VII. Other Time Frequency Distributions (II)

The trend of time-frequency analysis in recent years:

(1) S transform and its generalization

(2) Time-frequency reassignment and the synchosqueezing transform

(3) Time-variant signal expansion (Compressive sensing)

(4) Improvement for the Hilbert-Huang transform

VII-A S Transform

(Modification from the scaled Gabor transform) tt教 pages 82, 102 $\sigma = f^2$

$$S_x(t,f) = |f| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi (t-\tau)^2 f^2\right] \exp(-j2\pi f \tau) d\tau$$

closely related to the wavelet transform

advantages and disadvantages

f ↓ window width ↑ time resolution ↓ frequency resolution ↑

f ↑ ↓ ↑ ↓

[Ref] R. G. Stockwell, L. Mansinha, and R. P. Lowe, "Localization of the complex spectrum: the S transform," *IEEE Trans. Signal Processing*, vol. 44, no. 4, pp. 998–1001, Apr. 1996.

S transform 和 Gabor transform 相似。

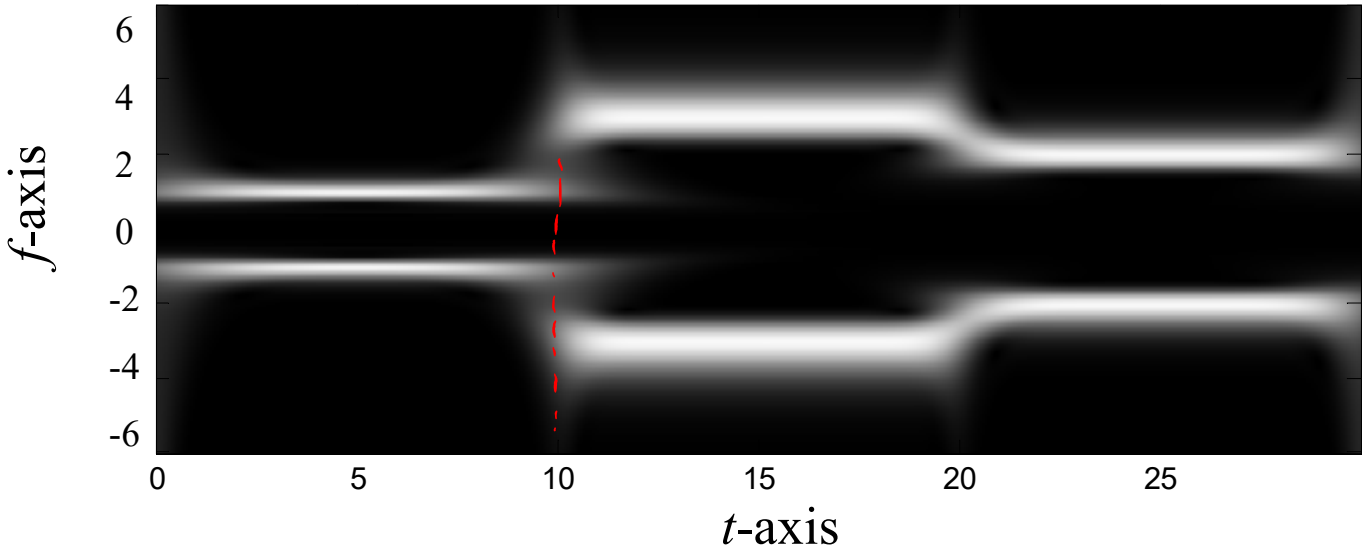
但是 Gaussian window 的寬度會隨著 f 而改變

$$w(t)=\exp\left[-\pi t^2\right] \qquad w(t)=|f|\exp\left[-\pi t^2 f^2\right]$$

低頻： worse time resolution, better frequency resolution

高頻： better time resolution, worse frequency resolution

The result of the S transform (compared with page 95)

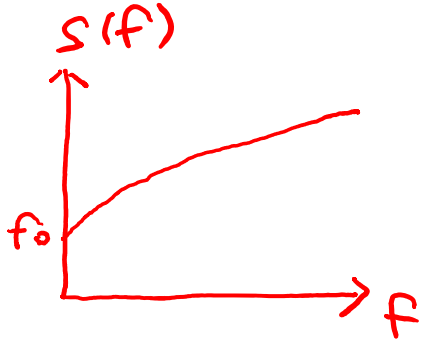


- General form

on page 210, $s(f) = f$

$$S_x(t,f) = |s(f)| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi (t-\tau)^2 s^2(f)\right] \exp(-j2\pi f \tau) d\tau$$

$s(f)$ increases with f



$s(f) = f_0 + f^{\alpha}$, $0 < \alpha < 1$

C. R. Pinnegar and L. Mansinha, "The S-transform with windows of arbitrary and varying shape," *Geophysics*, vol. 68, pp. 381-385, 2003.

Fast algorithm of the S transform

When f is fixed, the S transform can be expressed as a convolution form:

$$S_x(t,f) = |s(f)| \int_{-\infty}^{\infty} x(\tau) \exp\left[-\pi (t-\tau)^2 s^2(f)\right] \exp(-j2\pi f\tau)\, d\tau$$

$$S_x(t,f) = |s(f)| \left(x(t)\exp(-j2\pi ft) \underset{\substack{\text{convolution} \\ \text{along } t\text{-axis}}}{*} \exp\left[-\pi t^2 s^2(f)\right] \right)$$

(for every fixed f)

Remember: $g(t) * h(t) = \int g(\tau) h(t-\tau)\, d\tau$

Q: Can we use the FFT-based method on page 121 to implement the S transform?

VII-B Generalized Spectrogram

[Ref] P. Boggiatto, G. De Donno, and A. Oliaro, "Two window spectrogram and their integrals," *Advances and Applications*, vol. 205, pp. 251-268, 2009.

Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)G^*_{x,w_2}(t,f)$

$$G_{x,w_1}(t,f) = \int_{-\infty}^{\infty} w_1(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

$$G_{x,w_2}(t,f) = \int_{-\infty}^{\infty} w_2(t-\tau)x(\tau)e^{-j2\pi f\tau}d\tau$$

Original spectrogram: $w_1(t) = w_2(t)$

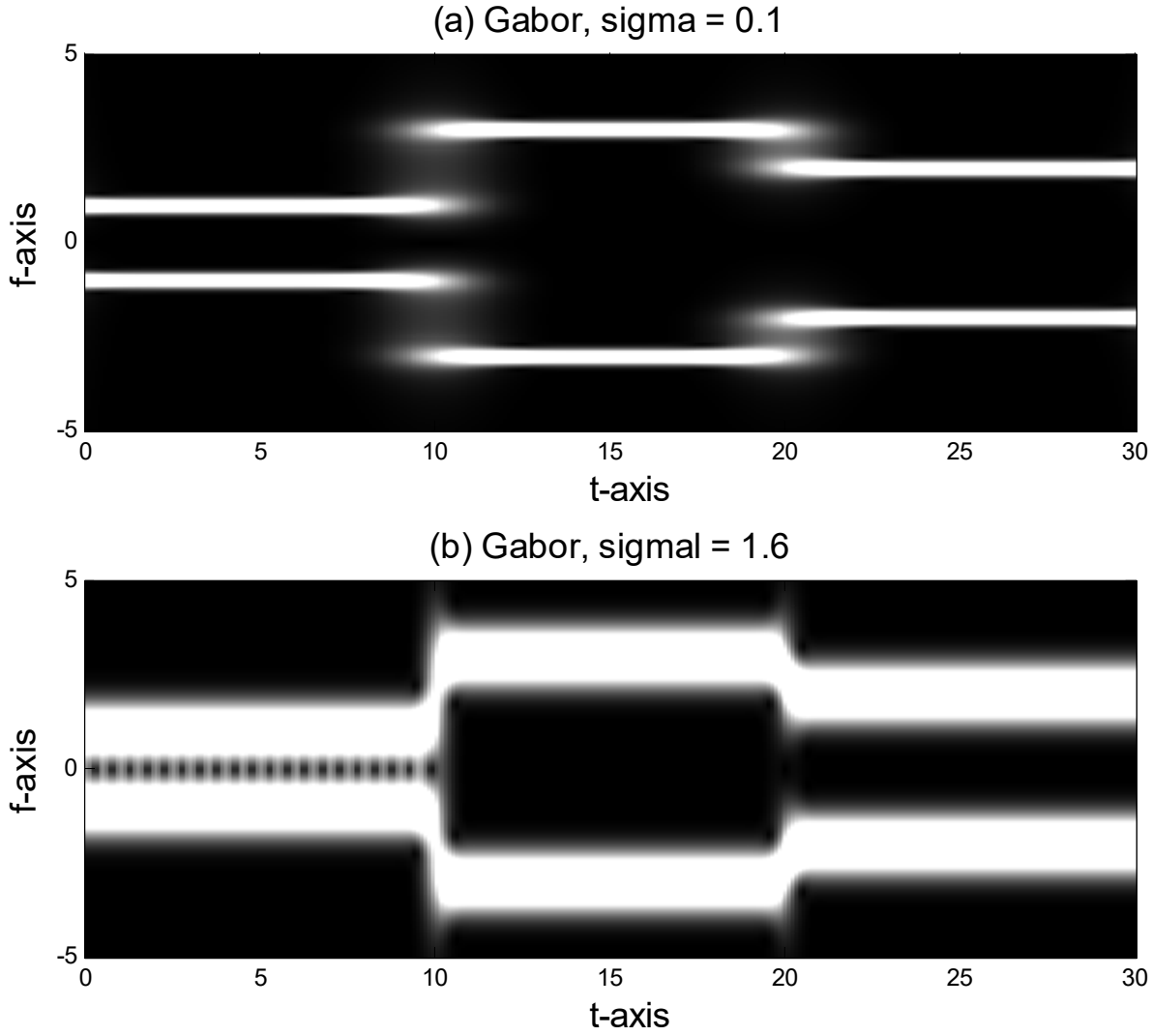
To achieve better clarity, $w_1(t)$ can be chosen as a wider window,
$w_2(t)$ can be chosen as a narrower window.

$x(t) = \cos(2\pi t)$ when $t < 10$,

$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(4\pi t)$ when $t \ge 20$

scaled Gabor
page 102

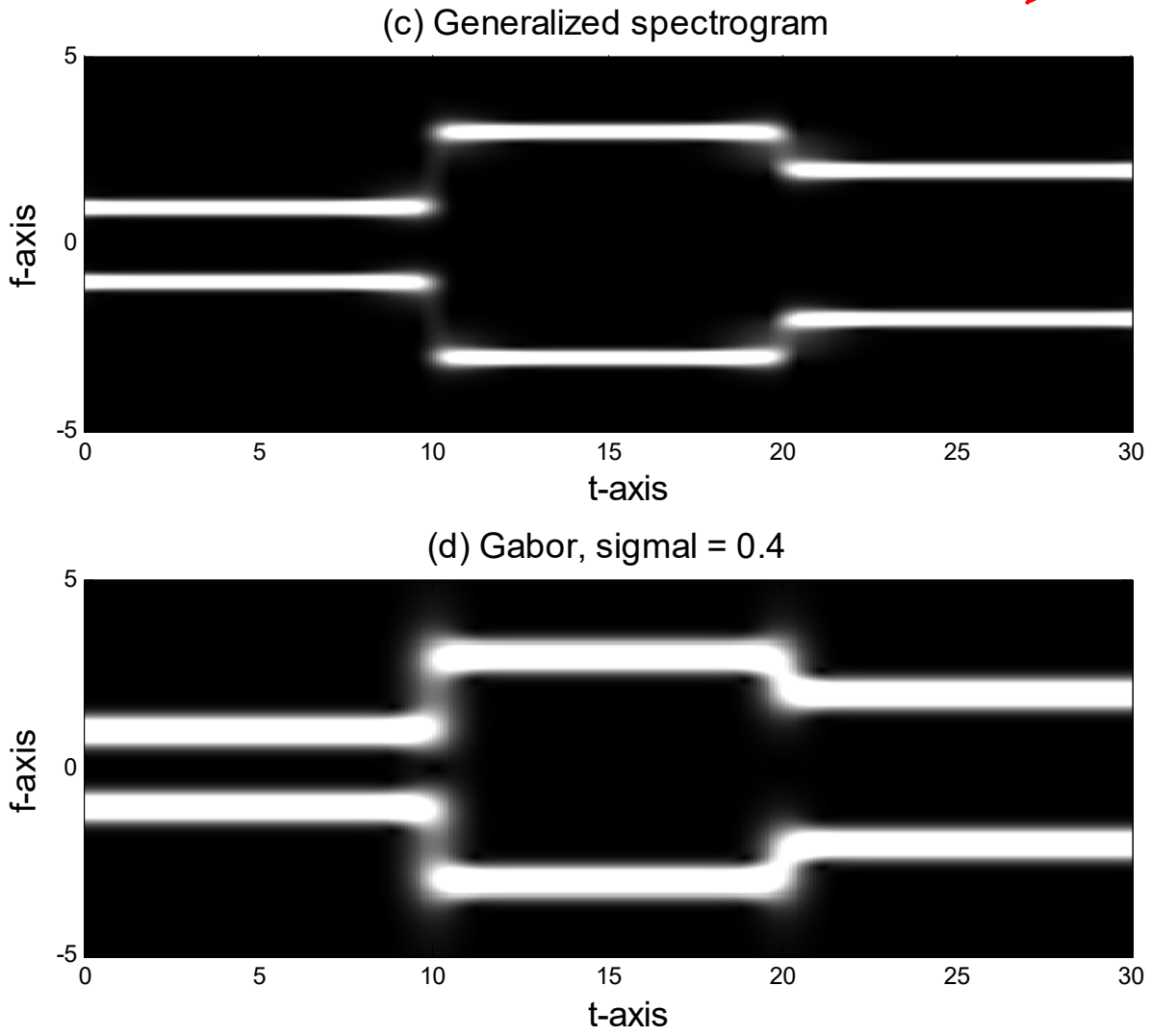


$x(t) = \cos(2\pi t)$ when $t < 10$,

$x(t) = \cos(6\pi t)$ when $10 \le t < 20$,

$x(t) = \cos(4\pi t)$ when $t \ge 20$

(a)X(b)



