III. Gabor Transform

III-A Definition

Standard Definition:

$$G_x(t,f)=\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau$$

Alternative Definitions:

$$G_{x,1}(t,f)=\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f(\tau-\frac{t}{2})} x(\tau)\,d\tau$$

$$G_{x,2}(t,f)=\sqrt[4]{2}\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau$$ ← normalization

$$G_{x,3}(t,\omega)=\int_{-\infty}^{\infty} e^{-(\tau-t)^2/2} e^{-j\omega\tau} x(\tau)\,d\tau$$

$$G_{x,4}(t,\omega)=\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(\tau-t)^2}{2}} e^{-j\omega(\tau-\frac{t}{2})} x(\tau)\,d\tau$$

Main Reference

• S. Qian and D. Chen, Sections 3-2 ~ 3-6 in *Joint Time-Frequency Analysis: Methods and Applications*, Prentice-Hall, 1996.

Other References

• D. Gabor, "Theory of communication", *J. Inst. Elec. Eng.*, vol. 93, pp. 429-457, Nov. 1946. (最早提出 Gabor transform)

• M. J. Bastiaans, "Gabor's expansion of a signal into Gaussian elementary signals," *Proc. IEEE*, vol. 68, pp. 594-598, 1980.

• R. L. Allen and D. W. Mills, *Signal Analysis: Time, Frequency, Scale, and Structure*, Wiley- Interscience.

• S. C. Pei and J. J. Ding, "Relations between Gabor transforms and fractional Fourier transforms and their applications for signal processing," *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4839-4850, Oct. 2007.

Note：

許多文獻把 Gabor transform 直接就稱作 short-time Fourier transform (STFT)，實際上， Gabor transform 是 STFT 當中的一個 special case.

III-B Approximation of the Gabor Transform

Although the range of integration is from $-\infty$ to ∞, due to the fact that

$$e^{-\pi a^2} < 0.00001 \qquad \text{when } |a| > 1.9143$$

$$e^{-a^2/2} < 0.00001 \qquad \text{when } |a| > 4.7985$$

the Gabor transform can be simplified as:

$$G_x(t,f) \approx \int_{t-1.9143}^{t+1.9143} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\, d\tau$$

$$G_{x,3}(t,\omega) = \sqrt{\frac{1}{2\pi}} \int_{t-4.7985}^{t+4.7985} e^{-\frac{(\tau-t)^2}{2}} e^{-j\omega(\tau-\frac{t}{2})} x(\tau)\, d\tau$$

III-C Why Do We Choose the Gaussian Function as a Mask

(1) Among all functions, the Gaussian function has the advantage that the area in time-frequency distribution is minimal.

(和其他的 STFT 相比，比較能夠同時讓 time-domain 和 frequency domain 擁有較好的清晰度)

$w(t)$ 太寬 $\rightarrow$ time domain 的解析度較差

$w(t)$ 太窄 $\rightarrow W(f) = FT[w(t)]$ 太寬 $\rightarrow$ frequency domain 的解析度較差

(2) Special relation between the Gaussian function and the rectangular function

(Note): 由於 Gaussian function 是 FT 的 eigenfunction，因此 Gabor transform 在 time domain 和 frequency domain 的性質將互相對稱

$$\int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j2\pi f t} dt = e^{-\pi f^2}$$

$$\int_{-\infty}^{\infty} e^{-t^2/2} e^{-j\omega t} dt = e^{-f^2/2}$$

according to $$\int_{-\infty}^{\infty} e^{-(at^2+bt)} dt = \sqrt{\pi / a} \cdot e^{b^2/4a}$$

M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 3rd Ed., 2009.

Gaussian function is also an eigenmode in optics, radar system, and other electromagnetic wave systems.

(will be illustrated in the 8th week)

Uncertainty Principle (Heisenberg, 1927)

For a signal $x(t)$, if $\sqrt{t}\,x(t) = 0$ when $|t| \longrightarrow \infty$, then

$$\sigma_t \, \sigma_f \geq 1/4\pi$$

where

$$\sigma_t^2 = \int (t-\mu_t)^2 P_x(t)\,dt \qquad \sigma_f^2 = \int (f-\mu_f)^2 P_X(f)\,df,$$

$$\mu_t = \int t\,P_x(t)\,dt, \qquad \mu_f = \int f\,P_X(f)\,df$$

$$P_x(t) = \frac{|x(t)|^2}{\int |x(t)|^2\,dt}, \qquad P_X(f) = \frac{|X(f)|^2}{\int |X(f)|^2\,df},$$

(Proof of Henseinberg's uncertainty principle):

