Selected Topics in Engineering Mathematics

Homework Assignment #5

(Due: June 18th)

Problems

1. (10 points) The matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 62/25 & 38/25 & 9/25 \\ 16/25 & -16/25 & 62/25 \end{bmatrix}. \tag{1}$$

Find the Jordan canonical form $\mathcal{J}$ of the matrix $\mathbf{A}$. The diagonal elements of $\mathcal{J}$ are sorted in the descending order:

$$[\mathcal{J}]_{1,1} \geq [\mathcal{J}]_{2,2} \geq [\mathcal{J}]_{3,3}\,. \tag{2}$$

2. (10 points) Let $\mathbf{X} \in \mathbb{C}^{2\times 2}$. Solve the system of equations

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \mathbf{X} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} + \mathbf{X} = \begin{bmatrix} 0 & 8 \\ -1 & 22 \end{bmatrix}. \tag{3}$$

for the matrix $\mathbf{X}$.
Hint: Take the vectorization operator $\mathrm{vec}(\cdot)$ on both sides of (3).

3. (10 points) The matrix $\mathbf{B}$ is defined as

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}. \tag{4}$$

Find the matrix power $\mathbf{B}^{10}$.

4. (20 points) Let $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{H}}$ be the SVD of the matrix $\mathbf{A}$. We assume that $[\mathbf{U}]_{1,n} > 0$ for all n. Determine the matrices $\mathbf{U}$, $\boldsymbol{\Sigma}$, and $\mathbf{V}$ for the following matrices:

   (a) (10 points)

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}. \tag{5}$$

(b) (10 points)

$$\mathbf{A} = \begin{bmatrix} 1 \\ \jmath \end{bmatrix}, \tag{6}$$

where $\jmath = \sqrt{-1}$.

5. (10 points) We consider the matrix $\mathbf{M}$ as follows:

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{7}$$

Determine the matrix p-norms

$$\|\mathbf{M}\|_1, \quad \|\mathbf{M}\|_2, \quad \|\mathbf{M}\|_\infty, \tag{8}$$

and the nuclear norm

$$\|\mathbf{M}\|_*. \tag{9}$$

6. (10 points) Let the data vectors be

$$\mathbf{x}_m \triangleq \begin{bmatrix} m & \sin\left(\frac{m\pi}{4M}\right) \end{bmatrix}, \tag{10}$$

where $m = 1, 2, \ldots, M$. We suppose that $M = 20$. We can derive a regression line in $\mathbb{R}^2$ based on (10) and the PCA with $L = 1$. Use MATLAB or Python to plot these data vectors and the this regression line. The code should be handed out by NTUCool.
Hint: The horizontal axis is $[\mathbf{x}]_1$ and the vertical axis is $[\mathbf{x}]_2$.

7. (10 points) Let $\mathbf{x} \in \mathbb{C}^2$. We consider the system of equations

$$\begin{bmatrix} 1 & 0 \\ \jmath & \jmath \\ 0 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \tag{11}$$

where $\jmath = \sqrt{-1}$. Determine the LS solution $\mathbf{x}_{\mathrm{LS}}$ to (11).

8. (10 points) The matrix $\mathbf{A}$ is defined as

$$\mathbf{A} \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}. \tag{12}$$

Determine the pseudo-inverse of $\mathbf{A}$.

9. (10 points) Find the optimal solution to the optimization problem

$$\underset{\mathbf{x} \in \mathbb{C}^3}{\text{minimize}} \quad \|\mathbf{x}\|_2^2 \tag{13}$$

$$\text{subject to} \quad 3 + \begin{bmatrix} 2 & \jmath & 1 \end{bmatrix} \mathbf{x} = 0, \tag{14}$$

where $\jmath = \sqrt{-1}$.