Generalized spectrogram: $SP_{x,w_1,w_2}(t,f) = G_{x,w_1}(t,f)\,G^{*}_{x,w_2}(t,f)$

Further Generalization for the spectrogram:

$$SP_{x,w_1,w_2}(t,f) = G^{\alpha}_{x,w_1}(t,f)\,\overline{G^{\beta}_{x,w_2}(t,f)}$$

or

$$SP_{x,w_1,w_2}(t,f) = \left|G_{x,w_1}(t,f)\right|^{\alpha}\left|G_{x,w_2}(t,f)\right|^{\beta}$$

VII-C Reassignment Method

再分配

After computing the time-frequency distribution, we can use the following way to make the energy even more concentrated.

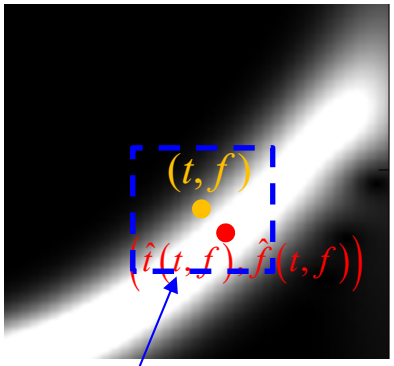
(1) First, estimate the offset.

expected value of t within the window

$$\hat{t}(t,f)=\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u\cdot\varphi(u-t,v-f)\cdot X(u,v)dudv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(u-t,v-f)\cdot X(u,v)dudv}$$

expected value of f within the window

$$\hat{f}(t,f)=\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v\cdot\varphi(u-t,v-f)\cdot X(u,v)dudv}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(u-t,v-f)\cdot X(u,v)dudv}$$



$\varphi(u-t,v-f)$ window centered at (t,f)

$X(t,f)$: time-frequency analysis (STFT, WDF…) of $x(t)$,

$\varphi(u,v)=1$ when $|u|,|v|<B$

$\varphi(u,v)=0$ otherwise

(2) Then, shift the time frequency distribution at (t,f) to $\left(\hat{t}(t,f),\hat{f}(t,f)\right)$

(2) Then, shift the time frequency distribution at (t, f) to $\left(\hat{t}(t, f), \hat{f}(t, f)\right)$

$$\hat{X}(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(t_1, f_1) \delta\left(t - \hat{t}(t_1, f_1)\right) \delta\left(f - \hat{f}(t_1, f_1)\right) dt_1 df_1$$

References

[1] F. Auger and P. Flandrin, “Improving the readability of time-frequency and time-scale representations by the reassignment method,” *IEEE Trans. Signal Processing*, vol. 43, issue 5, pp. 1068-1089, May 1995.

[2] F. Auger, P. Flandrin, Y.T. Lin, S. McLaughlin, S. Meignen, T. Oberlin, and H.T. Wu, “Time-frequency reassignment and synchrosqueezing: An overview,” *IEEE Signal Processing Magazine*, vol. 30, issue 6, pp. 32-41, 2013.

PS: 感謝 2017 年修課的盧德晏同學

Synchrosqueezing Transform

$$\hat{X}(t,f)=\int_{-\infty}^{\infty} X(t,f_1)\,\delta\left(f-\hat{f}(t,f_1)\right)df_1$$

It only moves along f-axis.

(simplification form of the reassignment method)

C. Li and M. Liang, "A generalized synchrosqueezing transform for enhancing signal time–frequency representation," *Signal Processing*, vol. 92, issue 9, 2264-2274, 2012.

T. Oberlin, S. Meignen, and V. Perrier. "The Fourier-based synchrosqueezing transform," in *IEEE Int. Conf. Acoustics, Speech and Signal Processing*, pp. 315-319, 2014.

VII-D Basis Expansion Time-Frequency Analysis

就如同 $|\varphi_m(t)| = 1$

- Fourier series: $\varphi_m(t) = \exp(j2\pi f_m t)$, $x(t) \approx \sum_{m=1}^{M} a_m \exp(j2\pi f_m t)$

if $x(t) = x(t+T)$

$$a_m = \frac{\langle x(t), \varphi_m(t) \rangle}{\langle \varphi_m(t), \varphi_m(t) \rangle} = \frac{1}{T}\int_0^T x(t)\exp(-j2\pi f_m t)dt \qquad f_m = m/T$$

when $T \to \infty$, Fourier series becomes the Fourier transform

部分的 Time-Frequency Analysis 也是意圖要將 signal 表示成如下的型態

$$x(t) \approx \sum_{m=1}^{M} a_m \varphi_m(t)$$

orthogonal $\langle \varphi_m(t), \varphi_n(t) \rangle = 0$ if $m \neq n$

$\to \langle x(t), \varphi_n(t) \rangle = \sum_{m=1}^{M} a_m \langle \varphi_m(t), \varphi_n(t) \rangle$

$= a_n \langle \varphi_n(t), \varphi_n(t) \rangle$

並且要求在 M 固定的情形下，

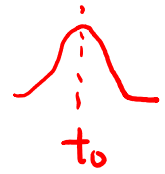
$$\text{approximation error} = \int_{-\infty}^{\infty} \left| x(t) - \sum_{m=1}^{M} a_m \varphi_m(t) \right|^2 dt$$ 為最小

將 $\varphi_m(t)$ 一般化，不同的 basis 之間不只是有 frequency 的差異

(1) Three Parameter Atoms

$$x(t) \approx \sum a_{t_0,f_0,\sigma} \varphi_{t_0,f_0,\sigma}(t)$$

$$\varphi_{t_0,f_0,\sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}} \exp(j2\pi f_0 t) \exp(-\frac{\pi(t-t_0)^2}{\sigma^2})$$



3 parameters: t_0 controls the central time
f_0 controls the frequency
σ controls the scaling factor

[Ref] S. G. Mallat and Z. Zhang, "Matching pursuits with time-frequency dictionaries," *IEEE Trans. Signal Processing*, vol. 41, no. 12, pp. 3397-3415, Dec. 1993.

Since $\varphi_{t_0,f_0,\sigma}(t)$ are not orthogonal, $a_{t_0,f_0,\sigma}$ should be determined by a matching pursuit process.

(2) Four Parameter Atoms (Chirplet)

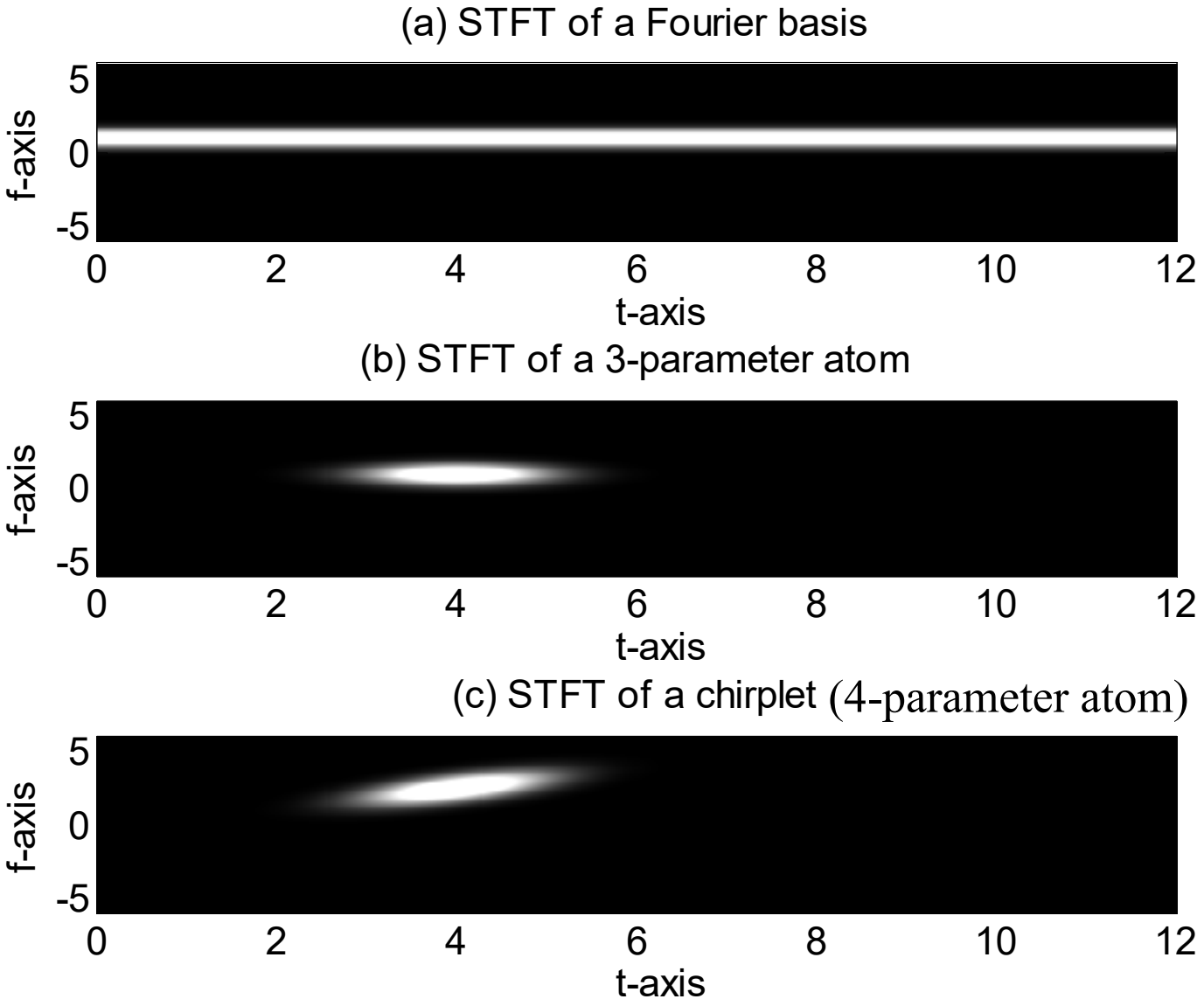
$$x(t) \approx \sum a_{t_0,f_0,\sigma,\eta} \varphi_{t_0,f_0,\sigma,\eta}(t)$$

$$\varphi_{t_0,f_0,\sigma}(t) = \frac{2^{1/4}}{\sigma^{1/2}} \exp(j2\pi(f_0 t + \frac{\eta}{2} t^2) - \frac{\pi(t-t_0)^2}{\sigma^2})$$

4 parameters: t_0 controls the central time
f_0 controls the initial frequency
σ controls the scaling factor
η controls the chirp rate

[Ref] A. Bultan, "A four-parameter atomic decomposition of chirplets," *IEEE Trans. Signal Processing*, vol. 47, no. 3, pp. 731–745, Mar. 1999.

[Ref] C. Capus, and K. Brown. "Short-time fractional Fourier methods for the time-frequency representation of chirp signals," *J. Acoust. Soc. Am.* vol. 113, issue 6, pp. 3253-3263, 2003.