From simplification, we consider the case where $\mu_t = \mu_f = 0$

Then, use Parseval's theorem

$$\sigma_t^2 \sigma_f^2 = \frac{1}{4\pi^2} \frac{\int t^2 |x(t)|^2 \, dt}{\int |x(t)|^2 \, dt} \frac{\int |x'(t)|^2 \, dt}{\int |x(t)|^2 \, dt}$$

$$\int |x(t)|^2 \, dt = \int |X(f)|^2 \, df$$ if $X(f) = FT[x(t)]$

From Schwarz's inequality $\langle x(t), x(t)\rangle\langle y(t), y(t)\rangle \ge \left|\langle x(t), y(t)\rangle\right|^2$

$$\int t^2 |x(t)|^2\, dt \int |x'(t)|^2\, dt \ge \left(\left|\int t x^*(t)\frac{d}{dt}x(t)dt\right|^2 + \left|\int t x(t)\frac{d}{dt}x^*(t)dt\right|^2\right)/2$$

$$\ge \left|\int\left(t x^*(t)\frac{d}{dt}x(t) + t x(t)\frac{d}{dt}x^*(t)\right)dt\right|^2/4 \quad \text{(using } |a+b|^2 + |a-b|^2 \ge 2|a|^2\text{)}$$

$$= \left|\int t\frac{d}{dt}\left[x(t)x^*(t)\right]dt\right|^2/4 = \left|t x(t)x^*(t)\Big|_{-\infty}^{\infty} - \int x^*(t)x(t)\,dt\right|^2/4$$

$$= \left|\left[t x(t)x^*(t)\Big|_{t\to\infty} - t x(t)x^*(t)\Big|_{t\to-\infty}\right] - \int x^*(t)x(t)\,dt\right|^2/4$$

$$= \left|\int |x(t)|^2\, dt\right|^2/4$$

$$\sigma_t^2\sigma_f^2 \ge \frac{1}{16\pi^2} \Longrightarrow \sigma_t\sigma_f \ge \frac{1}{4\pi}$$

For Gaussian function

$$x(t) = e^{-\pi t^2} \qquad X(f) = e^{-\pi f^2}$$

$$\sigma_t^2 = \int (t-\mu_t)^2 P_x(t)\,dt = \frac{\int (t-\mu_t)^2 \,|\,x(t)\,|^2\,dt}{\int |\,x(t)\,|^2\,dt} = \frac{\int t^2 \,|\,x(t)\,|^2\,dt}{\int |\,x(t)\,|^2\,dt}$$

Since $\mu_t = 0$

$$\int_{-\infty}^{\infty} \left|x(t)\right|^2 dt = \int_{-\infty}^{\infty} e^{-2\pi t^2} dt = ?$$

use $\int_{-\infty}^{\infty} e^{-(at^2+bt)} dt = \sqrt{\pi / a} \cdot e^{b^2/4a}$

$$\int_{-\infty}^{\infty} t^2 \left|x(t)\right|^2 dt = \int_{-\infty}^{\infty} t^2 e^{-2\pi t^2} dt = 2\int_{0}^{\infty} t^2 e^{-2\pi t^2} dt = 2\frac{\Gamma(3/2)}{2(2\pi)^{3/2}} = \frac{1}{2^{5/2}\pi}$$

use $\int_{0}^{\infty} t^m e^{-at^2} dt = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$

$$\Gamma(1/2) = \sqrt{\pi} \qquad \Gamma(n+1) = n\Gamma(n), \qquad \Gamma(3/2) = \sqrt{\pi}\,/\,2$$

$$\sigma_t^2 = \frac{\int t^2 \, |x(t)|^2 \, dt}{\int |x(t)|^2 \, dt} = \frac{1}{4\pi}, \qquad \sigma_t = \sqrt{\frac{1}{4\pi}}$$

