}\end{bmatrix}$ shows the second equation in (62).

- Remarks on (65):
  - The matrices $\mathbf{A}$ and $\mathbf{A}^{\mathsf{H}}$ have the same singular values.
  - The left singular vectors of $\mathbf{A}$ become the right singular vectors of $\mathbf{A}^{\mathsf{H}}$.

Computation of the Singular Vectors

- If $\sigma_i \neq 0$, then (62) can be rewritten as

$$\mathbf{u}_i = \frac{\mathbf{A}\mathbf{v}_i}{\sigma_i}, \tag{66}$$

$$\mathbf{v}_i = \frac{\mathbf{A}^{\mathsf{H}}\mathbf{u}_i}{\sigma_i}. \tag{67}$$

- Implications of (66) and (67)
  - If the matrix $\mathbf{A}$, the non-zero singular values, and one set of singular vectors are provided, we can uniquely determine another set of singular vectors.

An Example of the SVD (1/3)

- Consider the matrix $\mathbf{A}$ on page 38. We obtain

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{A}\mathbf{A}^{\mathsf{H}} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

3x3

EVD on AA^H

- The characteristic equation

$$\det\left(\mathbf{A}\mathbf{A}^{\mathsf{H}} - \lambda \mathbf{I}\right) = -(\lambda - 8)(\lambda - 4)\lambda = 0.$$

- The eigenvalues and eigenvectors are

$$\lambda_1(\mathbf{A}\mathbf{A}^{\mathsf{H}}) = 8, \qquad \lambda_2(\mathbf{A}\mathbf{A}^{\mathsf{H}}) = 4, \qquad \lambda_3(\mathbf{A}\mathbf{A}^{\mathsf{H}}) = 0, \tag{68}$$

$\sigma_1 = \sqrt{8}$ $\sigma_2 = 2$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \qquad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}. \tag{69}$$

An Example of the SVD (2/3)

$$\begin{bmatrix} |\sigma_1|^2 & 0 & 0 \\ 0 & |\sigma_2|^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- From the definition of SVD on page 36, we obtain

$$\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1^* & 0 & 0 \\ 0 & \sigma_2^* & 0 \end{bmatrix}$$

$$\boxed{\sigma_1 \geq \sigma_2 \geq 0}$$

$$\mathbf{\Sigma} = \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}}_{3\times 2}. \tag{70}$$

- According to (60), the matrix $\mathbf{\Sigma}\mathbf{\Sigma}^{\mathsf{H}}$ contains the eigenvalues of $\mathbf{A}\mathbf{A}^{\mathsf{H}}$.

$$\mathbf{\Sigma}\mathbf{\Sigma}^{\mathsf{H}} = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{71}$$

$$\begin{cases} \sigma_1^2 = 8 \\ \sigma_2^2 = 4 \end{cases}$$

- Since $\sigma_1, \sigma_2 \geq 0$, we obtain

$$\sigma_1 = \sqrt{8}, \qquad \sigma_2 = 2. \tag{72}$$

An Example of the SVD (3/3)

- Substituting (69) and (72) into (67) yields

$$\mathbf{v}_1 = \frac{\mathbf{A}^{\mathsf{H}}\mathbf{u}_1}{\sigma_1} = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 & -1 & -1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \tag{73}$$

$$\mathbf{v}_2 = \frac{\mathbf{A}^{\mathsf{H}}\mathbf{u}_2}{\sigma_2} = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}. \tag{74}$$

Outline

1. Motivations
2. Jordan Canonical Form
   - Definition and Examples
   - The Integer Power of a Matrix
3. Singular Value Decomposition (SVD)
   - Definition and Properties
   - Matrix Norms and SVD
4. Principal Component Analysis (PCA)

$$\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{\Sigma}}\,\underline{\underline{V}}^H$$

$[\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_M]$

$\sigma_1, \sigma_2, \ldots, \sigma_q$

$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_N$

vector: l_1-norm, l_2-norm, l_∞-norm

matrix: Frobenius norm

Induced norm.