(a) STFT of a Fourier basis
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis
(b) STFT of a 3-parameter atom
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis
(c) STFT of a chirplet (4-parameter atom)
f-axis
5
0
-5
0
2
4
6
8
10
12
t-axis

(3) Prolate Spheroidal Wave Function (PSWF)

$$x(t) \cong \sum_{n,T,\Omega,t_0,f_0} a_{n,T,\Omega,t_0,f_0} \psi_{n,T,\Omega}(t-t_0)\exp(j2\pi f_0 t)$$

where $\psi_{n,T,\Omega}(t)$ is the prolate spheroidal wave function

[Ref] D. Slepian and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty-I," *Bell Syst. Tech. J.*, vol. 40, pp. 43-63, 1961.

Note (i) No function has finite supports in both time and frequency domains.

(ii) The PSWF satisfies $x(t) \cong 0 \quad for\ |t| > T$

(iii) For all functions that satisfy $x(t) \cong 0 \quad for\ |t| > T$

The PSWF can maximize $\dfrac{\int_{-\Omega}^{\Omega} |X(f)|^2\, df}{\int_{-\infty}^{\infty} |X(f)|^2\, df}$

Concept of the prolate spheroidal wave function (PSWF):

- FT: $X(f)=\int_{-\infty}^{\infty}\exp(-j2\pi f t)x(t)dt \quad , \quad x, f \in (-\infty, \infty).$

energy preservation property (Parseval's property)

$$\int_{-\infty}^{\infty}|X(f)|^2\,df=\int_{-\infty}^{\infty}|x(t)|^2\,dt$$

- finite Fourier transform (fi-FT):

$$X_{fi}(f)=\int_{-T}^{T}\exp(-j2\pi f t)x(t)dt$$

space interval: $t \in [-T, T]$, frequency interval: $f \in [-\Omega, \Omega]$

$$0 < \text{energy preservation ratio } = \frac{\int_{-\Omega}^{\Omega}|X_{fi}(f)|^2\,df}{\int_{-T}^{T}|x(t)|^2\,dt} < 1$$

The PWSF $\psi_{0,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}|X_{fi}(f)|^2\,df}{\int_{-T}^{T}|x(t)|^2\,dt}$

The PWSF $\psi_{0,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}\left|X_{fi}(f)\right|^2 df}{\int_{-T}^{T}\left|x(t)\right|^2 dt}$

Among the functions orthogonal to $\psi_{0,T,\Omega}$

$\psi_{1,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}\left|X_{fi}(f)\right|^2 df}{\int_{-T}^{T}\left|x(t)\right|^2 dt}$

Among the functions orthogonal to $\psi_{0,T,\Omega}$ and $\psi_{1,T,\Omega}$

$\psi_{2,T,\Omega}(t)$ can maximize $\dfrac{\int_{-\Omega}^{\Omega}\left|X_{fi}(f)\right|^2 df}{\int_{-T}^{T}\left|x(t)\right|^2 dt}$

and so on.

- Prolate spheroidal wave functions (PSWFs) are the continuous functions that satisfy: $\int_{-T}^{T} K_{F,\Omega}(t_1,t)\psi_{n,T,\Omega}(t)\,dt = \lambda_{n,T,\Omega}\psi_{n,T,\Omega}(t_1)$,

  where $K_{F,\Omega}(t_1,t) = \dfrac{\sin[2\pi\Omega(t_1-t)]}{\pi(t_1-t)}$

PSWFs are orthonormal and can be sorted according to the values of $\lambda_{n,T,\Omega}$'s:

$$\int_{-T}^{T} \psi_{m,T,\Omega}(t)\psi_{n,T,\Omega}(t)\,dt = \delta_{m,n}$$

$1 > \lambda_{0,T,\Omega} > \lambda_{1,T,\Omega} > \lambda_{2,T,\Omega} > \ldots\ldots\ldots\ldots > 0.$ (All of $\lambda_{n,T,\Omega}$'s are real)

附錄十： Compressive Sensing and Matching Pursuit 的觀念

Different from orthogonal basis expansion, which applies a complete and orthogonal basis set, compressive sensing is to use an over-complete and non-orthogonal basis set to expand a signal.

Example:

Fourier series expansion is an orthogonal basis expansion method:

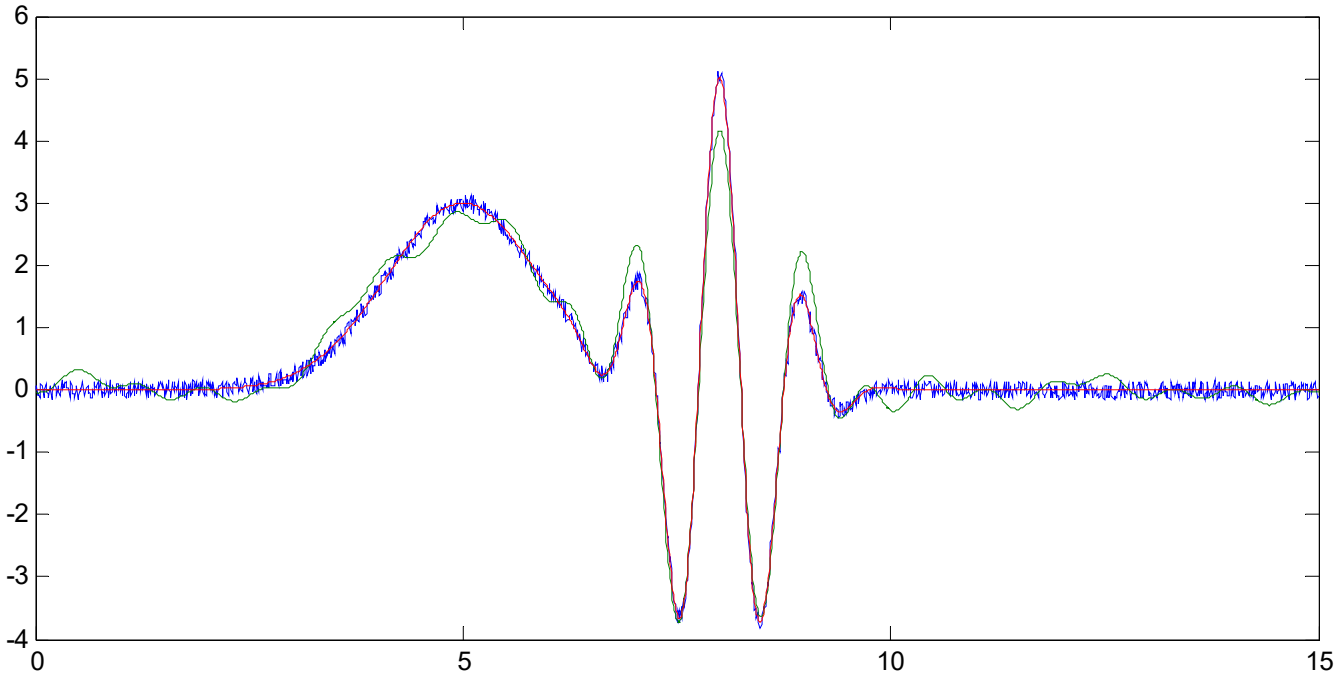
$$x(t) \approx \sum_{m=1}^{M} a_m \exp(j2\pi f_m t)$$

$$\int \exp(j2\pi f_m t)\overline{\exp(j2\pi f_n t)}\, dt = 0 \qquad \text{if } f_m \neq f_n$$

Three-parameter atom expansion, Four-parameter atom (chirplet) expansion, and PSWF expansion are over-complete and non-orthogonal basis expansion methods.

$$x(t) \approx \sum a_{t_0,f_0,\sigma} \varphi_{t_0,f_0,\sigma}(t)$$

$\varphi_{t_0,f_0,\sigma}(t)$ do not form a complete and orthogonal set.



For example, in the above figure, the blue line is the original signal

- When using three-parameter atoms, the expansion result is the red line

$$x(t) = 3e^{-0.2\pi(t-5)^2} + 2.5e^{-0.4\pi(t-8)^2+j2\pi t} + 2.5e^{-0.4\pi(t-8)^2-j2\pi t}$$

Only 3 terms are used and the normalized root square error is 0.39%

- When using Fourier series, if 31 terms are used, the expansion result is the green line and the normalized root square error is 3.22%

The problems that compressive sensing deals with:

Suppose that $b_0(t)$, $b_1(t)$, $b_2(t)$, $b_3(t)$ …….. form an over-complete and non-orthogonal basis set.

(Problem 1) We want to minimize $\|c\|_0$ （$\|\ \|_0$ 是 L_0 norm， $\|c\|_0$ 意指c_m 的值不為 0 的個數） such that

Sparse: smaller L_0 norm

$$x(t)=\sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $\|c\|_0$ such that

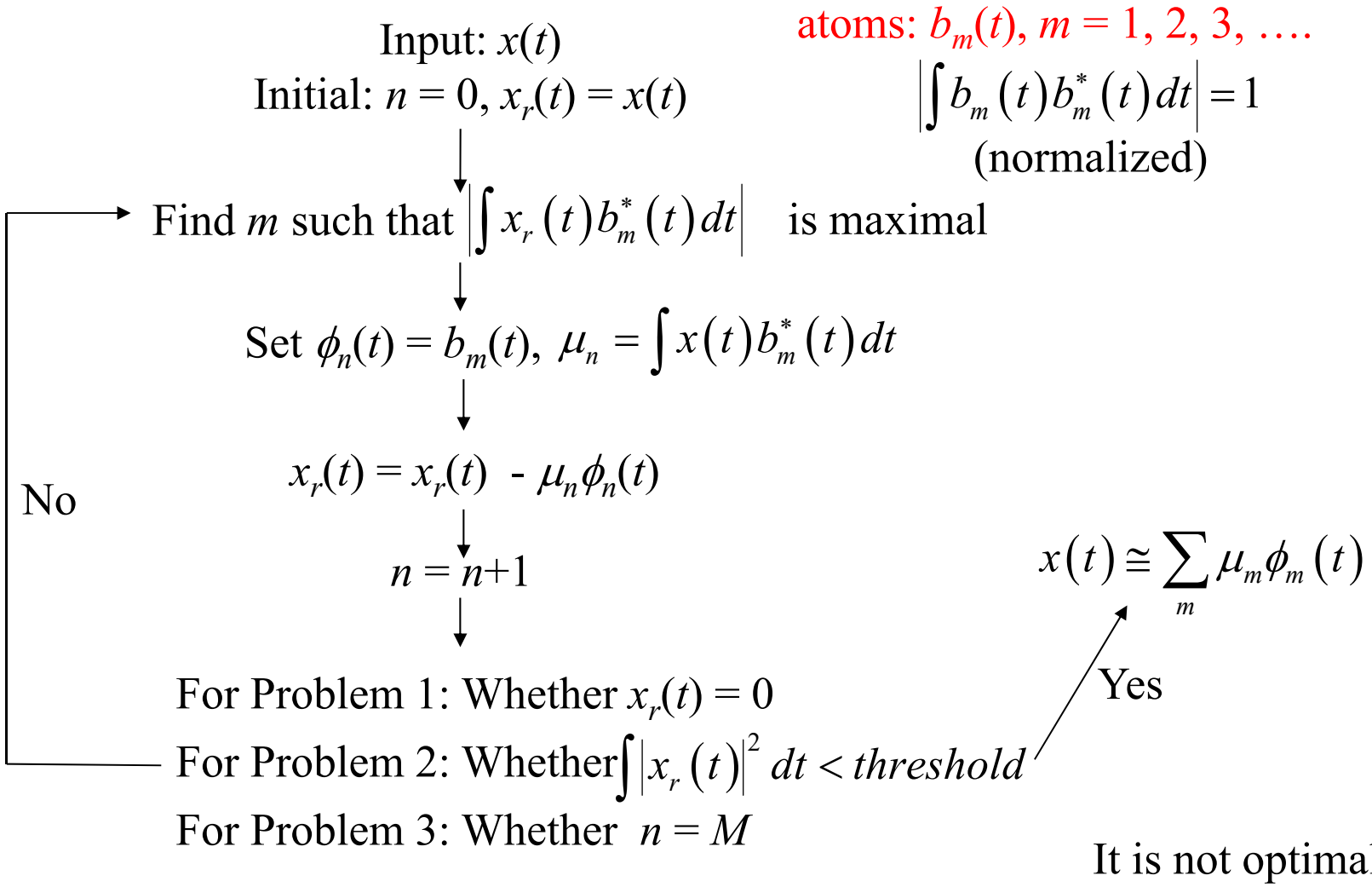
$$\int\left(x(t)-\sum_m c_m b_m(t)\right)^2 dt < threshold$$

(Problem 3) When $\|c\|_0$ is fixed to M, we want to minimize

$$\int\left(x(t)-\sum_{m=1}^{M} c_m b_m(t)\right)^2 dt$$

Question: How do we solve the optimization problems on page 231?