同理, $\sigma_f = \sqrt{\frac{1}{4\pi}}$

所以對 Gaussian function 而言，

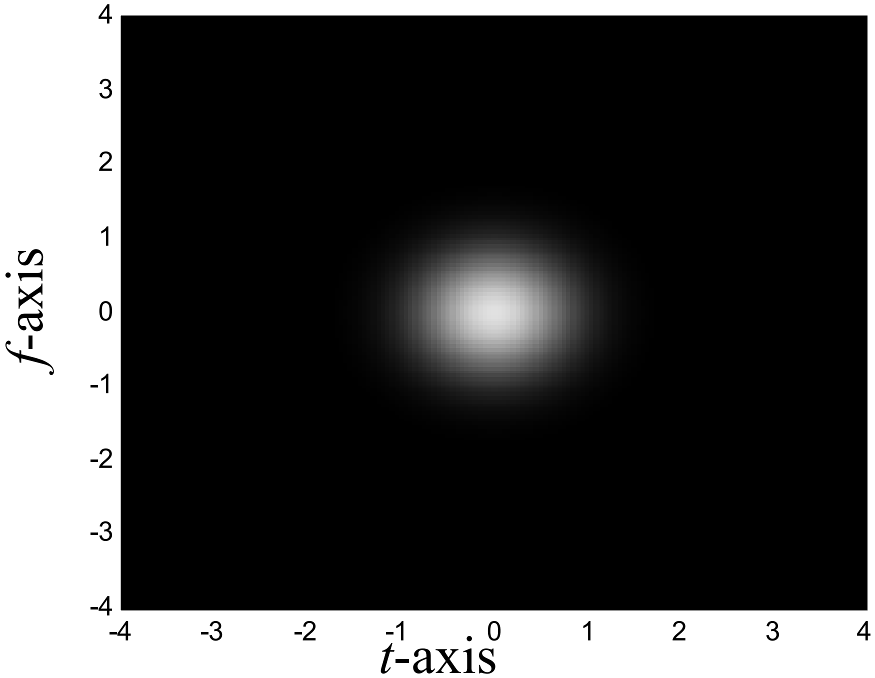
$$\sigma_t \sigma_f = \frac{1}{4\pi}$$

滿足下限

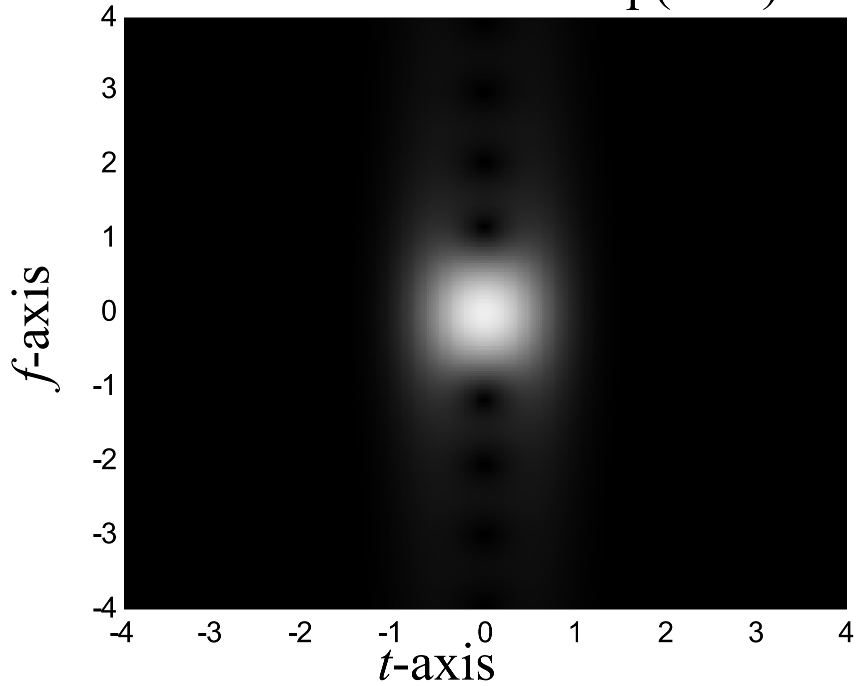
[工具書] M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 3rd Ed., 2009.

III-D Simulations

Gabor transform for
Gaussian function $\exp(-\pi t^2)$



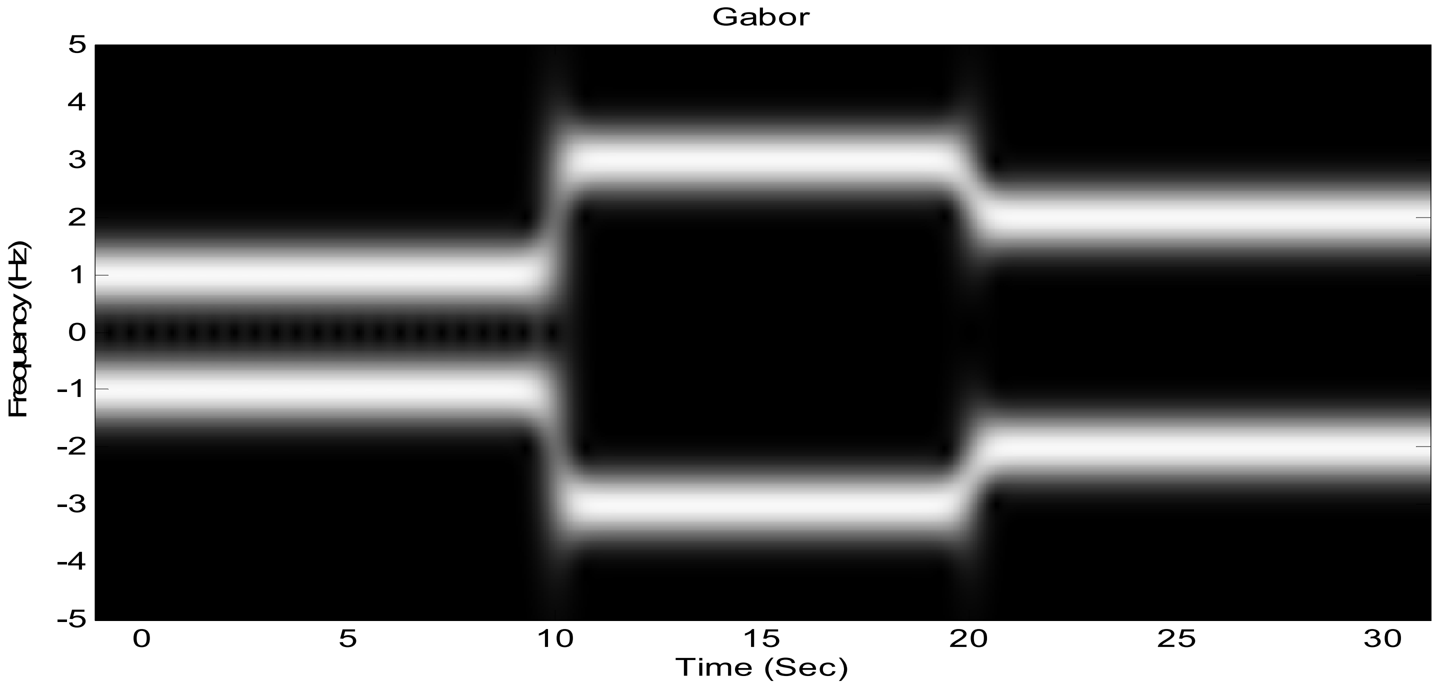
rec-STFT, B = 0.5 for
Gaussian function $\exp(-\pi t^2)$