The Operator Norm [GVL2013, pp. 72]

Entrywi

- The operator norm $\|\mathbf{A}\|_{\alpha,\beta}$ is defined as

"max"

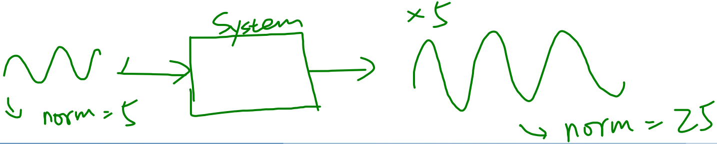
After operation

$$\|\mathbf{A}\|_{\alpha,\beta} \triangleq \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_{\beta}}{\|\mathbf{x}\|_{\alpha}}. \tag{75}$$

Before operation

- $\|\cdot\|_{\alpha,\beta}$ is subordinate to the vector norms $\|\cdot\|_{\alpha}$ and $\|\cdot\|_{\beta}$.

$$\underline{x} \xrightarrow{\text{operator}} \underline{\underline{A}}\,\underline{x}$$



The Matrix p-Norm

- By setting $\alpha = \beta = p$, the matrix p-norm is defined as

$$\|\mathbf{A}\|_p \triangleq \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}. \tag{76}$$

- According to (76), it can be shown that [HJ2013, pp. 344-345], [GVL2013, pp. 72]:

$$\|\mathbf{A}\|_1 = \max_{1 \leq j \leq N} \sum_{i=1}^{M} \left| [\mathbf{A}]_{i,j} \right|, \tag{77}$$

$$\|\mathbf{A}\|_\infty = \max_{1 \leq i \leq M} \sum_{j=1}^{N} \left| [\mathbf{A}]_{i,j} \right|. \tag{78}$$

- If $p = 2$, then $\|\mathbf{A}\|_2$ is the matrix 2-norm of $\mathbf{A}$.

(77)

$$\underline{A} = \begin{bmatrix} \boxed{\begin{matrix}1\\0\end{matrix}} & \boxed{\begin{matrix}3\\1\end{matrix}} & \boxed{\begin{matrix}5\\2\end{matrix}} \end{bmatrix}$$

$$\begin{array}{ccc} 1 & 4 & 7 \end{array}$$

$$\begin{cases} M=2 \\ N=3 \end{cases}$$

$j=1$

$|A_{1,1}| + |A_{2,1}|$

$$\max_{1\le j\le N} \sum_{i=1}^{M} |A_{i,j}|$$

$$= \max_{1\le j\le N} \underbrace{|A_{1,j}| + |A_{2,j}| + \dots + |A_{M,j}|}_{\shortparallel}$$

$$= \max\{1, 4, 7\} = 7$$

The Matrix Norms and the Singular Values

- Assume that $\mathbf{A} \in \mathbb{C}^{M \times N}$ has singular values (c.f. page 36)

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_q. \tag{79}$$

- Then, the matrix 2-norm and the Frobenius norm of $\mathbf{A}$ satisfy [GVL2013, pp. 77]:

$$\|\mathbf{A}\|_2 = \sigma_1, \tag{80}$$

$$\|\mathbf{A}\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_q^2}. \tag{81}$$

The Interpretation of Matrix Norms

- The matrix $\mathbf{A}$ is mapped to a vector $\boldsymbol{\sigma}$

singular values

$$\boldsymbol{\sigma} \triangleq \begin{bmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_r & \sigma_{r+1} & \dots & \sigma_q \end{bmatrix}^\mathsf{T}. \tag{82}$$

- Then, the matrix 2-norm and the Frobenius norm of $\mathbf{A}$ satisfy

$$\underbrace{\|\mathbf{A}\|_2}_{\text{matrix 2-norm}} = \underbrace{\|\boldsymbol{\sigma}\|_\infty}_{\text{vector } \infty\text{-norm}}, \quad = \sigma_1 \tag{83}$$

$$\underbrace{\|\mathbf{A}\|_F}_{\text{Frobenius norm}} = \underbrace{\|\boldsymbol{\sigma}\|_2}_{\text{vector 2-norm}}. \quad = \sqrt{\sigma_1^2 + \dots \sigma_q^2} \tag{84}$$

The Rank of a Matrix

- Based on the vector $\boldsymbol{\sigma}$, the rank of a matrix $\mathbf{A}$ satisfies

$$\operatorname{rank}(\mathbf{A}) = \underbrace{\|\boldsymbol{\sigma}\|_0}_{\ell_0 \text{ function}} = \operatorname{card}(\operatorname{supp}(\boldsymbol{\sigma})). \quad (85)$$

- The rank of $\mathbf{A}$ is the number of non-zero singular values.
- Low-rank optimization in signal processing

The Nuclear Norm

- Based on the vector $\boldsymbol{\sigma}$, the nuclear norm of a matrix $\mathbf{A}$ is defined as

$$\|\mathbf{A}\|_* = \underbrace{\|\boldsymbol{\sigma}\|_1}_{\text{vector 1-norm}} = \sum_{i=1}^{q} \sigma_i. \tag{86}$$

- The nuclear norm is viewed as a convex surrogate of the rank function.