Method 1: Matching Pursuit (Greedy Algorithm)



Method 2: Basis Pursuit

Change the L_0 norm into the L_1 norm

$$\|c\|_1 = |c_0| + |c_1| + |c_2| + \ldots\ldots$$

(Problem 1) We want to minimize $\|c\|_1$ such that

$$x(t) = \sum_m c_m b_m(t)$$

(Problem 2) We want to minimize $\|c\|_1$ such that

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt < threshold$$

(Problem 3) When $\|c\|_1 \le M$, we want to minimize

$$\int \left(x(t) - \sum_m c_m b_m(t) \right)^2 dt$$

Definitions of Generalized Norms

Norm (L_α norm): $\|x[n]\|_\alpha = \sqrt[\alpha]{\sum_{n=0}^{N-1} |x[n]|^\alpha}$

$|x[n]|^0 = 1$ if $x[n] \neq 0$

$0^0 = 0$

$\lim_{\alpha \to 0} (L_\alpha \text{ norm})^\alpha = K$ where K is the number of points such that $x[n] \neq 0$

$\sum_{n=0}^{N-1} |x[n]|^0$

(Physical meaning: The number of nonzero points)

L_1 norm: $\|x[n]\|_1 = \sum_{n=0}^{N-1} |x[n]|$

(Physical meaning: Sum of Amplitudes)

L_2 norm: $\|x[n]\|_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2}$

(Physical meaning: Distance)

Matching Pursuit: Zero order norm $\lim_{\alpha \to 0} (L_\alpha \text{ norm})^\alpha$

Basis Pursuit: First order norm L_1 norm

The L_α norm is convex if $\alpha \geq 1$.

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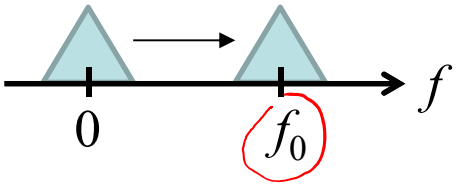
- W. He and T. Qu, "Audio lossless coding/decoding method using basis pursuit algorithm," IEEE Int. Conf. Acoustics, Speech and Signal Processing, pp. 552-555, May 2013. (Using basis pursuit for signal compression)

VIII. Motions on the Time-Frequency Distribution

Fourier spectrum 為 1-D form，只有二種可能的運動或變形：

- Modulation $e^{j2\pi f_0 t}x(t) \xrightarrow{FT} X(f-f_0)$



- Scaling $x(t/a) \xrightarrow{FT} |a|X(af)$

Time-frequency analysis 為 2-D，在 2-D 平面上有多種可能的運動或變形

(1) Horizontal shifting

(2) Vertical shifting

(3) Dilation

(4) Shearing

(5) Generalized Shearing

(6) Rotation

(7) Twisting

8-1 Basic Motions

(1) Horizontal Shifting

$$x(t-t_0) \rightarrow S_x(t-t_0, f)e^{-j2\pi f t_0} \quad \text{,STFT, Gabor}$$
$$\rightarrow W_x(t-t_0, f) \quad \text{,Wigner}$$

(2) Vertical Shifting modulation

$$e^{j2\pi f_0 t}x(t) \rightarrow S_x(t, f-f_0) \quad \text{,STFT,Gabor}$$
$$\rightarrow W_x(t, f-f_0) \quad \text{,Wigner}$$

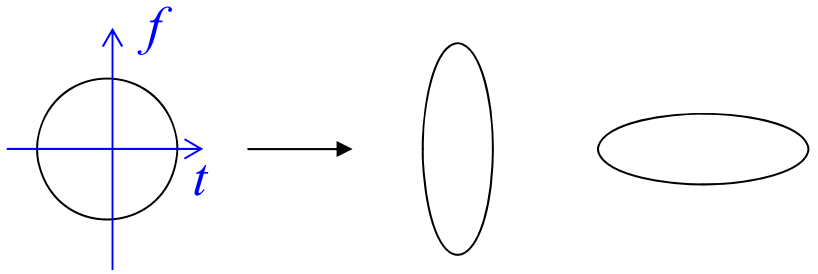
inst. freq. $= f_0$

(3) Dilation (scaling)

(the area is unchanged)

$$\frac{1}{\sqrt{|a|}} x(\frac{t}{a}) \to \approx S_x(\frac{t}{a}, af) \text{ ,STFT,Gabor}$$

$$\to W_x(\frac{t}{a}, af) \text{ ,WDF}$$



which one corresponds to $|a|>1$

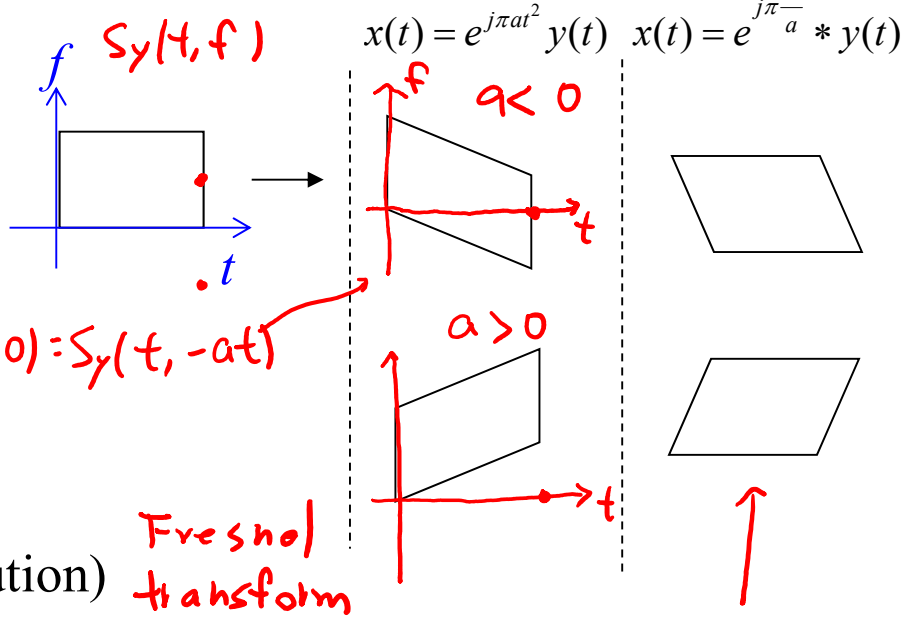
inst. freq. = at

(4) Shearing

$$x(t) = e^{j\pi a t^2} y(t)$$

$$S_x(t,f) \approx S_y(t, f-at) \text{ ,STFT,Gabor}$$

$$W_x(t,f) = W_y(t, f-at) \text{ ,WDF}$$



$S_x(t,0) = S_y(t,-at)$

$$x(t) = e^{j\pi \frac{t^2}{a}} * y(t) \quad (* \text{ means convolution})$$

$$S_x(t,f) \approx S_y(t-af, f) \text{ , STFT,Gabor}$$

$$W_x(t,f) = W_y(t-af, f) \text{ , WDF}$$

Fresnel transform

$e^{j\pi \frac{t^2}{\lambda z}} * y(t)$

$a = \lambda z$

Which one corresponds to a>0 ?

$e^{j\pi \frac{t^2}{a}}$ FT $\int_{-\infty}^{\infty} e^{j\pi \frac{t^2}{a} - j2\pi ft} dt = \sqrt{ja}\, e^{\frac{ja 4\pi^2 f^2}{4\pi}}$

page 88, $a \to -\frac{j\pi}{a}$, $b \to j2\pi f$ $= \sqrt{ja}\, e^{j\pi a f^2}$

$X(f) = \sqrt{ja}\, e^{j\pi a f^2} Y(f)$

How to do

Q:

$x(t) = e^{j\phi(t)} y(t)$

$\phi(t)$ is the 3rd order polynomial.

(Proof): When $x(t) = e^{j\pi at^2} y(t)$,

$$
\begin{aligned}
W_x(t,f) &= \int_{-\infty}^{\infty} x(t+\tau/2)\, x^*(t-\tau/2)\, e^{-j2\pi\tau f} \cdot d\tau \\
&= \int_{-\infty}^{\infty} e^{j\pi a(t+\tau/2)^2} e^{-j\pi a(t-\tau/2)^2} y(t+\tau/2)\, y^*(t-\tau/2)\, e^{-j2\pi\tau f} d\tau \\
&= \int_{-\infty}^{\infty} e^{j2\pi at\tau} y(t+\tau/2)\, y^*(t-\tau/2)\, e^{-j2\pi\tau f} d\tau \\
&= \int_{-\infty}^{\infty} y(t+\tau/2)\, y^*(t-\tau/2)\, e^{-j2\pi\tau(f-at)} d\tau \\
&= W_y(t, f-at)
\end{aligned}
$$

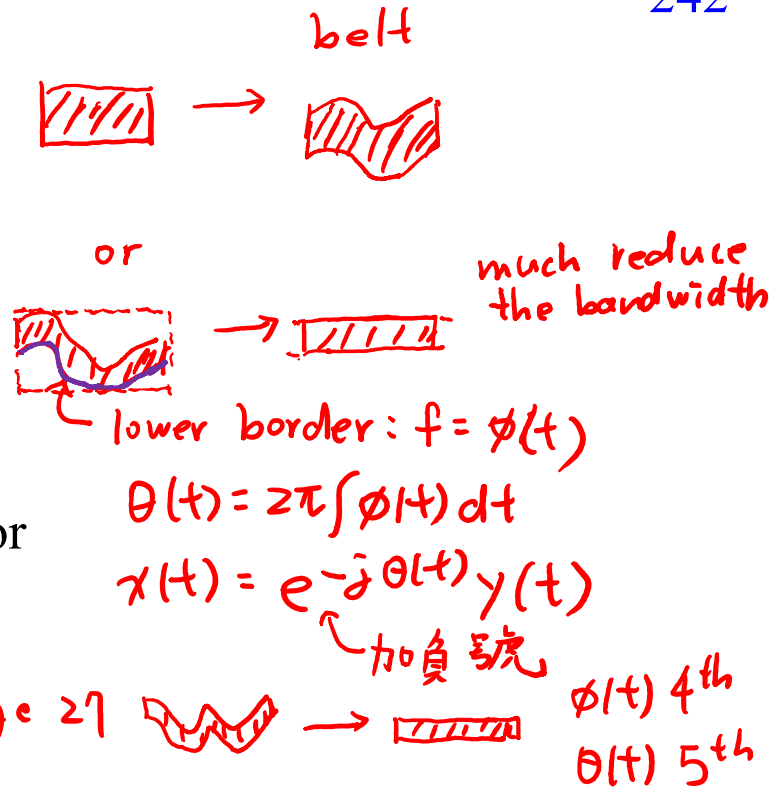
(5) Generalized Shearing

$$x(t) = e^{j\phi(t)} y(t) \quad \text{(What is its effect?)}$$

$$\phi(t) = \sum_{k=0}^{n} a_k t^k$$

$$S_x(t,f) \cong S_y\left(t, f - \sum_{k=1}^{n} k a_k t^{k-1}\right), \text{STFT, Gabor}$$

$$W_x(t,f) \cong W_y\left(t, f - \sum_{k=1}^{n} k a_k t^{k-1}\right), \text{WDF}$$



J. J. Ding, S. C. Pei, and T. Y. Ko, "Higher order modulation and the efficient sampling algorithm for time variant signal," *European Signal Processing Conference*, pp. 2143-2147, Bucharest, Romania, Aug. 2012.