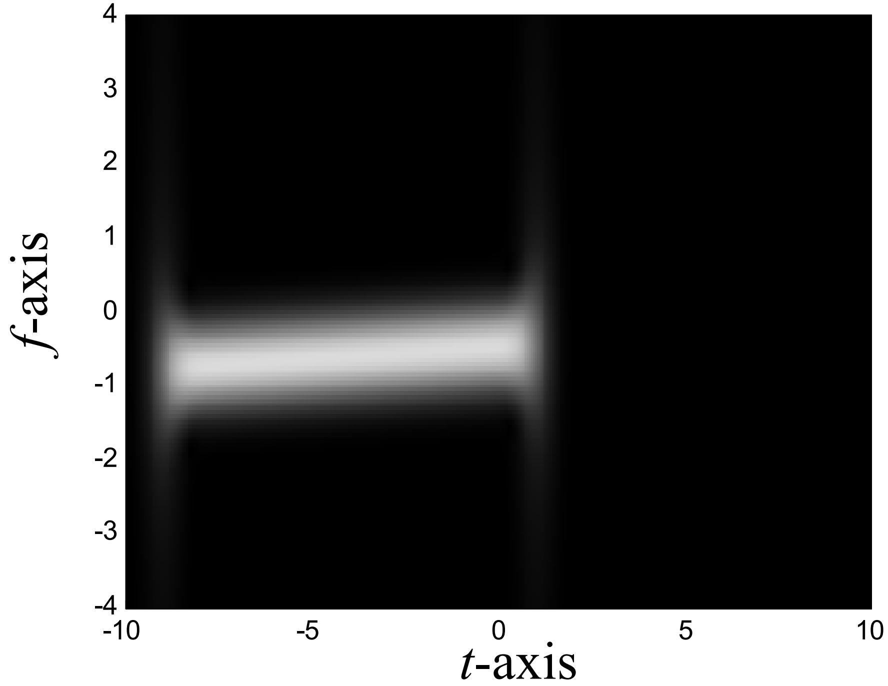
$x(t) = \cos(2\pi t)$ when $t < 10$,

$x(t) = \cos(6\pi t)$ when $10 \leq t < 20$,

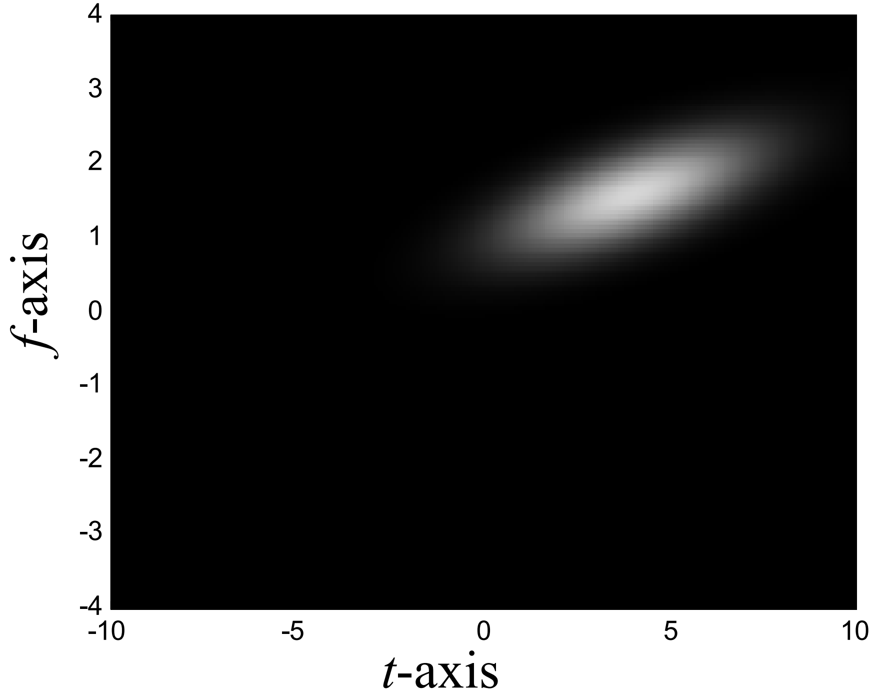
$x(t) = \cos(4\pi t)$ when $t \geq 20$



Gabor transform of $s(t)$



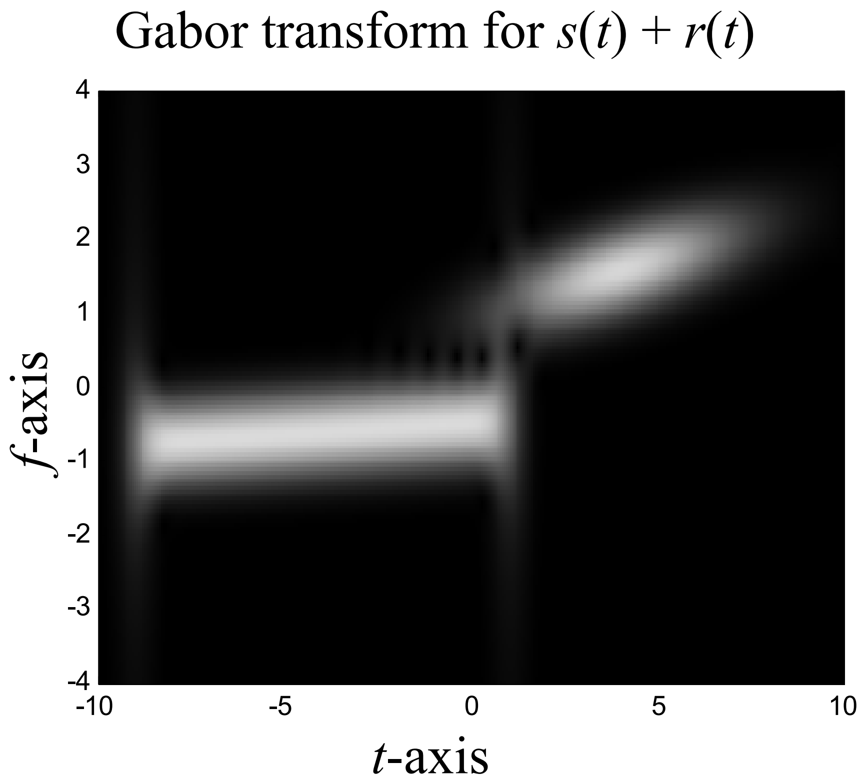
Gabor transform of $r(t)$



$s(t) = \exp\left(jt^2/10 - j3t\right)$ for $-9 \le t \le 1$,

$s(t) = 0$ otherwise,

$r(t) = \exp\left(jt^2/2 + j6t\right)\exp\left[-(t-4)^2/10\right]$


Gabor transform for s(t) + r(t)
f-axis
t-axis
4
3
2
1
0
-1
-2
-3
-4
-10
-5
0
5
10

III-E Properties of Gabor Transforms

$$G_x(t,f) = \int_{-\infty}^{\infty} e^{-j2\pi f\tau} e^{-\pi(\tau-t)^2} x(\tau)\,d\tau$$

(1) Integration property

When $k \neq 0$, $\int_{-\infty}^{\infty} G_x(t,f) e^{j2\pi ktf}\, df = e^{-\pi(k-1)^2 t^2} x(kt)$

When $k = 0$, $\int_{-\infty}^{\infty} G_x(t,f)\, df = e^{-\pi t^2} x(0)$

When $k = 1$, $\int_{-\infty}^{\infty} G_x(t,f) e^{j2\pi tf}\, df = x(t)$ (recovery property)

(2) Shifting property

If $y(t) = x(t - t_0)$, then $G_y(t,f) = G_x(t - t_0, f) e^{-j2\pi f t_0}$.

(3) Modulation property

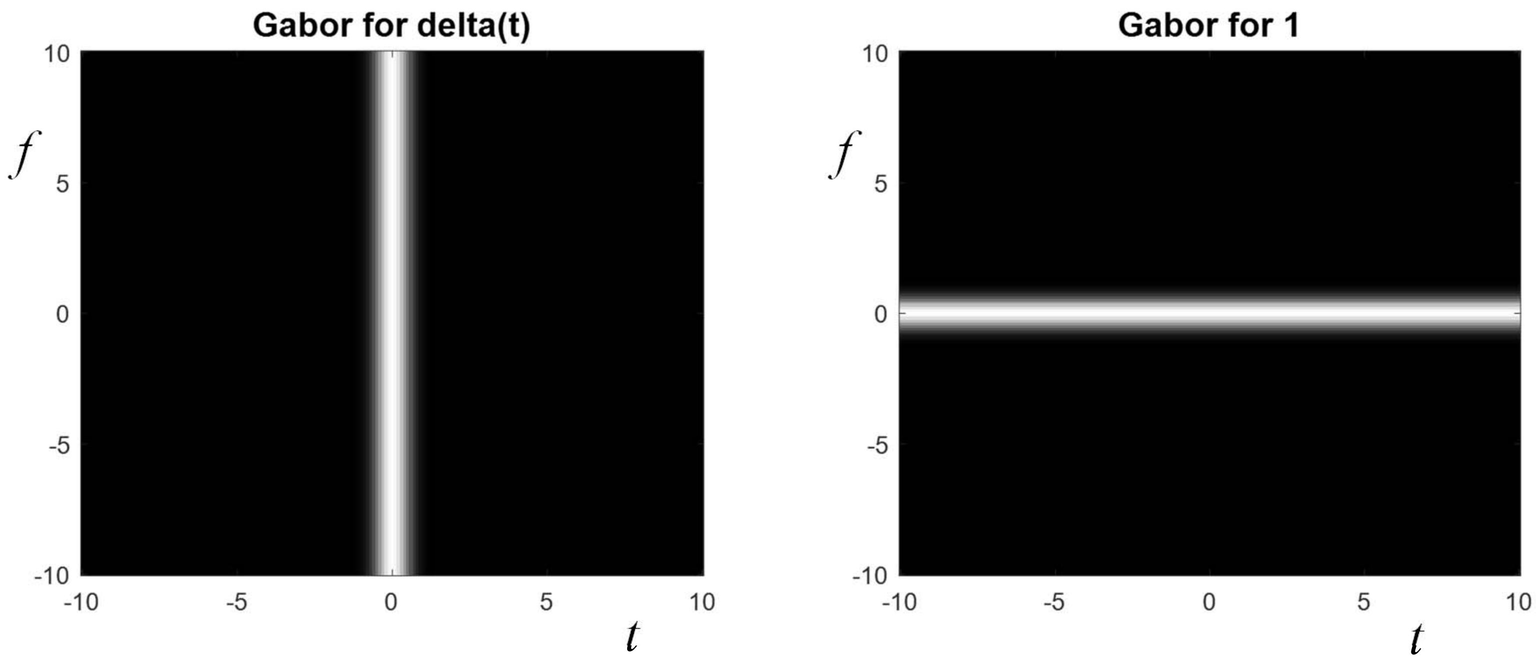
If $y(t) = x(t)\exp(j2\pi f_0 t)$, then $G_y(t,f) = G_x(t, f - f_0)$