J. J. Ding and C. H. Lee, "Noise removing for time-variant vocal signal by generalized modulation," *APSIPA ASC*, pp. 1-10, Kaohsiung, Taiwan, Oct. 2013

Q:

If $x(t) = h(t) * y(t)$ where $h(t) = IFT\left(\exp\left(j\sum_{k=0}^{n} a_k f^k \right) \right)$

then

$$S_x(t,f) \cong S_y(t + \frac{1}{2\pi}\sum_{k=1}^{n} k a_k f^{k-1}, \quad f) \text{ ,STFT,Gabor}$$

$$W_x(t,f) \cong W_y(t + \frac{1}{2\pi}\sum_{k=1}^{n} k a_k f^{k-1}, \quad f) \text{ ,WDF}$$

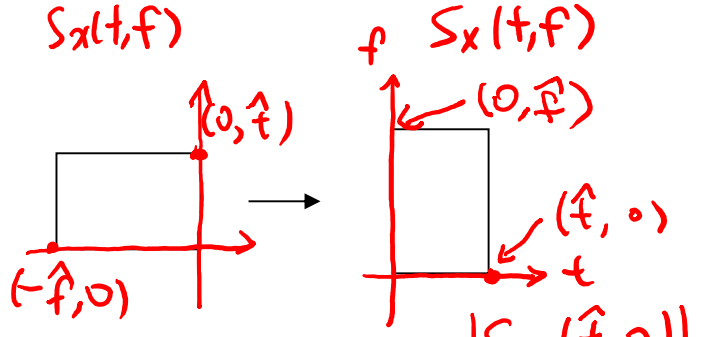
8-2 Rotation by $\pi/2$: Fourier Transform

$$X(f) = FT(x(t))$$

$$|S_X(t,f)| \approx |S_x(-f,t)| \quad \text{,STFT}$$

$$G_X(t,f) = G_x(-f,t)e^{-j2\pi ft} \quad \text{,Gabor}$$

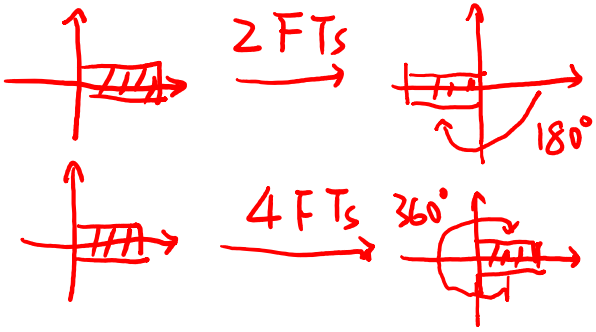
$$W_X(t,f) = W_x(-f,t) \quad \text{,WDF}$$



(clockwise rotation by 90°)

$|S_x(\hat{f},0)| \approx |S_x(0,\hat{t})|$

$|S_x(0,\hat{f})| \approx |S_x(-\hat{f},0)|$



$FT^2(x(t)) = x(-t)$

$FT^3(x(t)) = X(-f)$

$FT^4(x(t)) = x(t)$

Strictly speaking, the rec-STFT have no rotation property.

For Gabor transforms, if

$$G_x(t,f)=\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau \;,$$

$$G_X(t,f)=\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} X(\tau)\,d\tau \qquad X(f)=FT\left[x(t)\right]=\int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\,dt$$

then $G_X(t,f)=G_x(-f,t)e^{-j2\pi t f}$

(clockwise rotation by 90° for amplitude)

(Proof): $G_X(t,f)=\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau}\int_{-\infty}^{\infty} x(u) e^{-j2\pi\tau u}\,du\,d\tau$

$$=\int_{-\infty}^{\infty} x(u) e^{-\pi(\tau-t)^2}\left(\int_{-\infty}^{\infty} e^{-j2\pi\tau(f+u)}\,d\tau\right)du$$

$$=\int_{-\infty}^{\infty} x(u)\left(\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi\tau(f+u)}\,d\tau\right)du=\int_{-\infty}^{\infty} x(u)\left(FT\left(e^{-\pi(\tau-t)^2}\right)\Big|_{f\to f+u}\right)du$$

Since $\quad FT\left(e^{-\pi\tau^2}\right)=e^{-\pi f^2}, \quad FT\left(e^{-\pi(\tau-t)^2}\right)=e^{-j2\pi tf}e^{-\pi f^2}$

$$G_X(t,f)=\int_{-\infty}^{\infty} x(u) e^{-j2\pi t(f+u)} e^{-\pi(f+u)^2}\,du$$

$$=e^{-j2\pi tf}\int_{-\infty}^{\infty} x(u) e^{-j2\pi tu} e^{-\pi(u-(-f))^2}\,du = G_x(-f,t)e^{-j2\pi t f}$$

If we define the Gabor transform as

$$G_x(t,f) = e^{j\pi f t}\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\,d\tau \quad ,$$

and $G_X(t,f) = e^{j\pi f t}\int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} X(\tau)\,d\tau$

then $G_X(t,f) = G_x(-f,t)$

If $W_x(t,f)=\int_{-\infty}^{\infty} x(t+\tau/2)\cdot x^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$ is the WDF of $x(t)$,

$W_X(t,f)=\int_{-\infty}^{\infty} X(t+\tau/2)\cdot X^*(t-\tau/2)e^{-j2\pi\tau f}\cdot d\tau$ is the WDF of $X(f)$,

then $W_X(t,f)=W_x(-f,t)$

(clockwise rotation by 90°)

Question: Do other time-frequency distributions also have the similar rotation property?

- If $X(f) = IFT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{j2\pi f t} dt$, then

$$W_X(t,f) = W_x(f,-t), \qquad G_X(t,f) = G_x(f,-t) e^{j2\pi t f}$$

(counterclockwise rotation by 90°).

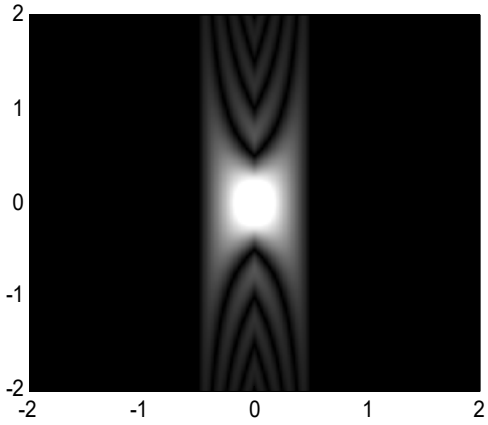
- If $X(f) = x(-t)$, then

$$W_X(t,f) = W_x(-t,-f) \ , \qquad G_X(t,f) = G_x(-t,-f).$$

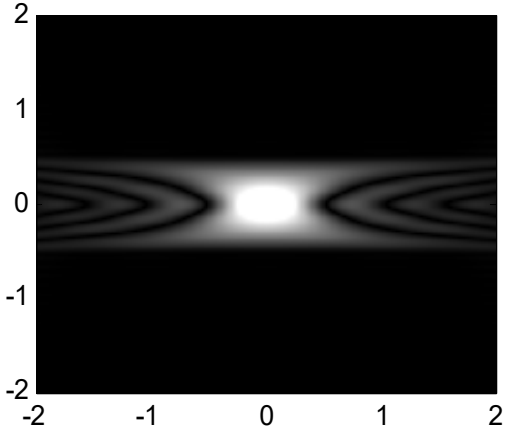
(rotation by 180°).

Examples: $x(t) = \Pi(t)$, $X(f) = FT[x(t)] = \mathrm{sinc}(f)$.

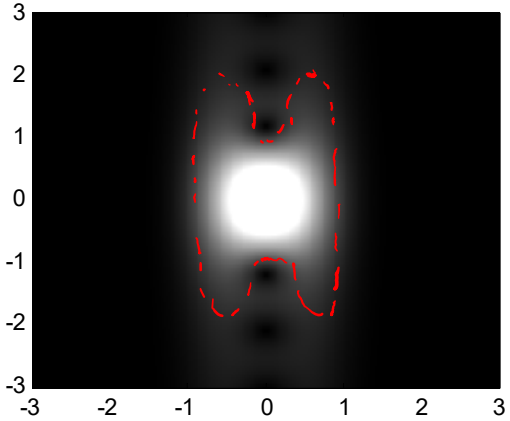
WDF of $\Pi(t)$



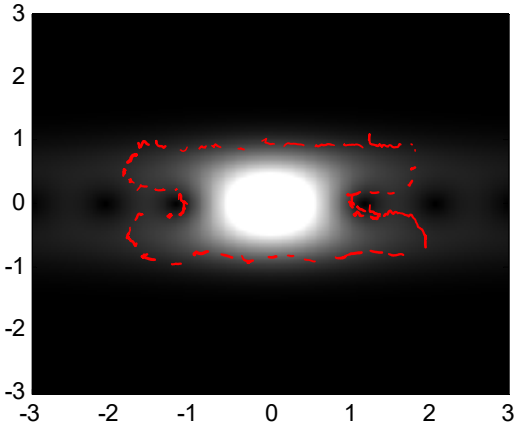
WDF of sinc(t)



Gabor transform of $\Pi(t)$



Gabor transform of sinc(t)



If a function is an eigenfunction of the Fourier transform,

$$\int_{-\infty}^{\infty} e^{-j2\pi f t} x(t)\,dt = \lambda x(f) \qquad \lambda = 1, -j, -1, j$$

then its WDF and Gabor transform have the property of

$$W_x(t,f) = W_x(f,-t) \qquad |G_x(t,f)| = |G_x(f,-t)|$$

(轉了 90º之後，和原來還是一樣)

Example: Gaussian function

$$\exp\left(-\pi t^2\right)$$

Hermite-Gaussian function

$$\phi_m(t) = \exp\left(-\pi t^2\right) H_m(t)$$

Hermite polynomials: $H_m(t) = C_m e^{2\pi t^2} \frac{d^m}{dt^m} e^{-2\pi t^2}$, C_m is some constant,

$$H_0(t) = 1 \qquad H_1(t) = t \qquad H_2(t) = 4\pi t^2 - 1$$

$$H_3(t) = 4\pi t^3 - 3t \qquad H_4(t) = 16\pi^2 t^4 - 24\pi t^2 + 3$$

$$\int_{-\infty}^{\infty} e^{-2\pi t^2} H_m(t) H_n(t) = D_m \delta_{m,n} \quad , D_m \text{ is some constant,}$$

$\delta_{m,n} = 1$ when $m = n$, $\delta_{m,n} = 0$ otherwise.

[Ref] M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, McGraw-Hill, 1990.

Hermite-Gaussian functions are eigenfunctions of the Fourier transform

$$\int_{-\infty}^{\infty} \phi_m(t) e^{-j2\pi f t} dt = (-j)^m \phi_m(f)$$

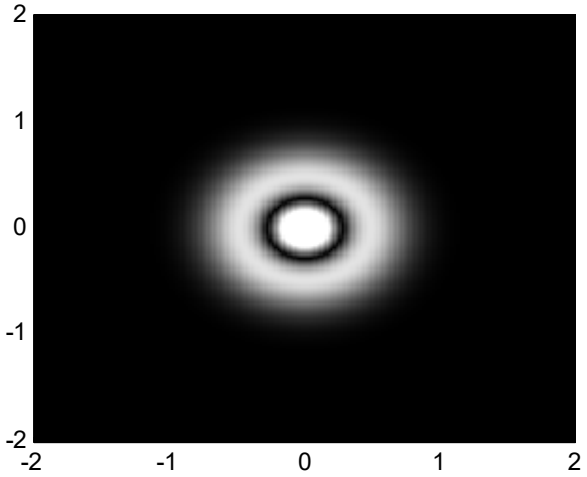
Any eigenfunction of the Fourier transform can be expressed as the form of

$$k(t) = \sum_{q=0}^{\infty} a_{4q+r} \phi_{4q+r}(t)$$

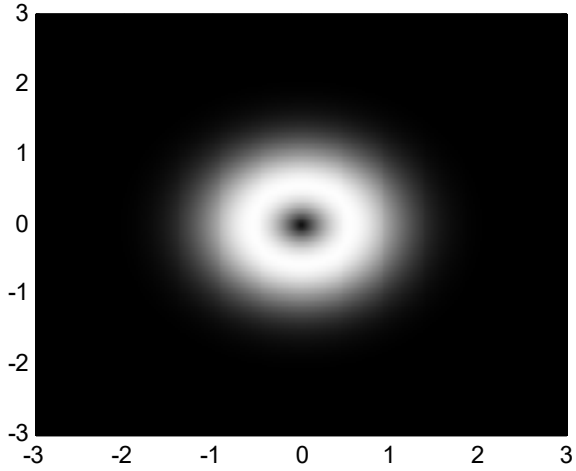
where $r = 0$, 1, 2, or 3,
a_{4q+r} are some constants