(4) Special inputs:

(a) When $x(\tau) = \delta(\tau)$, $G_x(t,f) = e^{-\pi t^2}$

(b) When $x(\tau) = 1$, $G_x(t,f) = e^{-j2\pi f t} e^{-\pi f^2}$

(symmetric for the time and frequency domains)



(5) Power decayed property

- If $x(t) = 0$ for $t > t_0$, then .

$$\int_{-\infty}^{\infty}\left|G_x(t,f)\right|^2 df < e^{-2\pi(t-t_0)^2}\int_{-\infty}^{\infty}\left|G_x(t_0,f)\right|^2 df$$

i.e., $\underset{(\text{fix } t,\text{ vary } f)}{\text{average of}}\left|G_x(t,f)\right|^2 < e^{-2\pi(t-t_0)^2}\times\underset{(\text{fix } t_0,\text{ vary } f)}{\text{average of}}\left|G_x(t_0,f)\right|^2$ for $t > t_0$.

(Proof):

$$G_x(t,f)=\int_{-\infty}^{t_0} e^{-\pi(\tau-t)^2}e^{-j2\pi f\tau}x(\tau)d\tau \quad G_x(t_0,f)=\int_{-\infty}^{t_0} e^{-\pi(\tau-t_0)^2}e^{-j2\pi f\tau}x(\tau)d\tau$$

Since $(\tau-t)^2 > (\tau-t_0)^2+(t_0-t)^2$ $\qquad e^{-\pi(t-\tau)^2} < e^{-\pi(t-t_0)^2}e^{-\pi(t_0-\tau)^2}$

$$G_x(t,f) < e^{-\pi(t-t_0)^2}G_x(t_0,f)$$

- If $X(f)=FT\left[x(t)\right]=0$ for $f > f_0$, then

$$\underset{(\text{fix } f,\text{ vary } t)}{\text{average of}}\left|G_x(t,f)\right|^2 < e^{-2\pi(f-f_0)^2}\times\underset{(\text{fix } f_0,\text{ vary } t)}{\text{average of}}\left|G_x(t,f_0)\right|^2 \text{ for } f > f_0.$$

(6) Linearity property

If $z(\tau) = \alpha\, x(\tau) + \beta\, y(\tau)$ and $G_z(t, f)$, $G_x(t, f)$ and $G_y(t, f)$ are their Gabor transforms, then

$$G_z(t, f) = \alpha\, G_x(t, f) + \beta\, G_y(t, f)$$

(7) Power integration property:

$$\int_{-\infty}^{\infty} \left|G_x(t, f)\right|^2 df = \int_{-\infty}^{\infty} e^{-2\pi(\tau-t)^2} \left|x(\tau)\right|^2 d\tau \approx \int_{u-1.9143}^{u+1.9143} e^{-2\pi(\tau-u)^2} \left|x(\tau)\right|^2 d\tau$$

(8) Energy sum property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_x(t, f)\, G_y^*(t, f)\, df\, dt = \int_{-\infty}^{\infty} x(\tau)\, y^*(\tau)\, d\tau$$

where $G_x(t, f)$ and $G_y(t, f)$ are the Gabor transforms of $x(\tau)$ and $y(\tau)$, respectively.

III-F Scaled Gabor Transforms

$$G_x(t,f)=\sqrt[4]{\sigma}\int_{-\infty}^{\infty}e^{-\sigma\pi(\tau-t)^2}e^{-j2\pi f\tau}x(\tau)\,d\tau$$

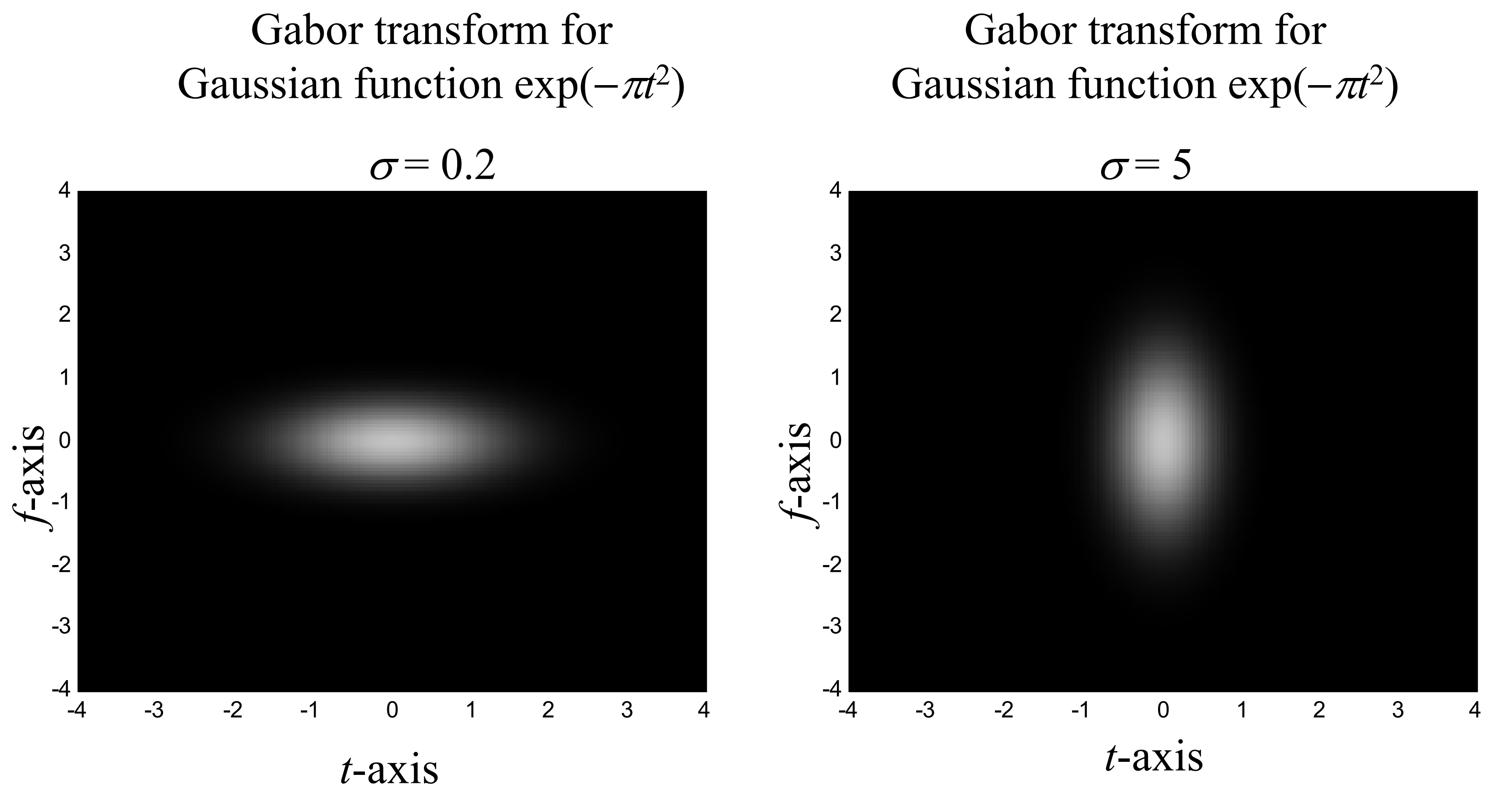
⇓ (finite interval form)

larger σ: higher resolution in the time domain

lower resolution in the frequency domain

smaller σ: higher resolution in the frequency domain

lower resolution in the time domain


Gabor transform for
Gaussian function exp(−πt²)
σ = 0.2
4
3
2
1
0
-1
-2
-3
-4
-4 -3 -2 -1 0 1 2 3 4
f-axis
t-axis
Gabor transform for
Gaussian function exp(−πt²)
σ = 5
4
3
2
1
0
-1
-2
-3
-4
-4 -3 -2 -1 0 1 2 3 4
f-axis
t-axis

處理對 time resolution 相對上比 frequency resolution 敏感的信號

(1) Using the generalized Gabor transform with larger σ

(2) Using other time unit instead of second

例如，原本　t (單位：sec)　　f (單位： Hz)

對聲音信號可以改成

t (單位： 0.1 sec)　f (單位：10 Hz)

III-G Gabor Transforms with Adaptive Window Width

For a signal,

when the instantaneous frequency varies fast $\rightarrow$ larger σ

when instantaneous frequency varies slowly $\rightarrow$ smaller σ

$$G_x(t,f) = \sqrt[4]{\sigma(t)} \int_{-\infty}^{\infty} e^{-\sigma(t)\pi(\tau - t)^2} e^{-j2\pi f \tau} x(\tau) d\tau$$

$\sigma(t)$ is a function of t

S. C. Pei and S. G. Huang, "STFT with adaptive window width based on the chirp rate," *IEEE Trans. Signal Processing*, vol. 60, issue 8, pp. 4065-4080, 2012.