$$\int_{-\infty}^{\infty} k(t) e^{-j2\pi f t} dt = (-j)^r k(f)$$

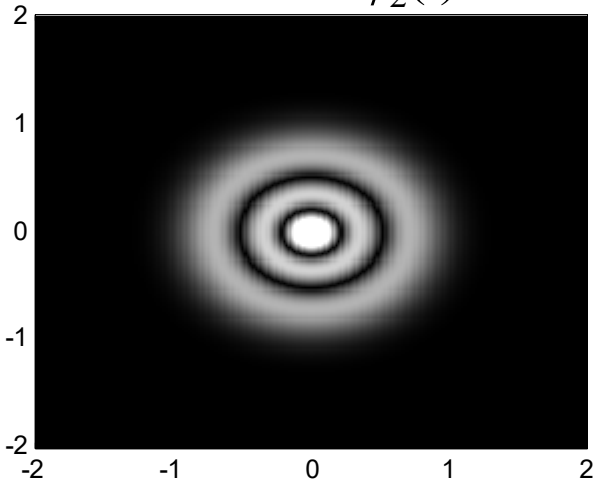
WDF for $\phi_1(t)$



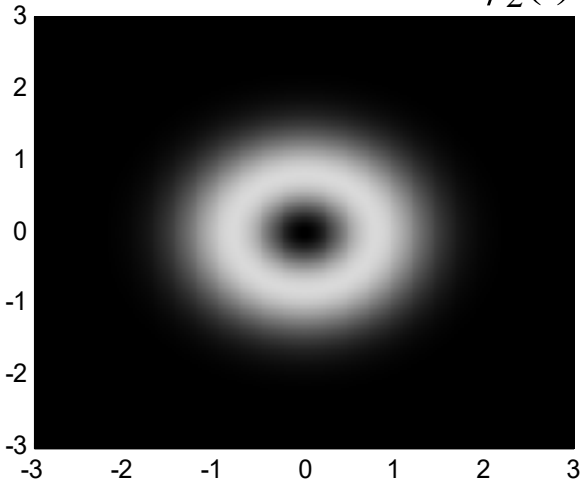
Gabor transform for $\phi_1(t)$



WDF for $\phi_2(t)$



Gabor transform for $\phi_2(t)$



Problem: How to rotate the time-frequency distribution by the angle other than $\pi/2$, π, and $3\pi/2$?

8-3 Rotation: Fractional Fourier Transforms (FRFTs)

$$X_\phi(u) = \sqrt{1 - j\cot\phi}\; e^{j\pi\cot\phi\cdot u^2} \int_{-\infty}^{\infty} e^{-j2\pi\cdot\csc\phi\cdot u\,t}\, e^{j\pi\cdot\cot\phi\cdot t^2}\, x(t)\,dt \;, \qquad \phi = 0.5a\pi$$

chirp mul. + scaled FT + chirp mul.

$a : \frac{\phi}{0.5\pi}$

When $\phi = 0.5\pi$, the FRFT becomes the FT.

Additivity property:

If we denote the FRFT as O_F^ϕ (i.e., $X_\phi(u) = O_F^\phi[x(t)]$)

then $O_F^\sigma\left\{O_F^\phi[x(t)]\right\} = O_F^{\phi+\sigma}[x(t)]$

Physical meaning: Performing the FT *a* times.

Another definition $$X_{\phi}(u)=\sqrt{\frac{1-j\cot\phi}{2\pi}}\,e^{j\frac{\cot\phi}{2}\cdot u^2}\int_{-\infty}^{\infty}e^{-j\csc\phi\cdot u\,t}\,e^{j\frac{\cot\phi}{2}\cdot t^2}\,x(t)dt$$

[Ref] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing*, New York, John Wiley & Sons, 2000.

[Ref] N. Wiener, "Hermitian polynomials and Fourier analysis," *Journal of Mathematics Physics MIT*, vol. 18, pp. 70-73, 1929.

[Ref] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *J. Inst. Maths. Applics.*, vol. 25, pp. 241-265, 1980.

[Ref] L. B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE Trans. Signal Processing*, vol. 42, no. 11, pp. 3084-3091, Nov. 1994.

[Ref] S. C. Pei and J. J. Ding, "Closed form discrete fractional and affine Fourier transforms," *IEEE Trans. Signal Processing*, vol. 48, no. 5, pp. 1338-1353, May 2000.

$$FT\left[x(t)\right] = X(f)$$

$$FT\left\{FT\left[x(t)\right]\right\} = x(-t)$$

$$FT\left(FT\left\{FT\left[x(t)\right]\right\}\right) = X(-f) = IFT\left[f(t)\right]$$

$$FT\left[FT\left(FT\left\{FT\left[x(t)\right]\right\}\right)\right] = x(t)$$

What happen if we do the FT non-integer times?

Physical Meaning:

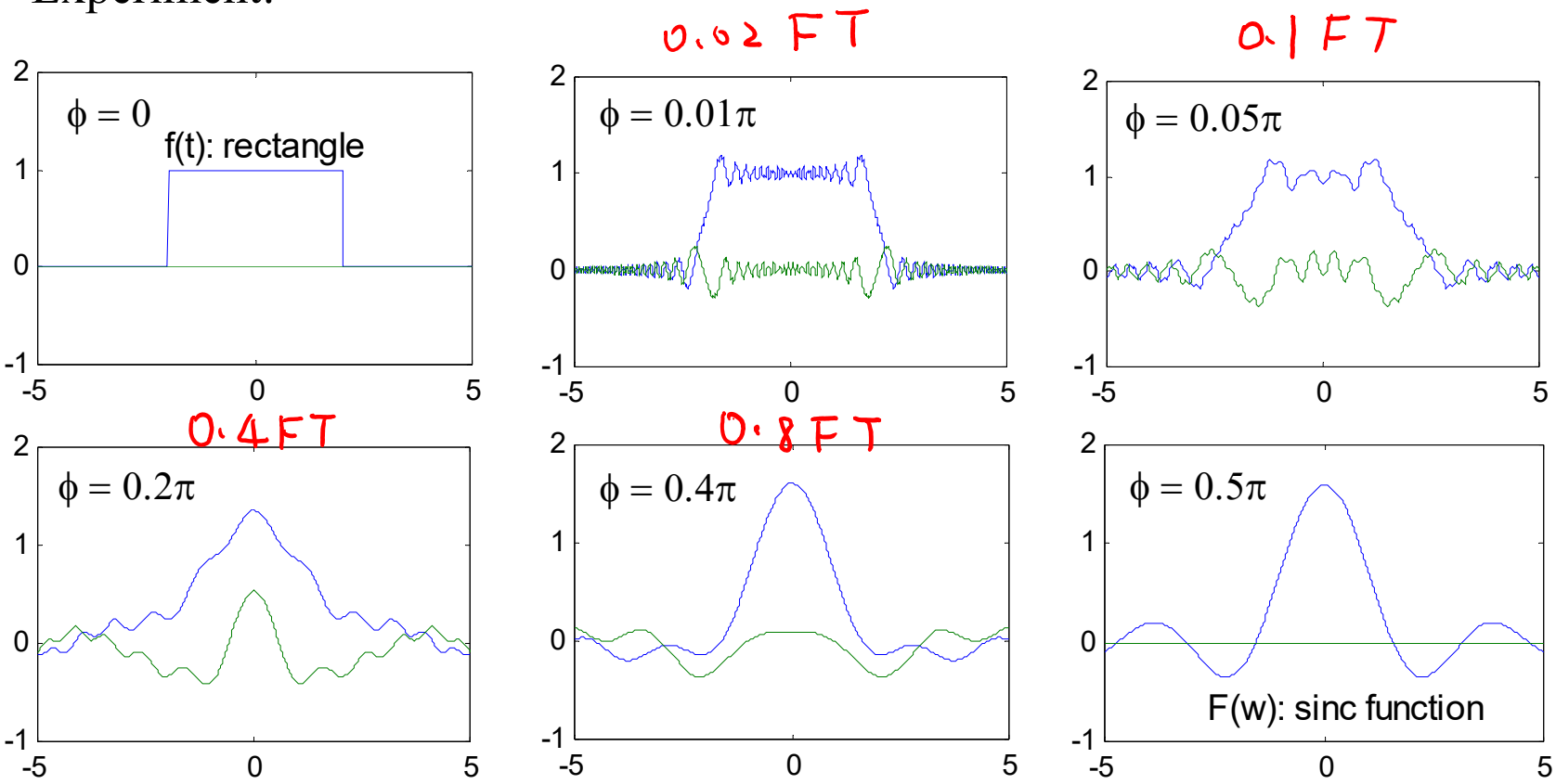
Fourier Transform: time domain $\rightarrow$ frequency domain

Fractional Fourier transform: time domain $\rightarrow$ fractional domain

Fractional domain: the domain between time and frequency

(partially like time and partially like frequency)

Experiment:



blue line: real part

green line: imaginary part

[Ref] L. B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE Trans. Signal Processing*, vol. 42, no. 11, pp. 3084-3091, Nov. 1994.

Time domain	Frequency domain	fractional domain
Modulation	Shifting	Modulation + Shifting
Shifting	Modulation	Modulation + Shifting
Differentiation	$\times j2\pi f$	Differentiation and $\times j2\pi f$
$\times -j2\pi f$	Differentiation	Differentiation and $\times -j2\pi f$

$$x(t-t_0) \xrightarrow{FT} \exp(-j2\pi f t_0) X(f)$$

$$x(t-t_0) \xrightarrow{\text{fractional FT}} \exp(j\varphi - j2\pi u t_0 \sin\phi) X(u - t_0\cos\phi)$$

$$\varphi = \pi t_0^2 \sin\phi\cos\phi$$

$$\frac{d}{dt}x(t) \xrightarrow{FT} j2\pi f\, X(f)$$

$$\frac{d}{dt}x(t) \xrightarrow{\text{fractional FT}} j2\pi u\, X(u)\sin\phi + \frac{d}{du}X(u)\cos\phi$$

[Theorem] The fractional Fourier transform (FRFT) with angle ϕ is equivalent to the **clockwise rotation operation** with angle ϕ for the Wigner distribution function (or for the Gabor transform)

FRFT with parameter ϕ = **with angle ϕ**

For the WDF

If $W_x(t, f)$ is the WDF of $x(t)$, and $W_{X\phi}(u, v)$ is the WDF of $X_\phi(u)$, ($X_\phi(u)$ is the FRFT of $x(t)$), then

$$W_{X_\phi}(u,v) = W_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)$$

For the Gabor transform (with standard definition)

If $G_x(t, f)$ is the Gabor transform of $x(t)$,

and $G_{X\phi}(u, v)$ is the Gabor transform of $X_\phi(u)$, then

$$G_{X_\phi}(u,v) = e^{j[-2\pi uv\sin^2\phi+\pi(u^2-v^2)\sin(2\phi)/2]} G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)$$

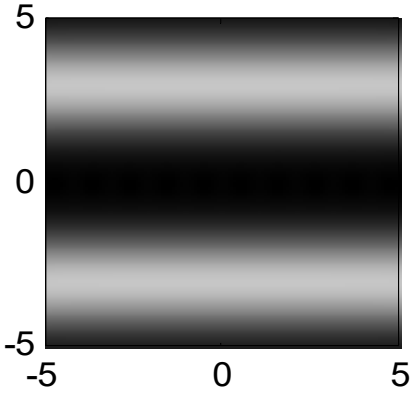
$$\left|G_{X_\phi}(u,v)\right| = \left|G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)\right|$$

For the Gabor transform (with another definition on page 246)

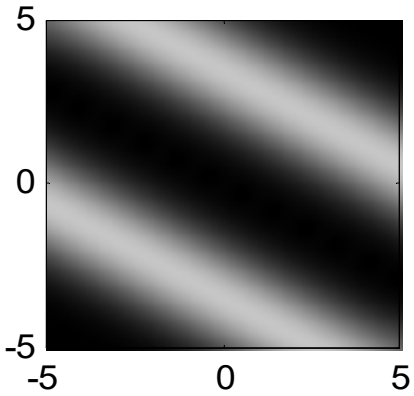
$$G_{X_\phi}(u,v) = G_x(u\cos\phi - v\sin\phi, u\sin\phi + v\cos\phi)$$

The Cohen's class distribution and the Gabor-Wigner transform also have the rotation property

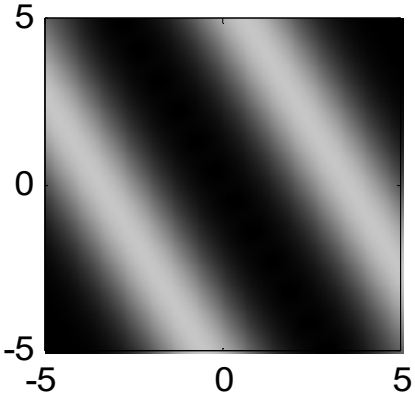
The Gabor Transform for the FRFT of a cosine function