附錄五：Matlab 寫程式的原則以及部分常用的指令

(1) 迴圈能避免就儘量避免

(2) 儘可能使用 Matrix 及 Vector operation

(3) 能夠不在迴圈內做的運算，則移到迴圈外

(4) 寫一部分即測試，不要全部寫完再測試 (縮小範圍比較容易 debug)

(5) 先測試簡單的例子，成功後再測試複雜的例子

註：作業 Matlab Program (or Python program) 鼓勵各位同學儘量用精簡而快速的方式寫。Program 執行速度越快，分數就越高。

一些重要的 Matlab 指令

(1) function: 放在第一行，可以將整個程式函式化

(2) tic, toc: 計算時間

tic 為開始計時，toc 為顯示時間

(3) find: 找尋一個 vector 當中不等於 0 的entry 的位置

範例：find([1 0 0 1]) = [1, 4]

find(abs([-5:5])<=2) = [4, 5, 6, 7, 8]

(因為 abs([-5:5])<=2 = [0 0 0 1 1 1 1 1 0 0 0])

(4) ′ : Hermitian (transpose + conjugation)， .′ : transpose

(5) imread: 讀圖

(註： 較老的 Matlab 版本 imread 要和 double 並用

A=double(imread('Lena.bmp'));

(6) image: 將圖顯示出來，

(i) 顯示灰階圖

image(A)   % A has the size of MxNx1

colormap(gray(256))

(ii) 顯示彩色圖，整數的情形

image(A)   % A has the size of MxNx3 and the entries are integer

(iii) 顯示彩色圖，非整數的情形

image(A)   % A has the size of MxNx3 and the entries are non-integer

(7) imshow, imagesc : 也可用來顯示圖

(8) imwrite: 製做圖檔

(9) aviread: 讀取 video 檔，限副檔名為 avi

(10) VideoReader: 讀取 video 檔

(11) VideoWriter: 製作 video 檔

(12) xlsread 或 readmatrix 或 readcell : 由 Excel 檔讀取資料

(13) xlswrite: 將資料寫成 Excel 檔

(14) dlmread: 讀取 *.txt 或其他類型檔案的資料

(15) dlmwrite: 將資料寫成 *.txt 或其他類型檔案

附錄六：寫 Python 版本程式可能會用到的重要指令

建議必安裝模組

```
pip install numpy
pip install scipy
pip install opencv-python
pip install matplotlib
```

(1) 定義函式：使用def

(2) 計算時間

```
import time
start_time = time.time()   #獲取當前時間
end_time = time.time()
total_time = end_time - start_time   #計算時間差來得到總執行時間
```

感謝2021年擔任助教的蔡昌廷同學

(3) 讀取圖檔、顯示圖檔、輸出圖檔

(方法一)

```
import cv2
image = cv2.imread(file_name)  #預設color channel為BGR
cv2.imshow('test', image)
# 若 image 的值非整數，要改成 cv2.imshow('test', image/255)
cv2.waitKey(0)
cv2.destroyAllWindows()
cv2.imwrite(file_name, image)   #需將color channel轉為BGR
```

(方法二)

```
import matplotlib.pyplot as plt
image = plt.imread(file_name)  #預設color channel為RGB
plt.imshow(image)
# 若 image 的值非整數，要改成 plt.imshow(image/255)
plt.show()
plt.imsave(file_name, image)   #需將color channel轉為RGB
```

(4) 尋找array中滿足特定條件的值的位置

(相當於 Matlab 的 find 指令)

```
import numpy as np
a = np.array([0, 1, 2, 3, 4, 5])
index = np.where(a > 3) # 回傳array([4, 5])
print(index)
    (array([4, 5], dtype=int64),)
index[0][0]
    4
index[0][1]
    5
```

$$\mathbf{A_1} = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & 5 \end{bmatrix}$$

```
A1= np.array([[1,3,6],[2,4,5]])
index = np.where(A1 > 3)
print(index)
    (array([0, 1, 1], dtype=int64), array([2, 1, 2], dtype=int64))
    (代表滿足 A1 > 3 的點的位置座標為 [0, 2], [1, 1], [1, 2]
[index[0][0], index[1][0]]
   [0, 2]
[index[0][1], index[1][1]]
   [1, 1]
[index[0][2], index[1][2]]
   [1, 2]
```

(5) Hermitian、transpose

```
import numpy as np
result = np.conj(matrix.T)     # Hermitian
result = matrix.T    # transpose
```

(6) 在 Python 當中讀取 Matlab 當中的 mat 檔

```
data = scipy.io.loadmat('***.mat')
y = np.array(data['y'])   # 假設 y 是 ***.mat 當中儲存的資料
```