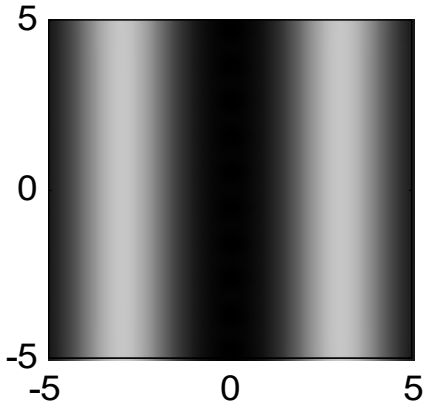
(a) $\phi = 0$



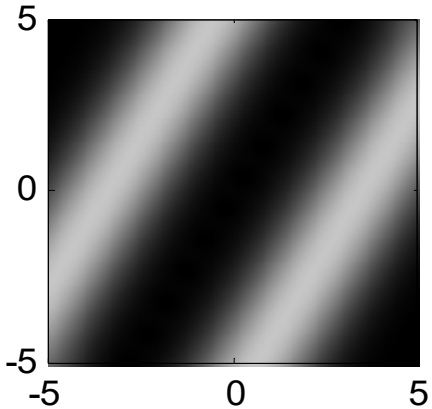
(b) $\phi = \pi/6$ ($\frac{1}{3}$ FT)



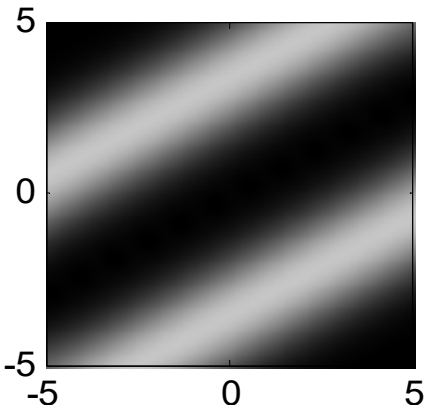
(c) $\phi = 2\pi/6$ ($\frac{2}{3}$ FT)



(d) $\phi = 3\pi/6$

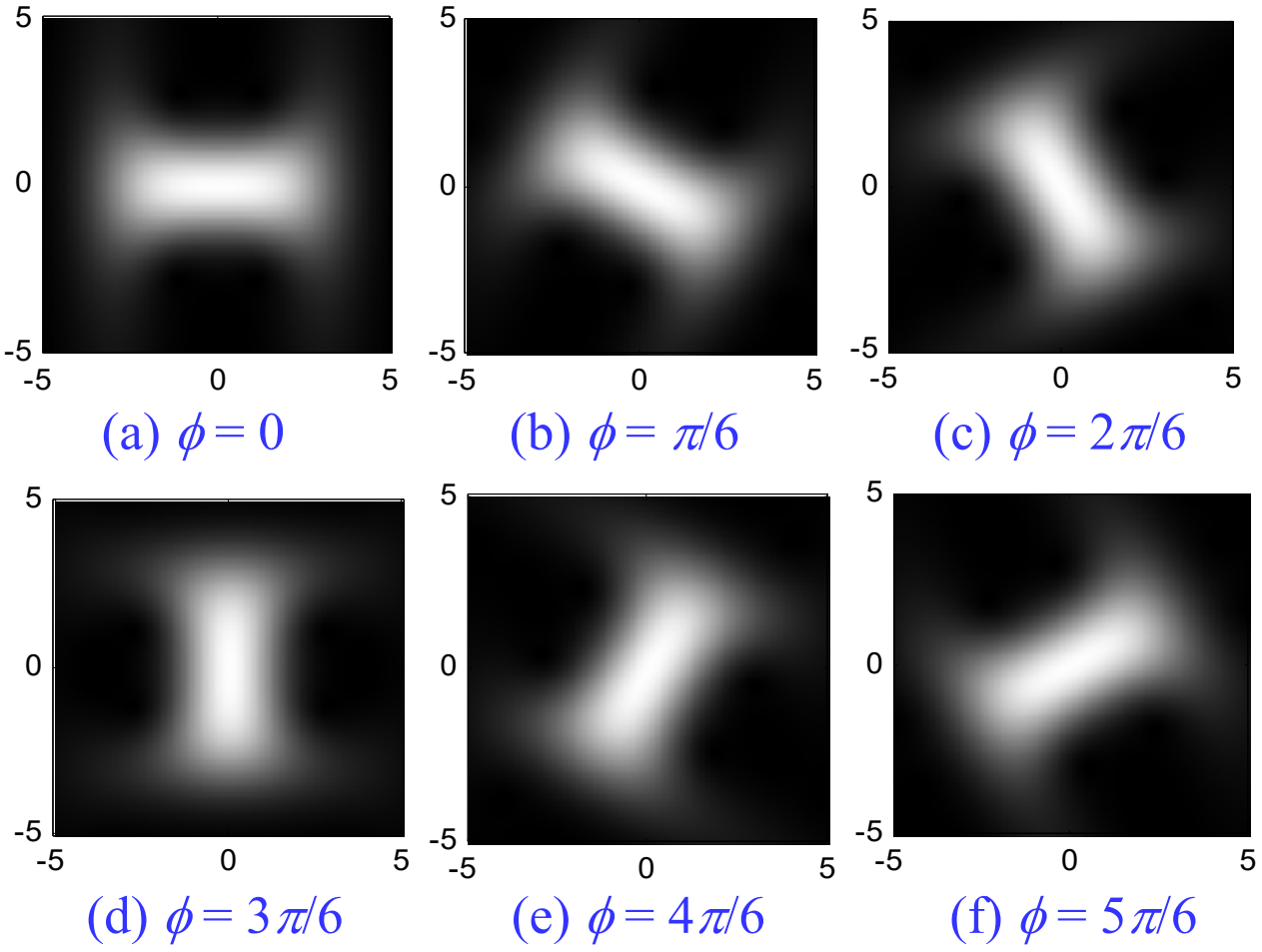


(e) $\phi = 4\pi/6$



(f) $\phi = 5\pi/6$

The Gabor Transform for the FRFT of a rectangular function.



(a) $\phi = 0$ (b) $\phi = \pi/6$ (c) $\phi = 2\pi/6$

(d) $\phi = 3\pi/6$ (e) $\phi = 4\pi/6$ (f) $\phi = 5\pi/6$

8-4 Twisting: Linear Canonical Transform (LCT)

完整

chirp mul. + scaled FT + chirp mul.

$$X_{(a,b,c,d)}(u) = \sqrt{\frac{1}{jb}}\, e^{j\pi \frac{d}{b}u^2} \int_{-\infty}^{\infty} e^{-j2\pi \frac{1}{b}ut}\, e^{j\pi \frac{a}{b}t^2}\, x(t)\,dt \qquad \text{when } b \neq 0$$

$$X_{(a,0,c,d)}(u) = \sqrt{d}\cdot e^{j\pi cd\,u^2}\, x(d\,u) \qquad \text{when } b = 0$$

$ad - bc = 1$ should be satisfied

Four parameters a, b, c, d

Additivity property of the WDF

If we denote the LCT by $O_F^{(a,b,c,d)}$, i.e., $X_{(a,b,c,d)}(u) = O_F^{(a,b,c,d)}[x(t)]$

then $$O_F^{(a_2,b_2,c_2,d_2)}\left\{O_F^{(a_1,b_1,c_1,d_1)}[x(t)]\right\} = O_F^{(a_3,b_3,c_3,d_3)}[x(t)]$$

where $$\begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

[Ref] K. B. Wolf, "*Integral Transforms in Science and Engineering*," Ch. 9: Canonical transforms, New York, Plenum Press, 1979.

If $W_{X_{(a,b,c,d)}}(u,v)$ is the WDF of $X_{(a,b,c,d)}(u)$, where $X_{(a,b,c,d)}(u)$ is the LCT of $x(t)$, then

$$W_{X_{(a,b,c,d)}}(u,v) = W_x(du - bv, -cu + av)$$

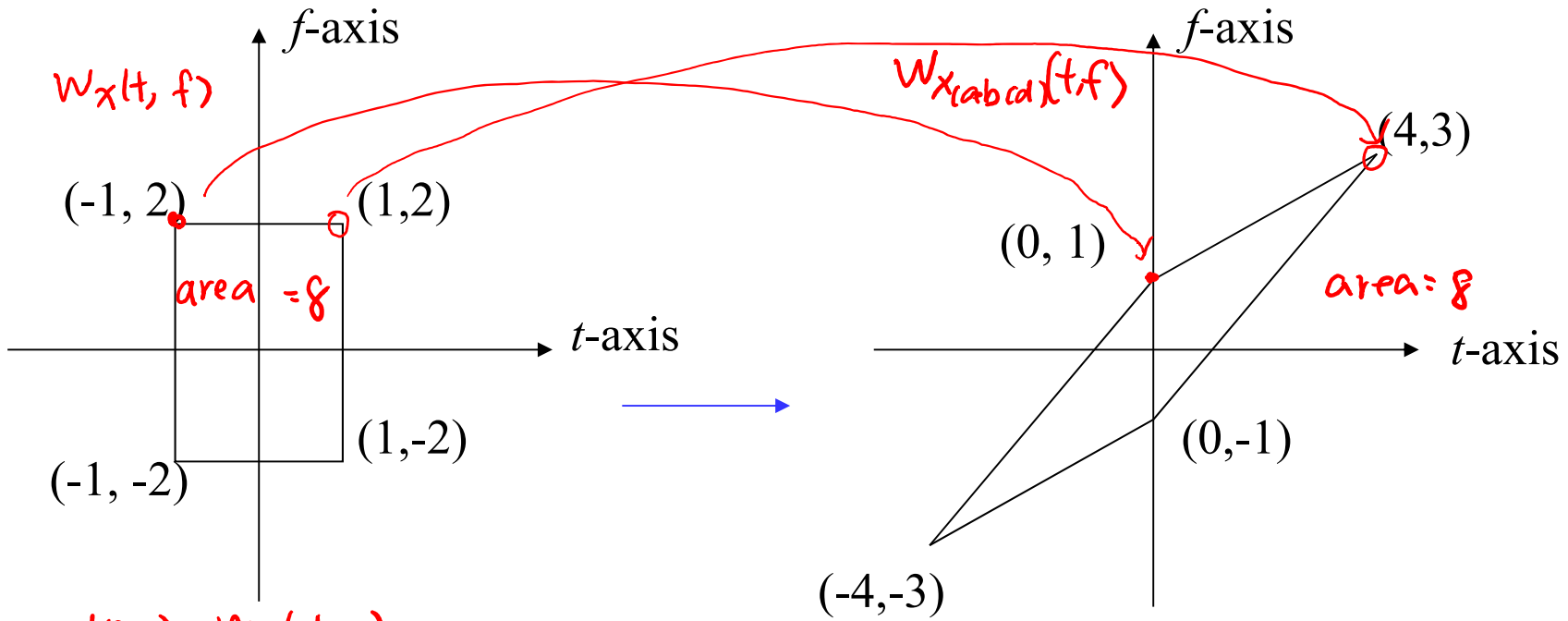
$$W_{X_{(a,b,c,d)}}(au + bv, cu + dv) = W_x(u,v)$$

LCT == twisting operation for the WDF

The Cohen's class distribution also has the twisting operation.

我們可以自由的用 LCT 將一個中心在 (0, 0) 的平行四邊形的區域，扭曲成另外一個面積一樣且中心也在 (0, 0) 的平行四邊形區域。

(0,0) 對應 (0,0)



$W_x(t, f)$

$W_{x[abcd]}(t,f)$

$W_{x[abcd]}(0,1) = W_x(-1,2)$

$\therefore$ When $(u,v) = (-1,2)$

$(au+bv, cu+dv) = (0,1)$

Similarly, $W_{x[abcd]}(4,3) = W_x(1,2)$

$\therefore$ When $(u,v) = (1,2)$

$(au+bv, cu+dv) = (4,3)$

$\begin{cases} -a+2b = 0 \\ a+2b = 4 \end{cases}$ $\begin{cases} -c+2d = 1 \\ c+2d = 3 \end{cases}$

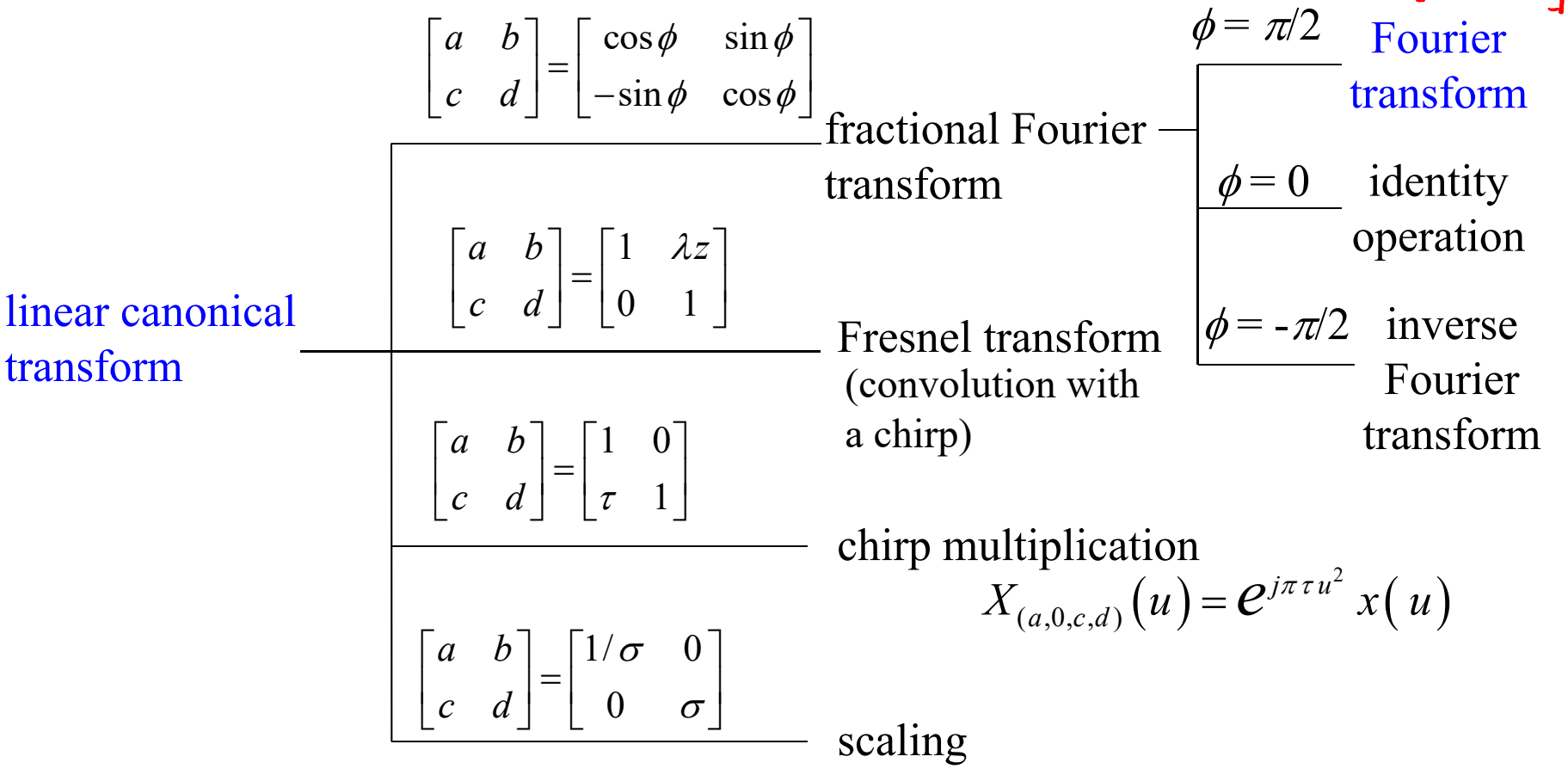
$a = 2, b = 1$ $c = 1, d = 1$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$X_{(a,b,c,d)}(u) = \sqrt{\frac{1}{jb}}\, e^{j\pi\frac{d}{b}u^2} \int_{-\infty}^{\infty} e^{-j2\pi\frac{1}{b}ut}\, e^{j\pi\frac{a}{b}t^2}\, x(t)dt \quad \text{when } b \neq 0$$

$$X_{(a,0,c,d)}(u) = \sqrt{d}\cdot e^{j\pi cdu^2}\, x(du) \quad \text{when } b = 0$$

$ad - bc = 1$ should be satisfied



附錄十一　Linear Canonical Transform 和光學系統的關係

(1) Fresnel Transform (電磁波在空氣中的傳播) = chirp convolution

$$U_o(x,y) = -\frac{i}{\lambda}\frac{e^{ikz}}{z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}\left[(x-x_i)^2+(y-y_i)^2\right]} U_i(x_i,y_i)\,dx_i dy_i$$

$\frac{a}{b}=\frac{1}{\lambda z}$, $\frac{1}{b}=\frac{1}{\lambda z}$, $\frac{d}{b}=\frac{1}{\lambda z}$

$k = 2\pi/\lambda$: wave number　　λ: wavelength　　z: distance of propagation

$\frac{k}{2z}=\frac{\pi}{\lambda z}$

$e^{j\frac{\pi}{\lambda z}x^2}\, e^{-j\frac{2\pi}{\lambda z}xx_i}\, e^{j\frac{\pi}{\lambda z}x_i^2}$

$b=\lambda z$, $a=d=1$ from $ad-bc=1$, $c=0$

$$U_o(x,y) = e^{ikz}\sqrt{\frac{1}{j\lambda z}}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}(y-y_i)^2}\sqrt{\frac{1}{j\lambda z}}\int_{-\infty}^{\infty} e^{j\frac{k}{2z}(x-x_i)^2} U_i(x_i,y_i)\,dx_i dy_i$$

(2 個 1-D 的 LCT)

$$e^{j\frac{k}{2z}x^2} * U_i(x,y_i) = e^{j\frac{\pi}{\lambda z}x^2} * U_i(x,y_i)$$

Fresnel transform 相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z \\ 0 & 1 \end{bmatrix}$

(2) Spherical lens, refractive index = n

multiplied by chirp

$$U_o(x,y) = e^{ikn\Delta} e^{-j\frac{k}{2f}\left[x^2+y^2\right]} U_i(x,y)$$

$-\frac{k}{2f} : \frac{-\pi}{\lambda f}$

f: focal length Δ: thickness of lens

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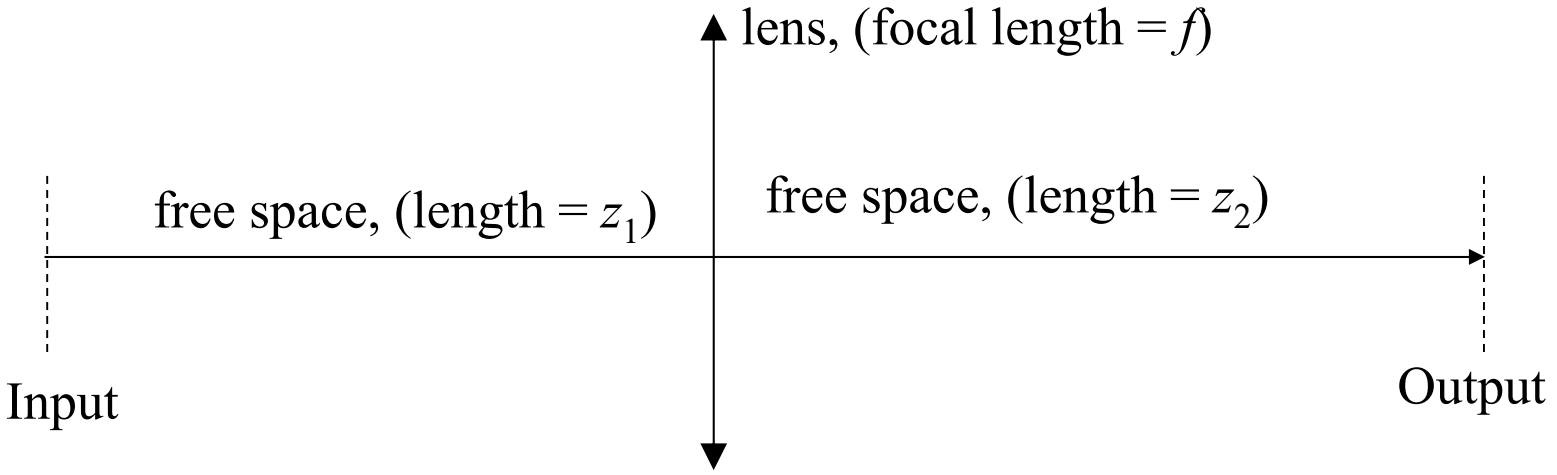
經過 lens 相當於 LCT $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix}$ 的情形

(3) Free spaces + Spherical lens



Input 和 output 之間的關係，可以用 LCT 表示

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \lambda z_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/\lambda f & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda z_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\frac{z_2}{f} & \lambda(z_1+z_2)-\frac{\lambda z_1 z_2}{f} \\ -\frac{1}{\lambda f} & 1-\frac{z_1}{f} \end{bmatrix}$$

LCT: $X_{(a,b,c,d)}(u)=\sqrt{\frac{1}{jb}}\,e^{j\pi\frac{d}{b}u^2}\int_{-\infty}^{\infty}e^{-j2\pi\frac{1}{b}ut}e^{j\pi\frac{a}{b}t^2}\,x(t)dt$

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}1-\frac{z_2}{f} & \lambda(z_1+z_2)-\frac{\lambda z_1 z_2}{f}\\ -\frac{1}{\lambda f} & 1-\frac{z_1}{f}\end{bmatrix}$$

$z_1=z_2=2f$ → 即高中物理所學的「倒立成像」

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}-1 & 0\\ -\frac{1}{\lambda f} & -1\end{bmatrix}$$

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$X(u)=\sqrt{\frac{1}{\lambda}}e^{j\pi\frac{1}{\lambda f}u^2}x(-u)$

$|X(u)|=|x(-u)|$

$z_1=z_2=f$ → Fourier Transform + Scaling

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}0 & \lambda f\\ -\frac{1}{\lambda f} & 0\end{bmatrix}$$

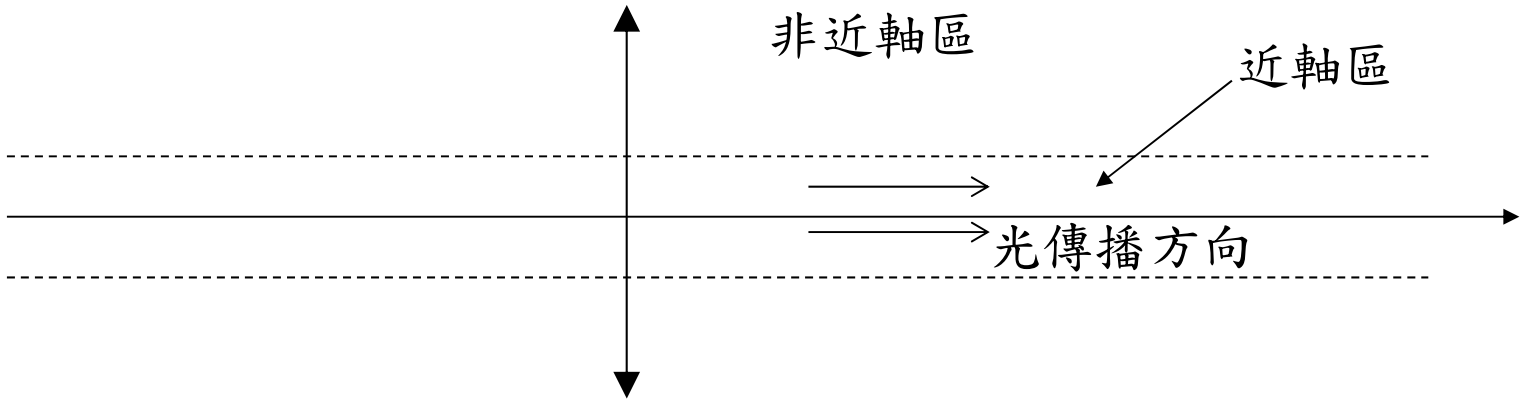
$=\begin{bmatrix}\lambda f & 0\\ 0 & \frac{1}{\lambda f}\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

$z_1=z_2$ → fractional Fourier Transform + Scaling

用 LCT 來分析光學系統的好處：

只需要用到 2×2 的矩陣運算，避免了複雜的物理理論和數學積分

但是 LCT 來分析光學系統的結果，只有在「近軸」的情形下才準確



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[2] L. M. Bernardo, “ABCD matrix formalism of fractional Fourier optics,” *Optical Eng.*, vol. 35, no. 3, pp. 732-740, March 1996.